Backstepping Control of a Wave PDE With Unstable Source Terms and Dynamic Boundary

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Abstract—This letter presents the design of an exponentially stabilizing controller for a one-dimensional wave partial differential equation. The control is acting on a Robin's boundary condition while the opposite boundary satisfies an unstable dynamic. The wave is also subject to unstable in-domain source terms. Closed-loop exponential stabilization is obtained via a full-state backstepping controller. The existence and uniqueness of this backstepping transformation is proven, using the method of successive approximations.

Index Terms—Distributed parameter systems.

I. INTRODUCTION

WE INVESTIGATE the feedback control problem of the following system

$$u_{tt}(x, t) = u_{xx}(x, t) + \lambda u_t(x, t) + \beta u(x, t) \tag{1}$$

$$u_x(1, t) = a_1 u_t(1, t) + a_2 u(1, t) + U(t) \tag{2}$$

$$u_{tt}(0, t) = b_1 u_t(0, t) + b_2 u_x(0, t) + b_3 u(0, t) \tag{3}$$

in which U is a scalar control input. The scalar values a_1, and a_2 are coefficients of the controlled boundary, λ is an in-domain velocity source term, β is an in-domain position source term, and the dynamical boundary coefficients are b_1, $b_2 \neq 0$, and b_3. All parameters are supposed to be known.

The main specificity of (1)-(3) consists in its boundary condition (3) (for information about wave PDEs and corresponding boundary conditions, see [5] and p. 30 in particular). This second-order dynamical boundary condition naturally arises when applying Newton second law of motion. Yet, it

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has seldom been considered in the literature. More specifically, one can refer to [6] for a study of a one-dimensional wave equation subject to a Wentzell boundary condition. System (1)-(3) can indeed be reformulated as such, replacing the derivative u_{tt} in (3) with the help of (1). One can also refer to [4] and [10] for higher order systems. However, this framework is not state-of-the-art and the stabilization of (1)-(3) remains an open problem.

In this letter, the proposed control design is based on backstepping. Note that there exists a fair amount of alternative control techniques. Indeed, most of the finite dimensional control tools have been extended to infinite dimensional systems through the theory of semigroup, e.g., frequency based control, LQR (see [2] for an introduction on semigroup theory and semigroup based-control techniques). Among other control strategies (see for example [9], in which the flatness-based tracking of a pure wave PDE ($\lambda = \beta = 0$) with two second-order dynamic boundary conditions is presented) backstepping is acknowledged for its relative simplicity for practitioners (see the monograph [7] which presents the main concepts of backstepping for infinite dimensional systems). It has already been applied in the literature in related yet different contexts. For example, we refer the reader to [14] for a treatment of a wave PDE with space-dependent in-domain velocity and position source terms ($\lambda(x)$, $\beta(x)$) associated with Dirichlet's boundary conditions (see references therein for further references about the control of the wave equation). Also, in [16], a pure wave PDE ($\lambda = \beta = 0$) with a first-order dynamic anti-damped boundary condition is considered. The controller and the boundary observer have both been designed using backstepping.

The framework of coupled PDE-ODEs can also be considered in order to deal with dynamic boundary conditions. For example Sagert *et al.* [13] express the system (1)-(3) (with $\beta = b_3 = a_1 = a_2 = 0$) as a wave PDE with Dirichlet's boundary conditions coupled with an ODE (Ordinary Differential Equation) to perform backstepping. But they need an additional assumption on the state space regularity. Another approach is to consider coupled hyperbolic PDE-ODE, as done in [8]. However, the reformulation of the wave PDE into the former framework, using Riemann invariants, may not be straightforward since it involves the position terms ($u(., t)$). Note that there seems to be no problem if $\beta = a_2 = b_3 = 0$.

For these reasons, we propose a more direct approach. As far as we know, there is no design stabilizing (1)-(3).

Our first objective is to improve our control design from [12], by allowing a change of in-domain viscous damping between the original system and the target system and thus giving an additional degree of freedom to the designer. Futhermore, in [12], the system under consideration does not involve any position source terms (case where $\beta = b_3 = a_1 = a_2 = 0$). As a result, the control problem considered here is different: in [12], the system is exponentially stabilized toward an attractor (by considering a functional of $\|u_t\|_{L_2}$, $\|u_x\|_{L_2}$, and $u_t(0, t)^2$), and in this letter the system is stabilized toward the origin.

The main contribution of this letter is the design of a backstepping control law which exponentially stabilizes the system (1)-(3). Standardly, we use a Volterra transformation mapping the original system into a stable target system and prove existence and uniqueness of this transformation using the method of successive approximations. The originality of our approach lies in our methodology to handle the dynamic boundary conditions, both in the establishment of the kernel integral equation and the method of successive approximations.

This letter is organized as follows. First, in Section II, the control law is defined and the corresponding stability result is stated. Then, Section III is dedicated to the proof of the stability result. Finally, we draw some perspectives for future works in Section IV.

Notation: In this letter, we consider the following norms

$$\|u\|_{L_2}^2 := \int_0^1 u(x)^2 dx, \ \|u\|_{H_1}^2 := \|u\|_{L_2}^2 + \int_0^1 u_x(x)^2 dx$$

$$|\Omega(.)|_\infty := \sup_{i=1...n} \{|\Omega_i(.)|\}, \ \|\Omega\|_{L_\infty} := \sup_{\substack{i=1...n \\ (x,y)\in[0,1]^2}} \{|\Omega_i(x, y)|\} \quad (4)$$

where Ω is a vector of n elements $\Omega_i \in L_\infty([0, 1]^2; \mathbb{R})$. For a function $k : (x, y) \mapsto k(x, y)$, $k'(x, x)$ is used to denote the total derivative of k evaluated at (x, x), i.e.,

$$k'(x, x) := \left.\frac{\partial k}{\partial x}\right|_{(x,x)} + \left.\frac{\partial k}{\partial y}\right|_{(x,x)}. \quad (5)$$

II. BACKSTEPPING STATE FEEDBACK CONTROL

The objective of this section is to present the backstepping transformation that maps (1)-(3), along with its corresponding feedback control law, into an exponentially stable target system. This target system is chosen as

$$w_{tt}(x, t) = w_{xx}(x, t) - \lambda_w w_t(x, t) - \beta_w w(x, t) \quad (6)$$
$$w_x(1, t) = -c_1 w_t(1, t) - c_2 w(1, t) \quad (7)$$
$$w_{tt}(0, t) = -d_1 w_t(0, t) + d_2 w_x(0, t) - d_3 w(0, t) \quad (8)$$

in which the boundary parameters c_1, c_2, d_1, d_2, and d_3 are positive constants. $\lambda_w > 0$ is an in-domain velocity source term and $\beta_w > 0$ is an in-domain position source term. As all these parameters are positive, this target system is exponentially stable (see Lemma 3 in Section III-D).

To solve the previously introduced control objective, the following backstepping transformation is selected

$$w(x, t) = u(x, t) - \int_0^x K^T(x, y)\chi(y, t)dy - \gamma(x)u_t(0, t) \quad (9)$$

in which K and χ are defined as

$$K(x, y) := \begin{bmatrix} k(x, y) \\ s(x, y) \\ m(x, y) \end{bmatrix}, \quad \chi(x, t) := \begin{bmatrix} u(x, t) \\ u_t(x, t) \\ u_x(x, t) \end{bmatrix} \quad (10)$$

and the kernel (K, γ) is chosen as the unique solution (see Lemma 2, Section III-B) of the following system

$$K_{xx}(x, y) - K_{yy}(x, y) = B_K K_y(x, y) + C_K K(x, y) \quad (11)$$
$$K'(x, x) = D_K K(x, x) + E_K \quad (12)$$
$$\begin{bmatrix} k_y(x, 0) \\ m_y(x, 0) \end{bmatrix} = H_K K(x, 0) \quad (13)$$
$$s_{xx}(x, 0) = b_2 s_y(x, 0) + H_S K(x, 0) \quad (14)$$
$$\begin{bmatrix} K(0, 0) \\ s_x(0, 0) \end{bmatrix} = J_K \quad (15)$$
$$\gamma(x) = \frac{s(x, 0)}{b_2} \quad (16)$$

in which

$$B_K^T := -(\Theta_2\Theta_1 + \Theta_1\Theta_2 + \lambda_w\Theta_2) \quad (17)$$
$$C_K^T := \Theta_1^2 + \lambda_w\Theta_1 + \beta_w I_3 \quad (18)$$
$$D_K^T := \frac{1}{2}(\Theta_2\Theta_1 + \Theta_1\Theta_2 + \lambda_w\Theta_2) \quad (19)$$
$$E_K^T := -\frac{1}{2}\begin{bmatrix} \beta + \beta_w & \lambda + \lambda_w & 0 \end{bmatrix} \quad (20)$$
$$H_K := \begin{bmatrix} \lambda_w & \beta - (\lambda_w + b_1)\frac{b_3}{b_2} & b_3 \\ 1 & \lambda - b_1 & b_2 \end{bmatrix} \quad (21)$$
$$H_S := \begin{bmatrix} 0 & \beta_w + b_3 + \lambda_w b_1 + b_1^2 & -(b_2\lambda_w + b_2 b_1) \end{bmatrix} \quad (22)$$
$$\Theta_1 := \begin{bmatrix} 0 & 1 & -1 \\ \beta & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Theta_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (23)$$
$$J_K := -\frac{1}{d_2}\begin{bmatrix} d_3 + b_3 \\ 0 \\ b_2 - d_2 \\ (b_1 + d_1)b_2 \end{bmatrix}. \quad (24)$$

The construction of (9) is inspired by [13] and by [16] for the integral terms. However, the kernel equations (11)-(16) are different from both papers. Indeed, (11) presents an additional source term which reveals troublesome to handle in the subsequent analysis. Moreover, note that the boundary condition (13) is similar to the one of [15]. However, the boundary condition in [15] is scalar, and contains in addition an integral term. The considered boundary (13) is, on the one hand, simpler as there is no integral term, but, on the other hand, it is a vectorial one coupled with another boundary condition. Yet, the recursive assumption we use in the method of successive approximations (Section III-B) is different from [15] and new, up to our knowledge.

Finally, the corresponding control law is

$$U(t) := K^T(1, 1)\chi(1, t) + \int_0^1 K_x^T(1, y)\chi(y, t)dy + \gamma'(1)u_t(0, t)$$
$$- (a_1 + c_1)u_t(1, t) - (a_2 + c_2)u(1, t) + c_2[\gamma(1)u_t(0, t)$$
$$+ \int_0^1 K^T(1, y)\chi(y, t)dy] + c_1\Big[[K^T(1, y)\Theta_2\chi(y, t)]_{y=0}^1$$
$$+ \int_0^1 \Big[K^T\Theta_1 - K_y^T\Theta_2\Big](1, y)\chi(y, t)dy + \gamma(1)\Theta_3\chi(0, t)\Big] \quad (25)$$

$$\Theta_3 := \begin{bmatrix} b_3 & b_1 & b_2 \end{bmatrix} \tag{26}$$

Theorem 1: Consider the closed-loop system consisting of the plant (1)-(3) with $b_2 \neq 0$, together with the control law (25) in which the kernel (K, γ) is the solution of (11)-(16). Define the functional (the L_2 and H_1 norms are defined in (4))

$$\Gamma(u, u_t) := \|u\|_{H_1}^2 + \|u_t\|_{L_2}^2 + u(1, t)^2 + u(0, t)^2 + u_t(0, t)^2 \tag{27}$$

There exist $\rho > 0$ and $R > 0$ such that

$$\Gamma(u(t), u_t(t)) \leqslant R\Gamma(u(0), u_t(0))e^{-\rho t}, \quad t \geqslant 0 \tag{28}$$

i.e., the closed-loop system is exponentially stable.

Note that the control law U (25) is expressed in terms of u, u_t and u_x but can also be written, using integration by parts, in terms of u and u_t only. Yet, it requires full-state measurement, which is seldom the case in practice. Future work will focus on relieving this constraint.

III. PROOF OF THEOREM 1

First, Section III-A presents the derivation of the backstepping kernel equations (11)-(16). Second, the existence and uniqueness of a solution to these equations is proven in Section III-B. In Section III-C, we analyze the invertibility of the backstepping transformation. The proof of the target system stability is established in Section III-D, through a Lyapunov analysis. Then, in Section III-E, the equivalence between the target system Lyapunov functional and the functional Γ defined in (27) is established. Finally, using all previous developments, we conclude the proof of Theorem 1 in Section III-F.

A. Backstepping Transformation

This section is devoted to the establishment of (11)-(16). Before stating the corresponding lemma, consider the vector $\chi(x, t)$ defined in (10). Using the system equation (1) together with the boundary condition (3), it follows that

$$\chi_t(x, t) = \Theta_1 \chi(x, t) + \Theta_2 \chi_x(x, t) \tag{29}$$

$$u_{tt}(0, t) = \Theta_3 \chi(0, t) \tag{30}$$

$$\chi_t(0, t) = \Theta_4 \underline{\chi}(0, t) \tag{31}$$

$$\chi(0, t) = \Theta_5 \underline{\chi}(0, t) \tag{32}$$

where Θ_1, Θ_2, and Θ_3 are defined in (23) and (26), and with

$$\underline{\chi}(0, t) := \begin{bmatrix} u(0, t) & u_t(0, t) & u_x(0, t) & u_{xt}(0, t) \end{bmatrix}^T \tag{33}$$

$$\Theta_4 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ b_3 & b_1 & b_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Theta_5 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{34}$$

Note that Θ_1 and Θ_2 such that (29) holds are not unique. The choice made in (23) however guarantees that $\Theta_2^2 = I_3$, a property which we use in the sequel.

Lemma 1: If (K, γ) in (9) is solution of (11)-(16), then the backstepping transformation (9) along the control law (25) map (1)-(3) into the target system (6)-(8).

Proof of Lemma 1: Consider the time derivative of (9)

$$w_t(x, t) = u_t(x, t) - \int_0^x K^T(x, y)\chi_t(y, t)dy - \gamma(x)u_{tt}(0, t) \tag{35}$$

Using (29), an integration by parts and (30), it holds

$$w_t(x, t) = u_t(x, t) - \left[K^T(x, y)\Theta_2\chi(y, t)\right]_{y=0}^x$$
$$- \int_0^x \left[K^T\Theta_1 - K_y^T\Theta_2\right](x, y)\chi(y, t)dy - \gamma(x)\Theta_3\chi(0, t) \tag{36}$$

Consider the time derivative of (36). Using (1), (29), the fact that $\Theta_2^2 = I_3$, and an integration by parts, one obtains

$$w_{tt}(x, t) = u_{xx}(x, t) + \lambda u_t(x, t) + \beta u(x, t)$$
$$- \left[K^T(x, y)\Theta_2\chi_t(y, t) + (K^T(x, y)\Theta_1\Theta_2 - K_y^T(x, y))\chi(y, t)\right]_{y=0}^x$$
$$- \int_0^x \left[(K^T\Theta_1 - K_y^T\Theta_2)\Theta_1 - (K_y^T\Theta_1\Theta_2 - K_{yy}^T)\right](x, y)\chi(y, t)dy$$
$$- \gamma(x)\Theta_3\chi_t(0, t) \tag{37}$$

Consider the first and second space derivatives of (9), which can be written as

$$w_x(x, t) = u_x(x, t) - K^T(x, x)\chi(x, t) - \int_0^x K_x^T(x, y)\chi(y, t)dy$$
$$- \gamma'(x)u_t(0, t) \tag{38}$$

$$w_{xx}(x, t) = u_{xx}(x, t) - \left(K'^T(x, x) + K_x^T(x, x)\right)\chi(x, t)$$
$$- K^T(x, x)\chi_x(x, t) - \int_0^x K_{xx}^T(x, y)\chi(y, t)dy$$
$$- \gamma''(x)u_t(0, t) \tag{39}$$

Gathering (9), (36), (37), and (39) the target system equation (6) holds if the kernel (K, γ) satisfies the following conditions

$$K^T(x, y)\Theta_1^2 - K_y^T(x, y)\Theta_2\Theta_1 - K_y^T(x, y)\Theta_1\Theta_2 + K_{yy}^T(x, y) - K_{xx}^T(x, y)$$
$$+ \lambda_w K^T(x, y)\Theta_1 - \lambda_w K_y^T(x, y)\Theta_2 + \beta_w K^T(x, y) = 0 \tag{40}$$

$$K^T(x, x)\Theta_2\chi_t(x, t) - \lambda\Theta_6\chi(x, t) - \beta\Theta_8\chi(x, t)$$
$$+ [K^T(x, x)\Theta_1\Theta_2 - K_y^T(x, x)]\chi(x, t) - \lambda_w\Theta_6\chi(x, t)$$
$$+ \lambda_w K^T(x, x)\Theta_2\chi(x, t) - \beta_w\Theta_8\chi(x, t)$$
$$- (K'^T(x, x) + K_x^T(x, x))\chi(x, t) - K^T(x, x)\chi_x(x, t) = 0 \tag{41}$$

$$\gamma(x)\Theta_3\chi_t(0, t) + (-K^T(x, 0)\Theta_1\Theta_2 + K_y^T(x, 0))\chi(0, t)$$
$$- K^T(x, 0)\Theta_2\chi_t(0, t) - \gamma''(x)\Theta_6\chi(0, t) - \lambda_w K^T(x, 0)\Theta_2\chi(0, t)$$
$$+ \lambda_w\gamma(x)\Theta_3\chi(0, t) + \beta_w\gamma(x)\Theta_6\chi(0, t) = 0 \tag{42}$$

$$\Theta_6 := \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \Theta_8 := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{43}$$

Following the definitions (17)-(18) of B_K and C_K, (40) can be rewritten as (11). Moreover, using (29), (41) can be reformulated as (12). Furthermore, using (31) and (32), one obtains that (42) is equivalent to the four following scalar conditions

$$\beta s(x, 0) - k_y(x, 0) + \lambda_w k(x, 0) + b_3 m(x, 0)$$
$$- \lambda_w b_3 \gamma(x) - b_1 b_3 \gamma(x) = 0 \tag{44}$$

$$-\beta_w\gamma(x) - s_y(x, 0) + \gamma''(x) + \lambda_w m(x, 0) - b_3\gamma(x)$$
$$- \lambda_w b_1 \gamma(x) + b_1 m(x, 0) - b_1^2\gamma(x) = 0 \tag{45}$$

$$k(x, 0) - m_y(x, 0) + \lambda s(x, 0) + b_2 m(x, 0)$$
$$- \lambda_w b_2 \gamma(x) - b_1 b_2 \gamma(x) + \lambda_w s(x, 0) = 0 \tag{46}$$

$$s(x, 0) - b_2\gamma(x) = 0 \tag{47}$$

which can be reformulated as (13), (14), and (16).

Now, the boundary condition (8) gives one last condition on the kernel. As, from (9), (36), (37), and (38),

$$
\begin{cases}
w_{tt}(0, t) = [\Theta_3\Theta_5 - \gamma(0)\Theta_3\Theta_4]\underline{\chi}(0, t) \\
w_t(0, t) = [\Theta_6 - \gamma(0)\Theta_3]\Theta_5\underline{\chi}(0, t) \\
w_x(0, t) = [\Theta_7 - K^T(0, 0) - \gamma'(0)\Theta_6]\Theta_5\underline{\chi}(0, t) \\
w(0, t) = [\Theta_8 - \gamma(0)\Theta_6]\Theta_5\underline{\chi}(0, t)
\end{cases} \quad (48)
$$

in which $\Theta_7 := \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, thus, a sufficient condition for (8) to hold is

$$
\begin{bmatrix} b_3 + d_3 \\ b_1 + d_1 \\ b_2 - d_2 \\ 0 \end{bmatrix} = -d_2 \begin{bmatrix} k(0, 0) \\ s(0, 0) \\ m(0, 0) \\ 0 \end{bmatrix} + \gamma(0) \begin{bmatrix} * \\ * \\ * \\ b_2 \end{bmatrix} - \gamma'(0) \begin{bmatrix} 0 \\ d_2 \\ 0 \\ 0 \end{bmatrix} \quad (49)
$$

which, using (47), is (15).

Finally the control law (25) can be obtained from (7) with (9), (36), (38), and using (2). ∎

B. Existence and Uniqueness of the Solution of (11)–(15)

The PDE (11)-(15) presents two different types of boundary conditions at $(x, 0)$, namely (13) and (14). The key idea in the sequel is to address each condition separately. First, consider the following change of variable

$$
G(\zeta, \eta) = e^{\frac{1}{2}B_K(\zeta - \eta)} K(\zeta + \eta, \zeta - \eta) \quad (50)
$$

The idea behind (50) is, on one hand, to gather the high order derivatives $(K_{xx} - K_{yy})$ in (11) into one term $(G_{\eta\zeta})$, with $x = \zeta + \eta$ and $y = \zeta - \eta$, and, on the other hand, to cancel the term in K_y in (11), which is done with $G = e^{\frac{1}{2}B_K y} K$.

Consider also the following definitions

$$
\phi_{k,m} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \phi_s := \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad (51)
$$

$$
K^{k,m} := \phi_{k,m}K, \ G^{k,m} := \phi_{k,m}G, \ K^s := \phi_s K, \ G^s := \phi_s G \quad (52)
$$

$$
G = \phi_{k,m}^T G^{k,m} + \phi_s^T G^s \quad (53)
$$

Proposition 1: The two following statements are equivalent
(i) K is a solution of (11)-(15).
(ii) G is a solution of

$$
G_{\zeta\eta}(\zeta, \eta) = C_G(\zeta, \eta)G(\zeta, \eta) \quad (54)
$$

$$
G(\zeta, 0) = F_G(\zeta) \quad (55)
$$

$$
G_\zeta^{k,m}(\eta, \eta) = H_G G(\eta, \eta) + G_\eta^{k,m}(\eta, \eta) \quad (56)
$$

$$
G^s(\eta, \eta) = 2 \int_0^\eta \int_0^z \big[b_2(G_\zeta^s(\mu, \mu) - G_\eta^s(\mu, \mu))
$$
$$
+ H_{GS}G(\mu, \mu) \big] d\mu\, dz + J_{GS}(2\eta) \quad (57)
$$

in which

$$
C_G(\zeta, \eta) = e^{\frac{1}{2}B_K(\zeta - \eta)} \left(C_K - \frac{1}{4}B_K^2 \right) e^{-\frac{1}{2}B_K(\zeta - \eta)} \quad (58)
$$

$$
F_G(\zeta) = e^{\frac{1}{2}B_K\zeta} F_K(\zeta) \quad (59)
$$

$$
F_K(\zeta) = K(x, x) = e^{D_K x} J_{GK} + \int_0^x e^{D_K(x-s)} E_K ds \quad (60)
$$

$$
H_G = 2H_K + \phi_{k,m}B_K, \quad H_{GS} = 2H_S - b_2\phi_s B_K \quad (61)
$$

$$
J_{GS}(x) := \begin{bmatrix} 0 & 1 & 0 & x \end{bmatrix} J_K, \quad J_{GK} := \Theta_5 J_K \quad (62)
$$

Proof: (i) $\Rightarrow$ (ii). The establishment of (54) and (55) is straightforward. Besides, from the fact that

$$
s(x, 0) = \int_0^x \int_0^s s_{xx}(z, 0)dzds + xs_x(0, 0) + s(0, 0) \quad (63)
$$

using (14), (16), (62), and $z = 2\mu$, one gets

$$
s(x, 0) = \int_0^x \int_0^{\frac{s}{2}} [2b_2 s_y + 2H_s K](2\mu, 0)d\mu\, ds + J_{GS}(x) \quad (64)
$$

From (50), it holds

$$
G_\zeta(\eta, \eta) - G_\eta(\eta, \eta) = 2K_y(2\eta, 0) + B_K K(2\eta, 0) \quad (65)
$$

Thus multiplying (65) by ϕ_s defined in (51), one writes

$$
2s_y(2\eta, 0) = G_\zeta^s(\eta, \eta) - G_\eta^s(\eta, \eta) - \phi_s B_K G(\eta, \eta) \quad (66)
$$

Replacing (66) into (64), for $x = 2\eta$, and $2z = s$, one obtains

$$
G^s(\eta, \eta) = \int_0^\eta 2 \int_0^z \big[b_2(G_\zeta^s(\mu, \mu) - G_\eta^s(\mu, \mu))
$$
$$
- \phi_s B_K G(\mu, \mu)) + 2H_S G(\mu, \mu) \big] d\mu\, dz + J_{GS}(2\eta) \quad (67)
$$

and (57) holds. Finally, multiplying (65) by $\phi_{k,m}$, one gets (56).

(ii) $\Rightarrow$ (i). This implication follows from the previous one and the fact that the change of variable (50) is bijective (as the composition of two bijective changes of variable). ∎

Lemma 2: Equations (54)-(57) has a unique solution G

$$
G(\zeta, \eta) = \sum_{n=0}^\infty (\Delta G)^n(\zeta, \eta) \quad (68)
$$

given as the cumulative sum of the following sequence, which exponentially converges towards zero,

$$
(\Delta G)^{n+1}(\zeta, \eta) = \mathscr{F}\big((\Delta G)^n(\zeta, \eta)\big) \quad (69)
$$

$$
(\Delta G)^0(\zeta, \eta) = \phi_{k,m}^T\phi_{k,m}[2F_G(\eta) - F_G(0)] + \phi_s^T[J_{GS}(2\eta)
$$
$$
- 2b_2\eta\phi_s F_G(0)] + F_G(\zeta) - F_G(\eta) \quad (70)
$$

in which $\phi_{k,m}$, ϕ_s are defined in (51), and

$$
\mathscr{F}(G(\zeta, \eta)) := \int_\eta^\zeta \int_0^\eta C_G(\zeta, \eta)G(s, z)dzds
$$
$$
+ \phi_{k,m}^T\left[-\int_0^\eta H_G G(s, s)ds + 2\phi_{k,m}\int_0^\eta \int_0^s C_G(\zeta, \eta)G(s, z)dzds \right]
$$
$$
+ \phi_s^T\left[\int_0^\eta \int_0^z 4b_2 \int_0^\mu \phi_s C_G(\zeta, \eta)G(\mu, s)ds\,d\mu\,dz + \int_0^\eta 4b_2\phi_s G(z, 0)dz \right.
$$
$$
\left. - 2b_2\int_0^\eta \phi_s G(z, z)dz + \int_0^\eta 2\int_0^z H_{GS}G(\mu, \mu)d\mu\,dz \right] \quad (71)
$$

Proof: Let us first notice that

$$
G(\zeta, \eta) = \int_\eta^\zeta \int_0^\eta G_{\zeta\eta}(s, z)dzds + G(\eta, \eta) + G(\zeta, 0) - G(\eta, 0) \quad (72)
$$

On the one hand, the integration of $G_{\zeta\eta}(\zeta, \eta)$ with respect to η from 0 to ζ gives

$$
\int_0^\zeta G_{\zeta\eta}(\zeta, s)ds = G_\zeta(\zeta, \zeta) - G_\zeta(\zeta, 0) \quad (73)
$$

using the fact that

$$
G_\zeta(\mu, \mu) - G_\eta(\mu, \mu) = 2G_\zeta(\mu, \mu) - G'(\mu, \mu) \quad (74)
$$

and (52), it holds

$$
\big[G_\zeta^s - G_\eta^s \big](\mu, \mu) = \phi_s \left[\int_0^\mu G_{\zeta\eta}(\mu, s)ds + 2G_\zeta(\mu, 0) - G'(\mu, \mu) \right] \quad (75)
$$

Therefore from (57), and (54), using (75) one gets

$$G^s(\eta, \eta) = 4b_2 \int_0^\eta \int_0^z \int_0^\mu \phi_s C_G(\mu, s) G(\mu, s) ds d\mu dz \quad (76)$$

$$+ \int_0^\eta 4b_2 G^s(z, 0) dz - 4b_2 \eta G^s(0, 0) - 2b_2 \int_0^\eta G^s(z, z) dz$$

$$+ 2b_2 \eta G^s(0, 0) + 2 \int_0^\eta \int_0^z H_{GS} G(\mu, \mu) d\mu dz + J_{GS}(2\eta)$$

On the other hand, and, according to (56), one can get

$$\frac{d\, G^{k,m}(\zeta, \zeta)}{d\zeta} = -H_G G(\zeta, \zeta) + 2G_\zeta^{k,m}(\zeta, \zeta) \quad (77)$$

Gathering both previous equations along with (52) and (54), it holds

$$G^{k,m}(\zeta, \zeta) = -\int_0^\zeta H_G G(s, s) ds + 2G^{k,m}(\zeta, 0) \quad (78)$$

$$+ 2\phi_{k,m} \int_0^\zeta \int_0^s C_G(s, z) G(s, z) dz ds - G^{k,m}(0, 0)$$

Finally, using (54) in (72), and using (53), it holds

$$G(\eta, \eta) = \phi_{k,m}^T G^{k,m}(\eta, \eta) + \phi_s^T G^s(\eta, \eta) \quad (79)$$

thus with (77) and (78), one gets

$$G(\zeta, \eta) = \mathscr{F}(G(\zeta, \eta)) + (\Delta G)^0(\zeta, \eta) \quad (80)$$

in which $(\Delta G)^0(\zeta, \eta)$ and $\mathscr{F}(.)$ are defined in (70) and (71).

Now, we use a standard iterative method to prove that (80) has a unique solution. The initial guess is $(\Delta G)^0$ defined in (70), and initiates the induction (69) for $n \in \mathbb{N}$. Define $M \in \mathbb{R}^+$ as an upper bound of $(\Delta G)^0$. Using $|\cdot|$ and $\|\cdot\|$ defined in (4), and the fact that $\eta \leqslant 1$, one gets

$$|(\Delta G)^0(\zeta, \eta)|_\infty \leqslant \|J_{GS}\|_{L_\infty} + (5 + 2b_2)\|F\|_{L_\infty} =: M \quad (81)$$

Assume that there a scalar exists κ such that

$$\kappa \geqslant (5 + 8|b_2|)\|C_G\|_{L_\infty} + 2|H_G|_\infty + 8|b_2| + 4|H_{GS}|_\infty \quad (82)$$

and the following inequality is true for a given rank $n \in \mathbb{N}$

$$|(\Delta G)^n(\zeta, \eta)|_\infty \leqslant M\kappa^n \frac{\zeta^n + \eta^n}{n!} \quad (83)$$

From (69), it follows

$$|(\Delta G)^{n+1}(\zeta, \eta)|_\infty \leqslant \|C_G\|_{L_\infty} \int_\eta^\zeta \int_0^\eta |(\Delta G)^n(s, z)|_\infty dz ds$$

$$+ |H_G|_\infty \int_0^\eta |(\Delta G)^n(s, s)|_\infty ds$$

$$+ 2\|C_G\|_{L_\infty} \int_0^\eta \int_0^s |(\Delta G)^n(s, z)|_\infty dz ds$$

$$+ 4|b_2| \, \|C_G\|_{L_\infty} \int_0^\eta \int_0^z \int_0^\mu |(\Delta G)^n(\mu, s)|_\infty ds d\mu dz$$

$$+ 4b_2 \int_0^\eta |(\Delta G)^n(z, 0)|_\infty dz + 2|b_2| \int_0^\eta |(\Delta G)^n(z, z)|_\infty dz$$

$$+ 2|H_{GS}|_\infty \int_0^\eta \int_0^z |(\Delta G)^n(\mu, \mu)|_\infty d\mu dz \quad (84)$$

Consequently, using the recursive hypothesis (83), it holds

$$|\Delta G^{n+1}(\zeta, \eta)|_\infty \leqslant \frac{M\kappa^n}{(n+1)!} \Big(\|C_G\|_{L_\infty} \big(\eta \zeta^{n+1} + \zeta \eta^{n+1} + 4\eta^{n+2} \big)$$

$$+ 8|b_2|\eta^{n+3} \big) + 2|H_G|_\infty \eta^{n+1} + 8|b_2|\eta^{n+1} + 4|H_{GS}|_\infty \eta^{n+2} \Big)$$

From the fact that $\zeta, \eta \leqslant 1$, and (82), one concludes

$$|(\Delta G)^{n+1}(\zeta, \eta)|_\infty \leqslant \frac{M\kappa^{n+1}}{(n+1)!} \big(\zeta^{n+1} + \eta^{n+1} \big) \quad (85)$$

As the estimate (83) is proven for $n = 0$ in (83), it follows by induction that $(\Delta G)^n$ exponentially converges and that the solution of (54)-(57) is given by (68). ∎

Remark 1: It is worth pointing out that the upper bound we consider in the induction assumption (83) is not standard. Indeed [14], [15], among others, introduce the bound $\frac{(\zeta + \eta)^n}{n!}$, which cannot be used here. Indeed the integration of this bound for the specific boundaries (56)-(57) would cause the recursive method to fail. For example, in order to perform the iterative method, one needs to integrate $\int_0^\eta \int_0^s |(\Delta G)^n(z, s)|_\infty dz\, ds$ (which appear in (71) and in (84)). However, using the previous upper bound, one has

$$\int_0^\eta \int_0^s \frac{(z + s)^n}{n!} dz ds = \big(2^{n+1} - 1 \big) \frac{\eta^{n+2}}{(n+2)!} \quad (86)$$

The 2^{n+1} term is too large to be compatible with the recursive hypothesis. This is why, instead, we suggest to consider (83), which is a smaller bound.

C. Invertibility of the Backstepping Transformation

Let us denote Π the map that transforms the system (1)-(3) with $U(t)$ defined in (25) into the target system (6)-(8).

$$\Pi: \qquad \mathtt{H} \to \mathtt{H}$$
$$(q_1, q_2, q_3, q_4, q_5) \mapsto (z_1, z_2, z_3, z_4, z_5) \quad (87)$$

with

$$z_1(x) = q_1(x) - \int_0^x K^T(x, y) \begin{bmatrix} q_1 \\ q_2 \\ q_1' \end{bmatrix}(y) dy - \gamma(x) q_5 \quad (88)$$

$$z_2(x) = q_2(x) - \begin{bmatrix} q_1(x) & q_2(x) & q_1'(x) \end{bmatrix} \Theta_2 K(x, x)$$

$$- \int_0^x \big[K^T(x, y)\Theta_1 - K_y^T(x, y)\Theta_2 \big] \begin{bmatrix} q_1 \\ q_2 \\ q_1' \end{bmatrix}(y)) dy \quad (89)$$

$$+ (k(x, 0) - \gamma(x) b_3) q_4 + (m(x, 0) - \gamma(x) b_1) q_5$$

$$z_3 = q_3 - \int_0^1 \begin{bmatrix} q_1 & q_2 & q_1' \end{bmatrix}(y) K(x, y) dy - \gamma(1) q_5 \quad (90)$$

$$z_4 = q_4, \quad z_5 = q_5 \quad (91)$$

where K is defined in (10) as the unique solution of (11)-(15), γ in (16), and with Θ_1 and Θ_2 defined in (23).

The existence of the inverse map Π^{-1} can then be obtained by simply replacing λ, β, a_i and b_i with respectively λ_w, β_w, c_i and d_i in the previous analysis.

D. Target System Exponential Stability

Consider the target system defined in (6)-(8) and the following Lyapunov functional candidate

$$V(w, w_t) = \int_0^1 \begin{bmatrix} w & w_t & w_x \end{bmatrix} P_0 \begin{bmatrix} w & w_t & w_x \end{bmatrix}^T + w(1) P_1 w(1)$$

$$+ \begin{bmatrix} w(0) & w_t(0) \end{bmatrix} P_2 \begin{bmatrix} w(0) \\ w_t(0) \end{bmatrix} \quad (92)$$

in which $P_0 \in \mathbb{R}^{3\times3}$, $P_1 \in \mathbb{R}$ and $P_2 \in \mathbb{R}^{2\times2}$ are symmetric positive definite matrices.

Lemma 3: For all $\lambda_w > 0$, $\beta_w > 0$, $c_i > 0$, and $d_i > 0$, there exist P_0, P_1 and P_2 symmetric positive definite matrices such that, it holds for a suitable $\rho > 0$

$$\dot{V}(w, w_t) \leqslant -\rho V(w, w_t) \tag{93}$$

in which V is defined in (92).

Proof of Lemma 3: Considering

$$P_0 = \begin{bmatrix} \beta_w & \sigma & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}, \quad P_1 = c_2, \quad P_2 = \begin{bmatrix} \frac{d_3 + \sigma d_1}{d_2} & \frac{\sigma}{d_2} \\ * & \frac{1}{d_2} \end{bmatrix} \tag{94}$$

with

$$0 < \sigma < \min\left\{ \sqrt{\beta_w}, \ \lambda_w, \ \frac{4\beta_w \lambda_w}{\lambda_w^2 + 4\beta_w}, \ \frac{4c_2}{c_1}, \ d_1 \right\} \tag{95}$$

which is well-defined as β_w, λ_w, c_1, c_2, and d_1 are positive. With (95), the matrices P_0 and P_2 are positive definite. Computing the derivative of V along the solution (92) leads to (6)-(8)

$$\dot{V}(w, w_t) = -\int_0^1 \begin{bmatrix} w & w_t & w_x \end{bmatrix} Q_0 \begin{bmatrix} w & w_t & w_x \end{bmatrix}^T$$
$$- \begin{bmatrix} w(1) & w_t(1) \end{bmatrix} Q_1 \begin{bmatrix} w(1) \\ w_t(1) \end{bmatrix} - \begin{bmatrix} w(0) & w_t(0) \end{bmatrix} Q_2 \begin{bmatrix} w(0) \\ w_t(0) \end{bmatrix} \tag{96}$$

in which

$$Q_0 = \begin{bmatrix} 2\beta_w \sigma & \lambda_w \sigma & 0 \\ * & 2\lambda_w - 2\sigma & 0 \\ * & * & 2\sigma \end{bmatrix} \tag{97}$$

$$Q_1 = \begin{bmatrix} 2c_2 \sigma & c_1 \sigma \\ * & 2c_1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} \frac{2\sigma d_3}{d_2} & 0 \\ * & \frac{2d_1}{d_2} - \frac{2\sigma}{d_2} \end{bmatrix} \tag{98}$$

From (95) the matrices Q_0, Q_1, and Q_2, are positive definite. Therefore, (93) holds. ∎

This Lyapunov functional candidate (92) is inspired by [14]. We have added the P_1 and P_2 terms to handle the dynamic boundaries under consideration.

E. Stability in Terms of Γ

To conclude on the exponential stability of (1)-(3) under the control law (25), the equivalence between $V(w, w_t)$ in (92) and $\Gamma(u, u_t)$ in (27) is proven in the following lemma.

Lemma 4: There exist $\mu_1, \mu_2 > 0$ such that

$$\Gamma(u, u_t) \leqslant \mu_1 V(w, w_t) \tag{99}$$
$$V(w, w_t) \leqslant \mu_2 \Gamma(u, u_t) \tag{100}$$

Proof: Only the establishment of (100) is detailed, as similar arguments hold for (99) using the inverse backstepping transformation (Section III-C). From (9), (36), and (38), there exist c_1, c_2, c_3, $c_4 > 0$ such that

$$\|w\|_{L_2}^2 \leqslant c_1 \left[\|u\|_{H_1}^2 + \|u_t\|_{L_2}^2 + u_t(0, t)^2 \right] \tag{101}$$

$$w(1, t)^2 \leqslant c_2 \left[\|u\|_{H_1}^2 + \|u_t\|_{L_2}^2 + u_t(0, t)^2 + u(1, t)^2 \right] \tag{102}$$

$$\|w_t\|_{L_2}^2 \leqslant c_3 \left[\|u\|_{H_1}^2 + \|u_t\|_{L_2}^2 + u(0, t)^2 + u_t(0, t)^2 \right] \tag{103}$$

$$\|w_x\|_{L_2}^2 \leqslant c_4 \left[\|u\|_{H_1}^2 + \|u_t\|_{L_2}^2 + u_t(0, t)^2 \right] \tag{104}$$

From (101)-(104) and the fact that $w(0, t) = u(0, t)$ and $w_t(0, t) = u_t(0, t)$ (as $\gamma(0) = 0$ from (15)-(16)), one obtains (100). ∎

F. Final Proof of Theorem 1

Using Lemma 2, there exists a unique solution to (11)-(16). From Lemma 1, this solution defines a backstepping transformation which maps the system (1)-(3) into (6)-(8) with the control law U defined in (25). Moreover, from Lemma 3, one gets that (6)-(8) is exponentially stable if all its parameters are positive. Therefore, using Lemma 4, one establishes (28).

IV. CONCLUSION

We have proposed a full-state feedback for a system where both the PDE and boundary conditions are unstable. In particular, we have generalized the control in [12], allowing a change of the in-domain viscous coefficient between the original and the target system. Our on-going works focus on the design of a collocated observer for the system (1)-(3).

A future direction of study could be to extend this design for a PDE such as $y_{tt}(x, t) = (a(x)y_x)_x(x, t) + \lambda(x)y_t(x, t) + \beta(x)y(x, t)$ with two second order boundaries as considered in [1], [3], and [11].

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