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Backstepping Observer Based-Control for an Anti-Damped Boundary Wave PDE in Presence of In-Domain Viscous Damping

Christophe Roman[1], Delphine Bresch-Pietri[1], Eduardo Cerpa[2], Christophe Prieur[1] and Olivier Sename[1]

***Abstract*— This paper presents a backstepping control design for a one-dimensional wave PDE with in-domain viscous damping, subject to a dynamical anti-damped boundary condition. Its main contribution is the design of an observer-based control law which stabilizes the wave PDE velocity, using only boundary mesurements. Numerical simulations on an oil-inspired example show the relevance of our result and illustrate the merits of this control design.**

I. INTRODUCTION

A large class of physical systems exhibits mechanical vibrations, which may induce stress and material fatigue. This is the case, e.g., for drilling facilities which suffer of angular velocity oscillations due to the downhole interaction with the rock (the so-called stick-slip phenomenon). Repetitive exposure to this phenomenon substantially decreases the productivity and sometimes leads to premature failure.

Usually, these dynamical phenomena are modeled as wave PDEs. Current studies on this topic may be classified in three types. First, some of them focus on the propagation phenomenon as [12], where a wave PDE with in-domain anti-damping and Dirichlet boundary is considered. Another trend of studies pays specific attention to the boundary conditions, e.g [13], where a pure wave PDE with an anti-stable boundary condition is considered. Here, we consider both dynamics and thus follow a third type approach.

The system under study is a one-dimensionnal wave PDE with in-domain viscous damping. The uncontrolled boundary is a second order anti-damped dynamics. For physical matters, as for the drilling application, only boundary velocity measurements are considered available for control. While several output feedback laws exist in the nominal case without in-domain damping [9], [1], [5], up to our knowledge, there exists only one design accounting for distributed viscous damping in [10]. However, this solution, which grounds on the backstepping methodology [6], is a full state feedback and requires a previous change of variables: the ODE and PDE are then decoupled, but only after performing a not straightforwardly invertible change of variables (spacial derivation of the state). Consequently, the obtained stability result is expressed in terms of space derivatives of the state.

[1]C. Roman, D. Bresch-Pietri, C. Prieur and O. Sename are with Univ. Grenoble Alpes, CNRS, GIPSA-lab, 11 rue des Mathématiques, 38000 Grenoble, France (christophe.roman@gipsa-lab.fr)

[2]Eduardo Cerpa is with Departemento de Matemática, Universidad Técnica Federico Santa Maria, Casilla 110-V, Valparaiso, Chile (eduardo.cerpa@usm.cl)

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In this paper, we aim at extending our approach in [8] which proved the robustness of the control law developed in [1], to small enough in-domain damping. This is consistent with the fact that this control we consider was originally designed for a pure wave PDE. The mains contributions of this paper are

- the development of a control law taking into account in-domain damping in a more explicit way than in [10] and generalizing the design of [1];
- the design of an observer using only boundary velocities measurements.

The paper is organized as follows. In Section II, we present the problem under consideration. Section III is devoted to the full state control design, while Section IV details the proof of convergence. Finally, an observer based control law is presented in Section V. We conclude with simulations illustrating our result on an oil-inspired drilling application in Section VI, and directions of future works.

Notation In this paper, $|\cdot|$ is the Euclidean norm and $\|u(\cdot)\|$ is the spatial L_2-norm of a function $[0,1] \ni x \mapsto u(x,\cdot)$. Moreover $\overset{a.e.}{=}$ stands for equal almost everywhere.

For a function $(x,y) \mapsto k(x,y)$, $k'(x,x)$ is used to denote the total derivative of $k(x,y)$ evaluated at (x,x), i.e.,

$$k'(x,x) = \left.\frac{\partial k(x,y)}{\partial x}\right|_{(x,x)} + \left.\frac{\partial k(x,y)}{\partial y}\right|_{(x,x)} \tag{1}$$

II. PROBLEM STATEMENT

Let us consider the following wave equation with in-domain viscous damping, subject to an anti-damping boundary, with actuation on the opposite boundary

$$u_{tt}(x,t) = u_{xx}(x,t) - 2\lambda u_t(x,t) \tag{2}$$
$$u_x(1,t) = U(t) \tag{3}$$
$$u_{tt}(0,t) = a q u_t(0,t) + a u_x(0,t) \tag{4}$$

in which $U(t)$ is the scalar control input, (u,u_t) is the system state, with $\big(u(\cdot,0),u_t(\cdot,0)\big) \in H_1(0,1) \times L_2(0,1)$, $a>0$ is a scalar constant. The in-domain viscous damping coefficient is $\lambda \geqslant 0$. The anti-damping coefficient is $q>0$. The parameters a, λ and q are supposed to be known constant values. The control objective is to stabilize the system velocity (i.e., $\|u_t\|$)

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and torque (i.e., $\|u_x\|$)[1]. Let us define the desired attractor

$$\mathscr{A} = \Big\{ (u,u_t) \in H_1(0,1) \times L_2(0,1) \;\Big|\; u_t(\cdot) \overset{a.e.}{=} 0,\ u(\cdot) \overset{a.e.}{=} C,\ C \in \mathbb{R} \Big\} \quad (5)$$

Our objective is to stabilize the system (2)-(4) towards the attractor $\mathscr{A}$.

Assumption 1: *The only measured quantities are the velocity boundaries, i.e., $u_t(0,t)$ and $u_t(1,t)$.*
This assumption will be used for the observer based control.

III. BACKSTEPPING STATE FEEDBACK

First the full-state feedback is presented. We consider in this section that the whole state is available from measurement, and in particular $u_t(\cdot,t)$ and $u_x(\cdot,t)$. The backstepping approach is used to design a control law able to stabilize system (2)-(4) towards the attractor $\mathscr{A}$ defined in (5). The objective is to find a control law $U(t)$ that matches (2)-(4) into

$$w_{tt}(x,t) = w_{xx}(x,t) - 2\lambda w_t(x,t) \quad (6)$$
$$w_x(1,t) = -w_t(1,t) \quad (7)$$
$$w_{tt}(0,t) = -a_w q_w w_t(0,t) + a_w w_x(0,t) \quad (8)$$

referred to as the Target system, which, assuming $a_w > 0$ and $q_w > 0$, is exponentially stable (see Section IV-A). The desired control law $U(t)$ is chosen as

$$\begin{aligned} U(t) = -u_t(1,t) + \frac{1}{(m(1,1)-1)} \Bigg[\int_0^1 \big(m_y(1,y) - s_x(1,y) \\ + 2\lambda(s(1,y) - g_y(1,y)) + g_{xy}(1,y)\big) u_t(y,t) dy \\ + \int_0^1 (s_y(1,y) - g_{yy}(1,y) - m_x(1,y)) u_x(y,t) dy \\ (m(1,0) + aqg(1,0) + g_x(1,0)) u_t(0,t) \Bigg] \end{aligned} \quad (9)$$

where s, m, and g are the kernels of the backstepping transformation displayed later (see (24)), and explicitly defined as

$$\begin{bmatrix} s(x,y) \\ m(x,y) \\ g(x,y) \end{bmatrix} = e^{H(y-x)} F \quad (10)$$

in which

$$F = \frac{1}{a_w} \begin{bmatrix} -(aq + a_w q_w) \\ a_w - a \\ 0 \end{bmatrix} \quad (11)$$

$$H = \begin{bmatrix} 0 & aq + 2\lambda & aq(aq+2\lambda) \\ 0 & a & a^2 q + 2a\lambda \\ 1 & 0 & a \end{bmatrix} \quad (12)$$

It is worth noticing that $1 - m(1,1) = \frac{a}{a_w}$ and cannot be zero since $a > 0$ here. Thus, the control law is always well defined. Computation of the control law requires the knowledge of $u_x(\cdot,t)$, $u_t(\cdot,t)$ and the velocity boundaries $u_t(0,t)$ and $u_t(1,t)$. Note that the derivatives of $s(x,y)$, $m(x,y)$, and $g(x,y)$ can be computed explicitly with (10).

Theorem 1: *Consider the closed-loop system consisting of the plant* (2)-(4)*, together with the control law* (9)-(12) *in which $a_w > 0$ and $q_w > 0$. Define the functional*

$$\Gamma(u,u_t) = u_t(0,t)^2 + \int_0^1 u_t(x,t)^2 dx + \int_0^1 u_x(x,t)^2 dx \quad (13)$$

Then there exist $\rho > 0$ and R such that, for all $t \geqslant 0$

$$\Gamma(u,u_t) \leqslant R\Gamma\big(u(.,0),u_t(.,0)\big) e^{-\rho t} \quad (14)$$

and therefore the closed-loop system for the the attractor $\mathscr{A}$ defined in (5) *is exponentially stable.*

For the case $\lambda = 0$, using the fact that a pure wave equation can be reformulated as two transport phenomena, one recovers the predictive control in [1] taking aside the adaptive part.

The control law (9) has a structure similar to the one proposed in [10] and thus requires knowledge of the same variables. Exact comparison of both controllers is a direction of future work as one cannot conclude, at a first glance at least, if they result in the same control law. Nevertheless, the design approach proposed here is more straightforward -we are considering another target system which does not require a preliminary change of variables- and the stability result (Theorem 1) is obtained with a more meaningful norm. Indeed the result from [10] is expressed in terms of $u_x(.,t)$, $u_{xx}(.,t)$, $u_{tx}(.,t)$ and $u_t(0,t)$.

IV. PROOF OF THEOREM 1

The proof of Theorem 1 is performed in three steps: first, the stability of the target system (see Lemma 1 in Section IV-A); second, the mapping between the original system and the target (Lemma 2 in Section IV-B), and the computation of the control law (see Section IV-B.5); finally, the stability in terms of the functional Γ in Section IV-C.

A. Stability of the target system

Since we are interested in the stabilization of the distributed velocity and torque, the Lyapunov functional is chosen as a function of w_x and w_t.

Lemma 1: *Consider the system* (6)-(8) *and the following Lyapunov function candidate*

$$V(w,w_t) = \frac{1}{2}\int_0^1 (w_x^2 + w_t^2) dx + \frac{1}{2a_w} w_t(0,t)^2 + \delta \int_0^1 (1+x) w_x w_t dx \quad (15)$$

in which $0 < \delta < \frac{1}{2}$. The function (15) *is positive definite with respect to the attractor $\mathscr{A}$ (i.e. $w_t(.,t)$, $w_x(.,t)$, and $w_t(0,t)$). Moreover*

$$\dot V(w,w_t) \leqslant -\min\left(\frac{\delta}{4},\ 2a_w q_w + \delta a_w\right) V(w,w_t) \quad (16)$$

if
$$\begin{cases} \delta < \min\left(\frac{8\lambda}{64\lambda^2 - 1}, \frac{1}{2}\right), & \text{for} \quad \lambda > \frac{1}{8} \\ \delta < \frac{1}{2}, & \text{otherwise} \end{cases} \quad (17)$$

[1]Note that these denominations (velocity and torque) are abusive, as the system is normalized and so variables do not have unit

Proof: Using Young's inequality and integrations by parts on the time derivative of (15), one gets

$$\dot{V}(w,w_t) \leqslant -(1-2\delta)w_t(1,t)^2 - 2\lambda\int_0^1 w_t^2 dx - q_w w_t(0,t)^2 - \frac{\delta}{2}(w_t(0,t)^2 + w_x(0,t)^2) - \frac{\delta}{2}\int_0^1 (w_t^2 + w_x^2)dx + 2\delta\lambda\int_0^1\left(\frac{w_x^2}{\varepsilon} + \varepsilon w_t^2\right)dx \quad (18)$$

in which $\varepsilon > 0$. Assuming that

$$\delta < \frac{1}{2}, \qquad \varepsilon \geqslant 8\lambda, \qquad \varepsilon \neq 1 \quad (19)$$

$$\delta \leqslant \frac{8\lambda}{\varepsilon^2 - 1} \quad \text{if} \quad \varepsilon > 1 \quad (20)$$

$$\delta \geqslant 0 \quad \text{if} \quad \varepsilon < 1 \quad (21)$$

then

$$\dot{V}(w,w_t) \leqslant -\frac{\delta}{4}\int_0^1 w_t^2 - \frac{\delta}{4}\int_0^1 w_x^2 - \left(q_w + \frac{\delta}{2}\right)w_t(0,t)^2 \quad (22)$$

One obtains, from (15), that

$$V(w,w_t) \leqslant \int_0^1 (w_x^2 + w_t^2)dx + \frac{1}{2a_w}w_t(0,t)^2 \quad (23)$$

Matching the last two inequalities, this concludes the proof of Lemma 1. ■

B. Backstepping transformation

Let us start this section by stating the following result. The remaining part of this section is devoted to its proof.

Lemma 2: *The backstepping transformation*

$$w(x,t) = u(x,t) - \int_0^x s(x,y)u_t(y,t)dy - \int_0^x m(x,y)u_x(y,t)dy - \int_0^x g(x,y)u_{xt}(y,t)dy \quad (24)$$

in which $s(x,y)$*,* $m(x,y)$*, and* $g(x,y)$ *have been defined in* (10)-(12)*, along with the control law* (9) *maps system* (2)-(4) *into system* (6)-(8).

One may be surprised by the form of this transformation, as the last integral in (24) adds only boundary terms. Indeed, using an integration by parts, one can rewrite the last integral as an integral term of $u_t(y,t)$ and boundary terms, $u_t(0,t)$ and $u_t(x,t)$. However, the existence and uniqueness of the kernel are more easily proven under this from. This idea of adding integral term was used in [13] with a third-order kernel in u, u_t, and u_x. Here, it is applied together with a third-order kernel but in u_x, u_t, and u_{xt}. It is worth noting that one may also choose to add an integral term in u in (24). However, due to the specific form of our target system, the corresponding kernel would be found equal to zero.

1) Computation of backstepping derivatives: To prove the existence of the kernel, the time derivative of (24) is computed, using integrations by parts and expressing $u_{tt}(\cdot,t)$ along with (2)

$$w_t(x,t) = u_t(x,t) + 2\lambda\int_0^x s(x,y)u_t(y,t)dy - [s(x,y)u_x(y,t)]_{y=0}^x + \int_0^x s_y(x,y)u_x(y,t)dy - \int_0^x m_y(x,y)u_{xt}(y,t)dy - [g(x,y)u_{xx}(y,t) - g_y(x,y)u_x(y,t)]_{y=0}^x - \int_0^x g_{yy}(x,y)u_x(y,t)dy + 2\lambda\int_0^x g(x,y)u_{xt}(y,t)dy \quad (25)$$

Similarly, one obtains the second order time derivative

$$w_{tt}(x,t) = u_{xx} - 2\lambda u_t(x,t) + 2\lambda\left[s(x,y)u_x(y,t)\right]_{y=0}^x - 2\lambda\int_0^x s_y(x,y)u_x(y,t)dy - 4\lambda^2\int_0^x s(x,y)u_t(y,t)dy - \left[s(x,y)u_{xt}(y,t) - s_y(x,y)u_t(y,t)\right]_0^x - \int_0^x s_{yy}(x,y)u_t(y,t)dy - [m(x,y)u_{xx}(y,t) - m_y(x,y)u_x(y,t)]_{y=0}^x - \int_0^x m_{yy}(x,y)u_x(y,t)dy + 2\lambda\int_0^x m(x,y)u_{xt}(y,t)dy - [g(x,y)u_{xxt}(y,t) - g_y(x,y)u_{xt}(y,t)]_0^x - \int_0^x g_{yy}(x,y)u_{xt}(y,t)dy + 2\lambda[g(x,y)u_{xx}(y,t) - g_y(x,y)u_x(y,t)]_0^x + 2\lambda\int_0^x g_{yy}(x,y)u_x(y,t)dy - 4\lambda^2\int_0^x g(x,y)u_{xt}(y,t)dy \quad (26)$$

Now, the first order space derivative of (24) can be computed as

$$w_x(x,t) = u_x(x,t) - s(x,x)u_t(x,t) - \int_0^x s_x(x,y)u_t(y,t)dy - m(x,x)u_x(x,t) - \int_0^x m_x(x,y)u_x(y,t)dy - g(x,x)u_{xt}(x,t) - \int_0^x g_x(x,y)u_{xt}(y,t)dy \quad (27)$$

and finally the second order space derivative of (24) is

$$w_{xx}(x,t) = u_{xx}(x,t) - s(x,x)u_{xt}(x,t) - (s'(x,x) + s_x(x,x))u_t(x,t) - \int_0^x s_{xx}(x,y)u_t(y,t)dy - m(x,x)u_{xx}(x,t) - (m'(x,x) + m_x(x,x))u_x(x,t) - \int_0^x m_{xx}(x,y)u_x(y,t)dy - g(x,x)u_{xxt}(x,t) - (g'(x,x) + g_x(x,x))u_{xt}(x,t) - \int_0^x g_{xx}(x,y)u_{xt}(y,t)dy \quad (28)$$

2) Kernel equations: As the considered backstepping transformation relies on Volterra integrals, we standardly solve the kernel equations on a triangle, i.e., $x \in [0,1]$, $y \in [0,x]$. The propagation phenomenon (6) fixes diagonal terms (e.g. $s(x,x)$), vertical terms (e.g. $s(x,0)$) and surface terms (e.g. $s(x,y)$). Moreover, the uncontrolled boundary fixes point-wise terms (e.g. $s(0,0)$). The control boundary fixes the control law. From (25)-(28), the propagation phenomenon (6) is verified if the following conditions are respected[2]

- Kernel surface terms (x,y)

$$\int u_t(y,t)dy: \quad s_{yy}(x,y) = s_{xx}(x,y) \quad (29)$$

$$\int u_x(y,t)dy: \quad m_{yy}(x,y) = m_{xx}(x,y) \quad (30)$$

$$\int u_{xt}(y,t)dy: \quad g_{yy}(x,y) = g_{xx}(x,y) \quad (31)$$

[2]the nomenclature used here is "factory term" " : " "condition"

- Kernel diagonal terms (x,x)

$$u_t(x,t): \quad s_y(x,x) = -s'(x,x) - s_x(x,x) \tag{32}$$
$$u_x(x,t): \quad m_y(x,x) = -m_x(x,x) - m'(x,x) \tag{33}$$
$$u_{xt}(x,t): \quad g_y(x,x) = -g_x(x,x) - g'(x,x) \tag{34}$$

- Kernel vertical terms $(x,0)$

$$u_t(0,t): \quad s_y(x,0) = (aq+2\lambda)m(x,0) + aq(aq+2\lambda)g(x,0) \tag{35}$$
$$u_x(0,t): \quad m_y(x,0) = am(x,0) + a(aq+2\lambda)g(x,0) \tag{36}$$
$$u_{xt}(0,t): \quad s(x,0) - g_y(x,0) + ag(x,0) = 0 \tag{37}$$

- Kernel point-wise terms $(0,0)$: To inspect these terms, note that the boundary condition (8) needs also to be verified. First one can get the following equation by expressing (24), (25), (26) and (27) for $x=0$, and using the uncontrolled boundary condition of the initial system (4)

$$w_{tt}(0,t) = a\,q\,u_t(0,t) + a\,u_x(0,t) \tag{38}$$
$$w_t(0,t) = u_t(0,t) \tag{39}$$
$$w_x(0,t) = u_x(0,t) - s(0,0)u_t(0,t) - m(0,0)u_x(0,t) - g(0,0)u_{xt}(0,t). \tag{40}$$

Then for the boundary condition (8) of the target system to be respected, one obtains the following conditions

$$u_t(0,t): \quad a_w\,s(0,0) = -(a\,q + a_w\,q_w) \tag{41}$$
$$u_x(0,t): \quad a_w\,m(0,0) = (a_w - a) \tag{42}$$
$$u_{tx}(0,t): \quad g(0,0) = 0 \tag{43}$$

3) Vector reformulation and explicit solving of the kernel equation: By denoting

$$S(x,y) = \begin{bmatrix} s(x,y) \\ m(x,y) \\ g(x,y) \end{bmatrix} \tag{44}$$

one can reformulate (32)-(37), and (41)-(43) as

$$S_{xx}(x,y) = S_{yy}(x,y) \tag{45}$$
$$S(x,x) = F \tag{46}$$
$$S_y(x,0) = HS(x,0) \tag{47}$$

in which F and H have been introduced in (11) and (12). As (45) is a pure wave equation, using Riemann invariants, there exist S^+ and S^- such that

$$S(x,y) = S^+(x+y) + S^-(x-y) \tag{48}$$

by expressing it for $y=x$ and using (46), one gets

$$S(x,x) = S^+(2x) + S^-(0) = F \tag{49}$$

and concludes that $S^+(x)$ is constant, so there exists $\widetilde{S}(x-y)$ such that

$$S(x,y) = \widetilde{S}(x-y) \tag{50}$$

and from (46)-(47), for all $x \in [0,1]$,

$$\widetilde{S}'(x) = -H\widetilde{S}(x) \tag{51}$$
$$\widetilde{S}(0) = F \tag{52}$$

which is a Cauchy problem. This proves the existence and uniqueness of the kernel. From (50)-(52) one can find (10).

4) Inversibility of the backstepping transformation: Let us consider the map $\Pi(u(x,t),u_t(x,t)) = (w(x,t),w_t(x,t))$ with $U(t)$ defined in (9). Π is the map that transforms the original system (2)-(4) into the target system (6)-(8). The existence of the inverse map Π^{-1} can be obtained by simply replacing a with a_w and q with q_w in the previous analysis. One gets the inversibility of the backstepping transformation straightforwardly.

5) Full state feedback control computation: We wish to show that the control law fixed by the boundary condition (7) and the backstepping transformation (24) can be expressed as (9). First, let us compute $w_x(1,t)$ and $w_t(1,t)$. Using integrations by parts on (27), and also (46), one obtains

$$w_x(1,t) = (1-m(1,1))u_x(1,t) - (g_x(1,1)+s(1,1))u_t(1,t) + g_x(1,0)u_t(0,t) + \int_0^1 (-s_x(1,y)+g_{xy}(1,y))u_t(y,t)dy - \int_0^1 m_x(1,y)u_x(y,t)dy \tag{53}$$

in which, using (10), $g_x(1,1)+s(1,1)=0$. Then, from (25), using integrations by parts and (37), one can write

$$w_t(1,t) = (1-m(1,1))u_t(1,t) + \int_0^1 (2\lambda s(1,y) + m_y(1,y) - 2\lambda g_y(1,y))u_t(y,t)dy + (g_y(1,1) - s(1,1)u_x(1,t) + (m(1,0)+aqg(1,0))u_t(0,t) + \int_0^1 (s_y(1,y) - g_{yy}(1,y))u_x(y,t)dy \tag{54}$$

which, using (10), allows to state that $g_y(1,1) - s(1,1) = 0$. Matching the expression (53) with (54), and using (3), one reaches the form of the control law (9).

C. Stability in terms of $\Gamma(t)$

To conclude on the exponential stability of (2)-(4) along with the control law (9), the equivalence between $V(w,w_t)$ in (15) and $\Gamma(u,u_t)$ in (13) remains to be proved. From the backstepping transformation (24), one can write

$$\|u_t\|^2 + \|u_x\|^2 \leqslant \alpha_1\|w_x\|^2 + \alpha_2\|w_t\|^2 + \alpha_3 w_t(0,t)^2 \tag{55}$$
$$u_t(0,t)^2 = w_t(0,t)^2 \tag{56}$$

in which α_1, α_3, and α_3 are positive constants. Thus, with (13), one obtains

$$\Gamma(u,u_t) \leqslant \alpha_1\|w_x\| + \alpha_2\|w_t\| + (\alpha_3+1)w_t(0,t)^2 \tag{57}$$

Consequently, from (15) and using Lemma 1, there exist $\mu_1 > 0$ and $\rho > 0$ such that

$$\Gamma(u,u_t) \leqslant \mu_1 V(w,w_t) \leqslant \mu_1 V(w(\cdot,0),w_t(\cdot,0))e^{-\rho t} \tag{58}$$

Then, from the inverse backstepping transformation investigated in Section IV-B.4, one can get

$$\|w_t\|^2 + \|w_x\|^2 \leqslant \alpha_4\|u_x\|^2 + \alpha_5\|u_t\|^2 + \alpha_6 u_t(0,t)^2 \tag{59}$$

in which α_4, α_5, and α_6 are positive constants. Therefore, one obtains the existence of $\mu_2 > 0$ such that

$$V(w,w_t) \leqslant \mu_2\Gamma(u,u_t) \tag{60}$$

Expressing (60) at time $t=0$ and matching it with (58) one finally reaches (14). This concludes the proof of Theorem 1.

V. OBSERVER BASED FEEDBACK

To provide a feedback law which can be implemented using only boundary measurements, we propose here to associate the full state feedback presented in the previous section with an observer. The following theorem is the extension of Theorem 1 in the case of observer-based control.

Theorem 2: *Consider the closed-loop system consisting of the plant* (2)-(4)*, the observer*

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) - 2\lambda \hat{u}_t(x,t) \quad (61)$$

$$\hat{u}_x(1,t) = U(t) - l_1(u_t(1,t) - \hat{u}_t(1,t)) \quad (62)$$

$$\hat{u}_{tt}(0,t) = aq\hat{u}_t(0,t) + a\hat{u}_x(0,t) - l_2(u_t(0,t) - \hat{u}_t(0,t)) \quad (63)$$

in which $l_1 > 0$ and $l_2 > aq$ and the control law

$$\begin{aligned} U(t) = -\hat{u}_t(1,t) + \frac{1}{(m(1,1)-1)} \Bigg[\int_0^1 & (2\lambda(s(1,y) - g_y(1,y)) \\ & + m_y(1,y) - s_x(1,y) + g_{xy}(1,y))\hat{u}_t(y,t)dy \\ & + (m(1,0) + aqg(1,0) + g_x(1,0))\hat{u}_t(0,t) \\ & + \int_0^1 (s_y(1,y) - g_{yy}(1,y) - m_x(1,y))\hat{u}_x(y,t)dy \Bigg] \end{aligned} \quad (64)$$

computed with (10)-(12) *in which $a_w > 0$ and $q_w > 0$. Define the functional*

$$\begin{aligned} \Gamma_{out}(u,u_t,\hat{u},\hat{u}_t) = & \, u_t(0,t)^2 + \int_0^1 u_t(x,t)^2 dx + \int_0^1 u_x(x,t)^2 dx \\ & + (\hat{u}_t - u_t)^2(0,t) + \int_0^1 \left[(\hat{u}_t - u_t)^2 + (\hat{u}_x - u_x)^2 \right](x,t)dx \end{aligned} \quad (65)$$

then there exist $\rho_{out} > 0$ and $R_{out} > 0$ such that, for all $t \geqslant 0$

$$\Gamma_{out}(u,u_t,\hat{u},\hat{u}_t) \leqslant R_{out} \Gamma_{out}(u,u_t,\hat{u},\hat{u}_t)\Big|_{(.,0)} e^{-\rho_{out} t} \quad (66)$$

and therefore the closed-loop system, for the attractor $\mathscr{A}$ defined in (5) *is exponentially stable.*

Proof: Consider the observation-error $\widetilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$, it can be shown that

$$\widetilde{u}_{tt}(x,t) = \widetilde{u}_{xx}(x,t) - 2\lambda \widetilde{u}_t(x,t) \quad (67)$$

$$\widetilde{u}_x(1,t) = -l_1 \widetilde{u}_t(1,t) \quad (68)$$

$$\widetilde{u}_{tt}(0,t) = -(l_2 - aq)\widetilde{u}_t(0,t) + a\widetilde{u}_x(0,t). \quad (69)$$

Following the same steps as in the proof of Lemma 1, the system (67)-(69) is exponentially stable with respect to $\widetilde{u}_x(.,t)$, $\widetilde{u}_t(.,t)$, and $\widetilde{u}_t(0,t)$ if $l_1 > 0$ and $l_2 > aq$. Then using Lemma 2, we can map the closed loop (2)-(4) with the control law (64) into the following plant

$$w_{tt}(x,t) = w_{xx}(x,t) - 2\lambda w_t(x,t) \quad (70)$$

$$w_x(1,t) = -w_t(1,t) - \widetilde{U}(t) \quad (71)$$

$$w_{tt}(0,t) = -a_w q_w w_t(0,t) + a_w w_x(0,t) \quad (72)$$

in which $\widetilde{U}(t)$ is the difference between (9) and (64). Following the same computations as the ones given in the proof of Lemma 1, and using the Lyapunov function (15) evaluated either for w or considering $\widetilde{u}$ instead of w, one can get the existence of $\eta_w > 0$ and $\eta_{\widetilde{u}} > 0$ such that, for any $\alpha > 0$

$$\begin{aligned} \dot{V}(w,w_t) + \alpha \dot{V}(\widetilde{u},\widetilde{u}_t) \leqslant & -\eta_w V(w,w_t) + \widetilde{U}(t)^2 \\ & - \alpha \eta_{\widetilde{u}} V(\widetilde{u},\widetilde{u}_t) \end{aligned} \quad (73)$$

Symbol	Description	Value
λ	In-domain damping coefficient	0.3
a	Uncontrolled boundary coefficient	0.6
q	Anti-damping coefficient	0.2
a_w	Uncontrolled boundary target coefficient	a
q_w	Damping target coefficient	q
l_1	Observer controlled boundary coefficient	0.01
l_2	Observer uncontrolled boundary coefficient	120% aq

Normalized coefficients (without unit) deduced from the physical model in [10] and [3]. The friction phenomenon used to compute q is described by the model in [14]

TABLE I

in which $V(w,w_t)$ is the Lyapunov function defined in (15) computed for the system (70)-(72) instead of system (6)-(8); $V(\widetilde{u},\widetilde{u}_t)$ is the Lyapunov function defined in (15) computed for the system (67)-(69) instead of system (6)-(8).

Using (9) and (64), applying Young and Cauchy Schwartz inequalities, there exists $\nu > 0$ such that

$$\widetilde{U}(t)^2 \leqslant \nu V(\widetilde{u},\widetilde{u}_t) \quad (74)$$

thus by choosing $\alpha < \frac{\nu}{\eta_{\widetilde{u}}}$, there exists η such that

$$\dot{V}(w,w_t) + \alpha \dot{V}(\widetilde{u},\widetilde{u}_t) \leqslant -\eta \left(V_w(w,w_t) + \alpha V(\widetilde{u},\widetilde{u}_t) \right) \quad (75)$$

Finally, one can prove the equivalence between $V(w,w_t) + \alpha V(\widetilde{u},\widetilde{u}_t)$ and $\Gamma_{out}(u,u_t,\hat{u},\hat{u}_t)$ using similar steps as in Section IV-C. This concludes the proof. ■

Note that the proper estimation of ν in (74) and thus η in (75) requires the computation of the backstepping kernel, and according, to Lemma 1, the maximal decreasing rate cannot be chosen for both the target system (6)-(8) and our observer (61)-(63).

VI. ILLUSTRATIVE EXAMPLE AND NUMERICAL SIMULATIONS

A. System definition

The wave equation (2)-(3) is a normalized linearized model for the stick-slip phenomenon [4] that occurs in drilling operation (e.g. [1], [2] and [10]). The drillstring angular oscillations can be modeled by a wave equation with a nonlinear boundary condition [11], accounting for the friction between the drillbit and the rock. Phenomenological expressions of this friction can be found ([14], [11] and [7]). Parameter values used in the simulation are given in Table I.

B. Numerical discretization and eigenvalues computation

First, a semi-discretization in space allows to rewrite the plant (2)-(4) under the state-space representation $\dot{X} = AX$ in which $X = \begin{bmatrix} u[1:n] & u_t[1:n] \end{bmatrix}^T$, with $A \in \mathbb{R}^{2n \times 2n}$ where n is the number of spatial points considered. Under this form, the eigenvalues of the target system (6)-(8), and the closed-loop system (2)-(4) with the control law (9), have been computed for $n = 30$ (see Figure 1). As expected, the control law efficiently fits the two previous systems eigenvalues (not perfectly due to numerical errors). One can observe that there is a couple eigenvalues very close to zero. Th corresponding eigenvectors are $u[i] = C \in \mathbb{R}$ and $u_t[i] = 0$ $\forall i = 1:n$. However, this is still consistent with our result; as the stability considered in Theorem 1 is towards the set $\mathscr{A}$ defined in (5), the vector space associated to this eigenvector

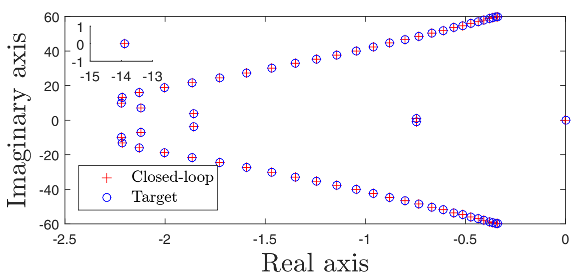

Fig. 1. Eigenvalues representation. The maximal relative error is $1.48e-5$.

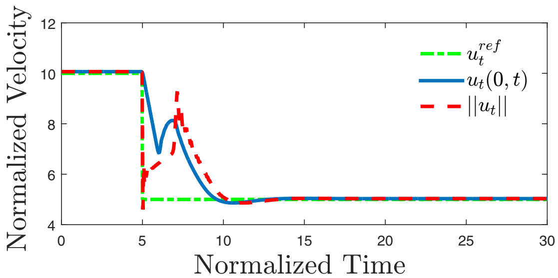

Fig. 2. Velocity for the full-state feedback. The stettling time -at 5%- for a reference step from 10 to 5 is about $\Delta t = 4.6$ for $u_t(0,t)$ and $\Delta t = 4.0$ for $\|u_t\|$

does not change the set $\mathscr{A}$. Indeed, we do not try to stabilize in u but in u_x and u_t.

C. State-feedback and observer based-control

Simulations have been performed with a tracking objective of u_t^{ref}. In details, let us consider the corresponding velocity tracking error $\check{u}_t(.,t) = u_t(.,t) - u_t^{\mathrm{ref}}$ and the torque tracking error $\check{u}_x(x,t) = u_x(x,t) + (q-2\lambda x)u_t^{\mathrm{ref}}$, $\forall x \in [0,1]$. One can obtain that $\check{u}(.,t)$ is solution of the system (2)-(4), and, apply Theorem 1 and 2 on $\check{u}(.,t)$ instead of $u(.,t)$, to perform tracking.

We consider that the system is initially at the equilibrium corresponding to u_t^{ref} and perform a reference step. The reference step is switched from $u_t^{ref} = 10$ to $u_t^{ref} = 5$ at $t = 5$.

First, the full state feedback case (i.e. system (3-5) with the control law (9)), solved in Theorem 1, is illustrated in Figure 2.

Figures 3 and 4 concern the observer-based control case (i.e. system (3-5) with the control law (67)), solved in Theorem 2. The observer starts at $t = 2$ and the reference step is switched from $u_t^{ref} = 10$ to $u_t^{ref} = 5$ at $t = 5$. Even if the transient behavior is different due to the presence of the observer, the closed-loop system still performs well and, moreover, the observer is efficient since the estimation errors converge to zero.

VII. CONCLUSION

An observer based-control law has been designed taking into account in-domain damping for a wave PDE. However, the target system and observer have a fixed maximal decreasing rate which is not satisfactory. Thus, future works will aim at finding alternative target plant and observer. In particular, to alleviate the decreasing rate limitation due to the in-domain damping, on-going work focuses on a target system with a different damping coefficient.

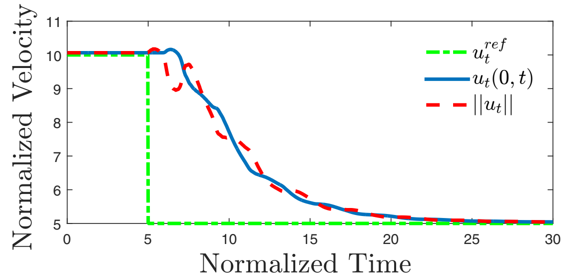

Fig. 3. Observer based-control. The settling time -at 5%- for a reference step from 10 to 5 is about $\Delta t = 14.0$ for $u_t(0,t)$ and $\Delta t = 13.9$ for $\|u_t\|$

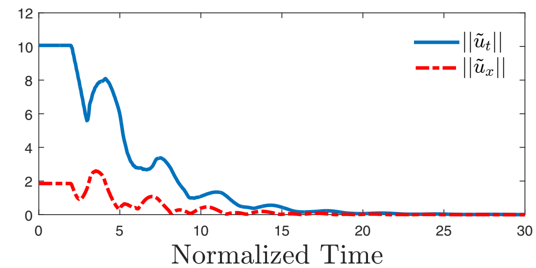

Fig. 4. Estimation errors. The observer is activated only from $t = 2$, the settling times are around $\Delta t = 11.8$ for $\|\tilde{u}_t\|$ and $\Delta t = 11.4$ for $\|\tilde{u}_x\|$.

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