Lecture Notes on Optimal State Feedback using LMIs (DRAFT)

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November 25, 2010

1 Introduction

TO BE COMPLETED.

2 Notation and preliminaries

This section summarizes well-known results on matrix theory, Lyapunov equations, and stochastic processes. Most statements are presented without proof and references are provided at the end of this document.

We use $\mathbb{R}$ for the real numbers, $\mathbb{C}$ for the complex numbers, $\mathbb{N}_0$ for the nonnegative integers.

2.1 Matrices

Given any matrix A, A^T refers to its transpose, A^H refers to its conjugate transpose, A_{*i} to its i^{th} column, A_{i*} to its i^{th} row, A_{ij} ti its $(i,j)^{th}$ element. If A is a square matrix, then $\operatorname{trace}\{A\}$ refers to its trace, $\det(A)$ to its determinant, A^{-1} to its inverse (if it exists). If $x \in \mathbb{C}$, then $\bar{x}$ stands for the conjugate of x, and $|x|$ for its magnitude.

Definition 1. *Given a square (possibly complex-valued) matrix A, we say that $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $Ax = \lambda x$, for some non-zero vector x. The vector x is the eigenvector associated to λ.* ■

Definition 2. *The spectrum of a matrix A is the set of all eigenvalues A. In symbols,* $\operatorname{spec}\{A\} \triangleq \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } A\}$. ■

Definition 3. *The spectral radius of a matrix A is the largest magnitude of an eigenvalue of A. In symbols,* $\rho(A) \triangleq \max_{\lambda \in \operatorname{spec}\{A\}} |\lambda|$. ■

Fact 1. *If A is a square matrix, then* $\operatorname{spec}\{A\} = \operatorname{spec}\left\{A^T\right\}$ *and* $\operatorname{spec}\{A\} = \overline{\operatorname{spec}\{A^H\}}$. ■

Fact 2. *If A is square and $\rho(A) < 1$, then:*

1. *$I - A$ is nonsingular and $(I-A)^{-1} = \sum_{i=0}^{\infty} A^k$.*

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2. *The sequence of vectors* $\{q(k); k \in \mathbb{N}_0\}$*, where*

$$q(k+1) = Aq(k) + b, \quad q(0) = q_o, \quad k \in \mathbb{N}_0, \tag{1}$$

converges to a limit that does not depend on q_o*. Moreover,* $\lim_{k\to\infty} q(k) = (I - A)^{-1}b$*.*

3. *The series* $\sum_{i=0}^{\infty} A^i Q A^{iT}$*, where* Q *is any real matrix, converges.* ■

Definition 4. *Given a matrix* A *with* n *columns, we define*

$$\operatorname{vec}\{A\} = \begin{bmatrix} A_{*1}^T & \cdots & A_{*n}^T \end{bmatrix}^T. \tag{2}$$

The operator $\operatorname{vec}\{\cdot\}$ *is usually referred to as the column stacking operator. We will use* $\operatorname{vec}^{-1}\{\cdot\}$ *to denote the corresponding inverse operator.* ■

Definition 5. *Given an* $n \times m$ *matrix* A *and any other matrix* B*, we define*

$$A \otimes B \triangleq \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nm} \end{bmatrix}. \tag{3}$$

$A \otimes B$ *is called Kronecker product between* A *and* B*.* ■

Fact 3. *Assume that* A*,* B *and* C *are matrices of appropriate dimensions. Then:*

1. $\operatorname{vec}\{ABC\} = \left(C^T \otimes A\right) \operatorname{vec}\{B\}$.

2. $\operatorname{spec}\{A \otimes B\} = \{\lambda_A \lambda_B \in \mathbb{C} : \lambda_A \in \operatorname{spec}\{A\}, \lambda_B \in \operatorname{spec}\{B\}\}$. ■

Definition 6. *We say that a matriz* M *is positive definite and write* $M > 0$ *(resp. positive-semidefinite and write* $M \geq 0$*) if and only* $M = M^H$ *and* $x^H M x > 0$ *(resp.* $x^H M x \geq 0$*) for every nonzero vector* x*.* ■

Fact 4. *Assume that* M *and* X *are matrices of appropriate dimensions.*

1. *If* M, N *and* X *are matrices of appropriate dimensions and* $X = X^H$*, then* $MXM^H \geq 0$*.*

2. $X > 0$ *if and only if* $X^{-1} > 0$*.*

3. *If* $M > 0$*, then* $X > 0$ *if and only if* $MXM > 0$*.* ■

Lemma 1 (Schur complement)**.** *If* $Q = Q^H$*,* $R = R^H$ *and* S *are matrices of appropriate dimensions, then*

$$\begin{bmatrix} R & S \\ S^H & Q \end{bmatrix} > 0. \tag{4}$$

if and only if $Q > 0$ *and* $R - SQ^{-1}S^H > 0$*.* ■

2.2 Lyapunov equations

A Lyapunov equation is an equation of the form

$$P = APA^T + Q, \tag{5}$$

where A and Q are given matrices of appropriate dimensions, and P is the matrix-valued unknown.

Theorem 1. *Consider* (5) *and assume that* A *and* Q *are given real matrices. Then:*

1. *If* $\rho(A) < 1$*, then there exists a unique solution to* (5) *given by*

$$P = \text{vec}^{-1}\left\{(I - A \otimes A)^{-1}\,\text{vec}\,\{Q\}\right\} = \sum_{i=0}^{\infty} A^i Q A^{iT}. \tag{6}$$

2. *Assume that* $Q > 0$*. Then, there exists* $P > 0$ *satisfying* (5) *if and only if* $\rho(A) < 1$*.*
3. *There exists* $P > 0$ *such that* $P > APA^T$ *if and only if* $\rho(A) < 1$*.*
4. *Assume that* Q *is symmetric, that* $\rho(A) < 1$*, and that* $P_o = AP_oA^T + Q$*. If* $P \geq APA^T + Q$ *(resp.* $P > APA^T + Q$*), then* $P \geq P_o$ *(resp.* $P > P_o$*).*

Proof:

1. Using Fact 3, (5) can be written as

$$(I - A \otimes A)\,\text{vec}\,\{P\} = \text{vec}\,\{Q\}\,. \tag{7}$$

Since $\rho(A) < 1$, $(I - A \otimes A)$ is nonsingular (see Facts 2 and 3) and we thus see that there exist a unique vec $\{P\}$ satisfying (7). This proves that there exists a unique solution to (5) given by the middle term in (6).

Since $\rho(A) < 1$, the infinite series in (6) converges (Fact 2). Therefore,

$$A\left(\sum_{i=0}^{\infty} A^i Q A^{iT}\right)A^T = \sum_{i=1}^{\infty} A^i Q A^{iT} = \sum_{i=0}^{\infty} A^i Q A^{iT} - Q, \tag{8}$$

which shows that the solution to (5) can be written as the infinite series in (6).

2. If $\rho(A) < 1$, then Part 1 implies that the solution to (5) satisfies $P = Q + \sum_{i=1}^{\infty} A^i Q A^{iT}$. Therefore, if $Q > 0$, then $P > 0$. To prove the converse, assume that x is an eigenvector associated to the eigenvalue λ of A^T (by definition, $x \neq 0$). Thus, if $Q > 0$ and P satisfies (5), then

$$x^H P x = x^H APA^T x + x^H Q x > |\lambda|^2\, x^H P x \quad \Rightarrow \quad (1 - |\lambda|^2)\, x^H P x > 0. \tag{9}$$

Since, in addition, $P > 0$, it follows that $|\lambda| < 1$ necessarily holds. Since λ is an arbitrary eigenvalue of A^T (and therefore of A), the result follows.

3. Assume that $\rho(A) < 1$ and pick an arbitrary $Q > 0$. Then, Part 2 implies that there exists $P > 0$ such that $P = APA^T + Q > APA^T$. On the other hand, if there exists $P > 0$ such that $P > APA^T$, then there exists $Q > 0$ such that $P = APA^T + Q$, and Part 2 implies that $\rho(A) < 1$.

4. We will prove the non-strict case only. The strict one follows similarly. Since $\rho(A) < 1$ and Q is symmetric, there exist unique $P_o \geq 0$ and $P \geq 0$ satisfying

$$P_o = \sum_{i=0}^{\infty} A^i Q A^{iT}, \quad P = \sum_{i=0}^{\infty} A^i \left(Q + N\right) A^{iT}, \tag{10}$$

where $N \geq 0$ is such that $P = APA^T + Q + N$. Therefore,

$$P - P_o = \sum_{i=0}^{\infty} A^i N A^{iT} \geq 0 \tag{11}$$

and the result follows. ■

The explicit expression for P given in (6) is not necessarily well suited for numerical calculations. (In Matlab, Lyapunov equations can be solved reliably by using the command `dlyap`.)

2.3 Stochastic processes

In this writing, we consider only real vector-valued discrete-time stochastic processes (also called discrete-parameter processes) with well defined probability density functions. A given stochastic process $\{x(k); x \in \mathbb{N}_0\}$ will be usually referred to as x. We also use $\mathcal{E}\{\cdot\}$ for the expectation operator.

The following notions are defined for stochastic processes, but can be readily particularized to random variables:

Definition 7. *If x is a process, then its mean and variance matrix at time instant k are defined by*

$$\mu_x(k) \triangleq \mathcal{E}\left\{x(k)\right\}, \quad P_x(k) \triangleq \mathcal{E}\left\{\left[x(k) - \mu_x(k)\right]\left[x(k) - \mu_x(k)\right]^T\right\}, \tag{12}$$

respectively. ■

Definition 8. *We say that x is a second order (stochastic) process if and only if $\mu_x(k)$ and $P_x(k)$ exist and are finite for every $k \in \mathbb{N}_0$ (and remain bounded as $k \to \infty$).* ■

Definition 9. *We say the processes x and y are uncorrelated if and only if*

$$\mathcal{E}\left\{\left[x(i) - \mu_x(i)\right]\left[y(j) - \mu_y(j)\right]^T\right\} = 0, \tag{13}$$

for every $i, j \in \mathbb{N}_0$. ■

Definition 10. *We say that a process x is a white noise sequence if and only if*

$$\mathcal{E}\left\{\left[x(i) - \mu_x(i)\right]\left[x(j) - \mu_x(j)\right]^T\right\} = 0, \tag{14}$$

for every $i, j \in \mathbb{N}_0$, $i \neq j$. ■

3 Analysis of LTI systems with white noise inputs

Consider a discrete-time linear time-invariant (LTI) system described by

$$x(k+1) = Ax(k) + Bd(k), \quad x(0) = x_o, \quad k \in \mathbb{N}_0, \tag{15a}$$

$$y(k) = Cx(k) + Dd(k), \tag{15b}$$

where x is the system state, d is an input, and y is an output. The signals x, d and y are allowed to have arbitrary dimensions, and the constant real matrices (A, B, C, D) are assumed to be of appropriate dimensions.

We will work under the following assumptions on x_o and d:

Assumption 1.

1. *The initial state x_o is a second order random variable having mean μ_o and variance matrix $P_o \geq 0$.*

2. *The input d is a second order white noise sequence uncorrelated with x_o, and having constant mean μ_d and constant variance matrix $P_d \geq 0$.* ■

We start our study of LTI systems excited with white noise by establishing simple recursive formulae for the mean and variance matrix of the state:

Lemma 2. *Consider the LTI system in* (15) *and suppose that Assumption 1 holds. Then, for every $k \in \mathbb{N}_0$,*

$$\mu_x(k+1) = A\mu_x(k) + B\mu_d, \quad \mu_x(0) = \mu_o, \tag{16}$$

$$P_x(k+1) = AP_x(k)A^T + BP_dB^T, \quad P_x(0) = P_o. \tag{17}$$

Proof: Taking expectation on both sides of (15a) yields

$$\mu_x(k+1) = \mathcal{E}\left\{Ax(k) + Bd(k)\right\} = A\mu_x(k) + B\mu_d, \tag{18}$$

where we exploited the linearity of $\mathcal{E}\left\{\cdot\right\}$. Similarly, (15a) yields

$$\begin{aligned} P_x(k+1) &= \mathcal{E}\left\{\left[x(k+1) - \mu_x(k+1)\right]\left[x(k+1) - \mu_x(k+1)\right]^T\right\} \\ &= \mathcal{E}\left\{\left[A\left(x(k) - \mu_x(k)\right) + B\left(d(k) - \mu_d(k)\right)\right]\left[A\left(x(k) - \mu_x(k)\right) + B\left(d(k) - \mu_d(k)\right)\right]^T\right\} \\ &= AP_x(k)A^T + BP_dB^T + A\mathcal{E}\left\{\left[x(k) - \mu_x(k)\right]\left[d(k) - \mu_d(k)\right]^T\right\}B^T \\ &\qquad + B\mathcal{E}\left\{\left[d(k) - \mu_d(k)\right]\left[x(k) - \mu_x(k)\right]^T\right\}A^T. \end{aligned} \tag{19}$$

Now, note that

$$x(k) = A^k x_o + \sum_{i=0}^{k-1} A^{k-1-i}Bd(i). \tag{20}$$

Therefore,

$$\begin{aligned} &\mathcal{E}\left\{\left[x(k) - \mu_x(k)\right]\left[d(k) - \mu_d(k)\right]^T\right\} \\ &\quad = \mathcal{E}\left\{\left[A^k\left(x_o - \mu_o\right) + \sum_{i=0}^{k-1} A^iB\left(d(i) - \mu_d(i)\right)\right]\left[d(k) - \mu_d(k)\right]^T\right\} \\ &\quad = A^k\mathcal{E}\left\{\left[x_o - \mu_o\right]\left[d(k) - \mu_d(k)\right]^T\right\} + \sum_{i=0}^{k-1} A^iB\mathcal{E}\left\{\left[d(i) - \mu_d(i)\right]\left[d(k) - \mu_d(k)\right]^T\right\}. \end{aligned} \tag{21}$$

Thus, if x_o and d are uncorrelated, then the first term in (21) is zero and, if d is white, then the second term in (21) is zero (note that $i < k$). The result thus follows. ■

A relevant question is whether or not the recursions in Lemma 2 converge as k grows unbounded. We show below that the answer to this question is very simple. Before doing so, we introduce a notion that is usually used to jointly refer to the convergence of (16) and (17) (see also [6,12]):

Definition 11. *The LTI system in* (15) *is mean square stable (MSS) if and only if, for any initial state x_o and input d satisfying Assumption 1,*

$$\mu_x \triangleq \lim_{k\to\infty} \mu_x(k), \quad P_x \triangleq \lim_{k\to\infty} P_x(k) \tag{22}$$

exist, are finite, and do not depend on the statistics of x_o. ■

Remark 1. *We will abuse the MSS acronym and use it to also refer to mean square stability.* ■

In other words, a system is MSS if and only if its state has finite stationary mean and finite stationary variance matrix whenever the initial state and (white) input have finite mean and variances.

We remark that, since (15) considers a random input d, the usual notions of asymptotic stability or bounded-input bounded-output (BIBO) stability (see, e.g., [8,18]) do not make sense. MSS extends the notion of BIBO stability in a natural way to the present situation where the inputs of interest are not (deterministically) bounded.

Theorem 2. *The LTI system in* (15) *is MSS if and only if $\rho(A) < 1$.*

Proof: Part 2 of Fact 2 and (16) imply that $\mu_x(k)$ converges to a limit than does not depend on μ_o if and only if $\rho(A) < 1$. The same reasoning can be applied to analyze the convergence of $P_x(k)$. To that end, use Fact 3 to rewrite (17) as

$$\operatorname{vec}\{P_x(k+1)\} = (A\otimes A)\operatorname{vec}\{P_x(k)\} + \operatorname{vec}\left\{BP_dB^T\right\}, \quad \operatorname{vec}\{P_x(0)\} = \operatorname{vec}\{P_o\}. \tag{23}$$

Since $\rho(A\otimes A) < 1$ if and only if $\rho(A) < 1$, Part 2 of Fact 2 implies that $P_x(k)$ converges to a limit that does not depend on P_o if and only if $\rho(A) < 1$. This completes the proof. ■

Theorem 2 shows that the notion of MSS is equivalent to that of asymptotic stability and, therefore, to most (basic) notions of stability for LTI systems. This is a key simplifying fact that facilitates the analysis and design of LTI systems in a stochastic framework.

We end this section by stating two simple consequences of Lemma 2 and Theorem 2. We start by giving closed form expressions for the stationary mean and the stationary variance matrix of the state in a MSS system:

Corollary 1. *Consider the setup and assumptions of Lemma 2. If the LTI system is MSS, then*

$$\mu_x = (I-A)^{-1}B\mu_d, \quad P_x = \operatorname{vec}^{-1}\left\{(I-A\otimes A)^{-1}\operatorname{vec}\left\{BP_dB^T\right\}\right\}, \tag{24}$$

where μ_x and P_x are as in Definition 11.

Proof: If the LTI system is MSS, then (16) implies that $\mu_x = A\mu_x + B\mu_d$ and the first equality in (24) follows. (Note that $\rho(A) < 1$ guarantees that $I - A$ is nonsingular and thus μ_x is well defined and unique, as expected.) Similarly, the second equality follows by proceeding as in the proof of Part 1 of Theorem 1. ■

Consistent with the comments in Section 2.2, we emphasize that the explicit expression for P_x in (24) is usually unsuitable for numerical calculations. In Matlab, P_x can be calculated by using `dlyap` to solve the equation $P_x = AP_xA^T + BP_uB^T$.

So far, we have only focused on the system state x. Next corollary gives expressions for the mean and variance of the output y in (15b):

Corollary 2. *Consider the setup and assumptions of Lemma 2. Then, for every $k \in \mathbb{N}_0$,*

$$\mu_y(k) = C\mu_x(k) + D\mu_d, \quad P_y(k) = CP_x(k)C^T + DP_dD^T. \tag{25}$$

Moreover, if the LTI system is MSS, then $\mu_y \triangleq \lim_{k\to\infty} \mu_y(k)$ and $P_y \triangleq \lim_{k\to\infty} P_y(k)$ exist, are finite, do not depend on the statistics of x_o, and satisfy

$$\mu_y = C\mu_x + D\mu_d, \quad P_y = CP_xC^T + DP_dD^T, \tag{26}$$

where expressions for μ_x and P_x are given in (24).

Proof: The result follows upon using the definition of MSS, Corollary 1, and by proceeding as in the proof of Lemma 2. ■

It is worth noting that, in general, the converse of the last statement of Corollary 2 is not true. That is, the existence of μ_y and P_y does not always imply MSS. For instance, the LTI system

$$x(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(k), \quad x(0) = x_o, \tag{27}$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k), \tag{28}$$

is not MSS, but μ_y and P_y exist whenever x_o and d satisfy Assumption 1. (The convergence of the mean and variance of the output y is equivalent to MSS if and only if (C, A) is detectable [13, 18].)

4 Minimum variance control in the state feedback case

We will now apply the results in the previous sections to solve a simple optimal control problem. Consider the LTI system

$$x(k+1) = Ax(k) + B_uu(k) + B_dd(k), \quad x(0) = x_o, \quad k \in \mathbb{N}_0, \tag{29a}$$

$$e(k) = Cx(k) + D_{ue}u(k) + D_{de}d(k), \tag{29b}$$

where x is the state, u is the control input, d is a disturbance, and e is an output. All signals are allowed to have arbitrary dimensions, and the constant real matrices $(A, B_u, B_d, C, D_{ue}, D_{de})$ are assumed to be of appropriate dimensions.

We assume that the state x can be measured (without noise) and focus on static control laws such that

$$u(k) = Kx(k), \quad \forall k \in \mathbb{N}_0, \tag{30}$$

where K is a constant real matrix of appropriate dimensions.

Problem 1. *Consider the LTI system in* (29)*, the control law in* (30)*, and suppose that Assumption 1 holds. Find*

$$J_{\text{opt}} \triangleq \inf_{K\in\mathcal{S}} \text{trace}\{P_e\}, \tag{31}$$

$$K_{\text{opt}} \triangleq \arg\inf_{K\in\mathcal{S}} \text{trace}\{P_e\}, \tag{32}$$

where $\mathcal{S}$ denotes the set of all real matrices such that $\rho(A+B_uK) < 1$, and $P_e \triangleq \lim_{k\to\infty} P_e(k)$ denotes the stationary variance matrix of e. ■

Remark 2. *Given Theorem 2 and Corollary 2, P_e exists for every $K \in \mathcal{S}$. We also note that $\mathcal{S}$ is nonempty if and only if the pair (A, B_u) is stabilizable [13, 18]. (If $\mathcal{S}$ is empty, then, as usual, we set $J_{\text{opt}} = +\infty$ [5].)* ■

Problem 1 is a minimum variance control problem, where one is interested in finding the static state feedback gain K that, within the class of all gains that render the resulting closed loop system MSS, minimizes the stationary variance of the output e. Problem 1 has received much attention in the literature [1, 13]. The traditional approach for solving this problem is based on Riccati equations [13]. Instead of describing that approach here, we will consider an alternative point of view, where J_{opt} and K_{opt} are characterized in terms of the solution to a convex optimization problem. To do so, we start by stating the following lemma, which play a central role when proving the main result of this section (see Theorem 3 below):

Lemma 3. *Consider the setup and assumptions of Problem 1. If $\Lambda > 0$ is a given matrix of appropriate dimensions, then the following two conditions are equivalent:*

1. $\rho(A + B_u K) < 1$ *and* $P_e < \Lambda$.
2. *There exists* $X > 0$ *such that*

$$X > (A + B_u K) X (A + B_u K)^T + B_d P_d B_d^T, \tag{33}$$
$$\Lambda > (C + D_{ue} K) X (C + D_{ue} K)^T + D_{de} P_d D_{de}^T. \tag{34}$$

Proof: When the control law (30) is used to control the system in (29), the resulting closed loop can be described by (cf. (15)).

$$x(k+1) = (A + B_u K) x(k) + B_d u(k), \quad x(0) = x_o, \quad k \in \mathbb{N}_0, \tag{35}$$
$$e(k) = (C + D_{ue} K) x(k) + D_{de} d(k). \tag{36}$$

Therefore, if $\rho(A + B_u K) < 1$, then the stationary variances of x and e exist, are unique, and satisfy

$$P_x = (A + B_u K) P_x (A + B_u K)^T + B_d P_d B_d^T, \tag{37}$$
$$P_e = (C + D_{ue} K) P_x (C + D_{ue} K)^T + D_{de} P_d D_{de}^T. \tag{38}$$

We now use these facts to prove the lemma.

- (1. $\Rightarrow$ 2.) If $\rho(A + B_u K) < 1$, then $P_x \geq 0$ is unique and satisfies

$$P_x = \sum_{i=0}^{\infty} (A + B_u K)^i B_d P_d B_d^T (A + B_u K)^{iT}. \tag{39}$$

  Define

$$X_\epsilon \triangleq \sum_{i=0}^{\infty} (A + B_u K)^i \left(B_d P_d B_d^T + \epsilon I \right) (A + B_u K)^{iT}, \tag{40}$$

  where $\epsilon \geq 0$. For every $\epsilon > 0$, $X_\epsilon > P_x \geq 0$ and

$$\begin{aligned} X_\epsilon &= (A + B_u K) X_\epsilon (A + B_u K)^T + B_d P_d B_d^T + \epsilon I \\ &> (A + B_u K) X_\epsilon (A + B_u K)^T + B_d P_d B_d^T. \end{aligned} \tag{41}$$

On the other hand, the condition $\Lambda > P_e$ can be rewritten as

$$\Lambda > P_e = (C + D_{ue}K)P_x(C + D_{ue}K)^T + D_{de}P_dD_{de}^T. \tag{42}$$

Since X_ϵ is continuous in ϵ, $X_0 = P_x$, and $X_\epsilon > P_x$, we conclude from (42) that there exists a sufficiently small $\epsilon > 0$ such that X_ϵ satisfies

$$\Lambda > (C + D_{ue}K)X_\epsilon(C + D_{ue}K)^T + D_{de}P_dD_{de}^T. \tag{43}$$

To complete this part of the proof, choose a sufficiently small $\epsilon > 0$ such that (43) holds, and set $X = X_\epsilon$.

- (2. $\Rightarrow$ 1.) If there exists $X > 0$ such that (33) holds, then

$$\begin{aligned} X &> (A + B_uK)X(A + B_uK)^T + B_dP_dB_d^T \\ &\geq (A + B_uK)X(A + B_uK)^T \end{aligned} \tag{44}$$

and thus $\rho(A + B_uK) < 1$. Therefore, P_x exists and satisfies (37). Using Part 4 of Theorem 1, it follows from (44) and (37) that $X > P_x$ and also that (see (34))

$$\begin{aligned} \Lambda &> (C + D_{ue}K)X(C + D_{ue}K)^T + D_{de}P_dD_{de}^T \\ &> (C + D_{ue}K)P_x(C + D_{ue}K)^T + D_{de}P_dD_{de}^T \\ &= P_e. \end{aligned} \tag{45}$$

The proof is now complete. ■

Theorem 3. *Consider the setup and assumptions of Problem 1, and define the following optimization problem in the matrix variables X, Λ and Z:*

$$\text{Find:} \quad \rho \triangleq \inf \operatorname{trace}\{\Lambda\} \tag{46}$$

$$\text{Subject to:} \quad X > 0, \quad \Lambda > 0, \tag{47}$$

$$\begin{bmatrix} X - B_dP_dB_d^T & AX + B_uZ \\ \star & X \end{bmatrix} > 0, \tag{48}$$

$$\begin{bmatrix} \Lambda - D_{de}P_dD_{de}^T & CX + D_{ue}Z \\ \star & X \end{bmatrix} > 0, \tag{49}$$

where $\star$ stands for terms that can be inferred by symmetry. Then:

1. *Problem 1 is feasible if and only if there exist X and Z such that the matrix inequality in (48) is feasible.*
2. *If Problem 1 is feasible, then $J_{\text{opt}} = \rho$ and $K_{\text{opt}} = Z_{\text{opt}}X_{\text{opt}}^{-1}$, where X_{opt} and Z_{opt} denote the optimal values of X and Z in the optimization problem defined by (46)–(49).*

Proof:

1. By definition, Problem 1 is feasible if and only if there exists K such that $\rho(A+B_uK) < 1$, (i.e., if and only if the set $\mathcal{S}$ is nonempty). It follows from the the proof of Lemma 3 that $\rho(A+B_uK) < 1$

is equivalent to the existence to $X > 0$ such that (33) holds. Thus, we need to show that (48) is equivalent to (33) (note that (48) implies$X > 0$). Using Lemma 1 and Fact 4, it follows that

$$X > 0 \text{ and } (33) \quad \Leftrightarrow \quad \begin{bmatrix} X - B_d P_d B_d^T & A + B_u K \\ \star & X^{-1} \end{bmatrix} > 0 \tag{50}$$

$$\Leftrightarrow \quad \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} X - B_d P_d B_d^T & A + B_u K \\ \star & X^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} > 0 \tag{51}$$

$$\Leftrightarrow \quad \begin{bmatrix} X - B_d P_d B_d^T & AX + B_u K X \\ \star & X \end{bmatrix} > 0, \tag{52}$$

The result now follows upon defining $Z \triangleq KX$.

2. Given Lemma 3 and the proof of Part 1 above, it suffices to show that (49) is equivalent to (34). To see this, we proceed as in Part 1 and use Lemma 1 to write

$$X > 0 \text{ and } (34) \quad \Leftrightarrow \quad \begin{bmatrix} \Lambda - D_{de} P_d D_{de}^T & C + D_{ue} K \\ \star & X^{-1} \end{bmatrix} > 0 \tag{53}$$

$$\Leftrightarrow \quad \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} X - B_d P_d B_d^T & A + B_u K \\ \star & X^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} > 0 \tag{54}$$

$$\Leftrightarrow \quad \begin{bmatrix} \Lambda - D_{de} P_d D_{de}^T & CX + D_{ue} K X \\ \star & X \end{bmatrix} > 0. \tag{55}$$

The result now follows upon defining $Z \triangleq KX$ and by combining the above with the proof of Part 1. ■

The matrix inequalities in (47)–(49) are linear matrix inequalities (LMIs) and, as such, they define convex constraints on X, Λ and Z [5]. Theorem 3 thus shows that Problem 1 is equivalent to solving a convex optimization problem, for which efficient algorithms exist (see, e.g., [9]).

In the statement of Problem 1, we have imposed that the control laws are of the static state feedback type. Interestingly, it can be shown that the solution to Problem 1 remains unchanged if one removes that assumption and focuses on the class of all (possibly nonlinear and time-varying) causal mappings between the state and the plant input (see, e.g., [3, 16]). Showing this fact requires, however, ideas far beyond those discussed here. On the other hand, one could also remove the assumption of having the state available for feedback. If, in that case, one focuses on LTI control laws, then a statement analogous of that in Theorem 3 can be made. We refer the reader to [7, 14, 15] for details.

5 Further reading

The results presented in Section 2.1 are standard. Good references on matrix theory are [2, 10, 11, 17]. Theorem 1 was adapted to the discrete-time case from [7]; see also [18] for related results. Good textbooks on stochastic processes (at a "user's level") include [1, 16]. In particular, the material in Section 3 has been taken from [16]. Section 4 has been adapted to the discrete-time case from [7]. Other relevant references include [4, 14, 15].

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