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Stabilization of the linear Kuramoto-Sivashinsky equation with a delayed boundary control ⋆

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Abstract: In this paper we stabilize the linear Kuramoto-Sivashinsky equation by means of a delayed boundary control. From the spectral decomposition of the spatial operator associated to the equation, we find that there is a finite number of unstable eigenvalues. After applying the Artstein transform to deal with the delay phenomenon, we design a feedback law based on the pole-shifting theorem to exponential stabilize the finite-dimensional system associated to the unstable eigenvalues. Then, thanks to the use of a Lyapunov function, we prove that the same feedback law exponential stabilize the original unstable infinite-dimensional system.

Keywords: Partial differential equations; time delay; global stability; feedback stabilization; transformations; pole assignment; Lyapunov function.

1. INTRODUCTION

The Kuramoto-Sivashinsky equation

$$z_t + z_{xxxx} + \lambda z_{xx} + z z_x = 0, \tag{1}$$

where $\lambda > 0$ is known as the anti-diffusion parameter, was originally derived by Kuramoto and Tsuzuki (1975) as a model of phase turbulance in the context of a reaction-diffusion system, and by Sivashinsky (1977) in the context of flame front propagation. In the later case $z(t,x)$ represents the perturbation of a flame front which propagates in a combustible mixture.

In this paper we consider the linear Kuramoto-Sivashinsky equation

$$z_t + z_{xxxx} + \lambda z_{xx} = 0. \tag{2}$$

In both (1) and (2) the presence of $\lambda > 0$ induces instability. Let $L > 0$ be the length of the domain and $D \geq 0$ be the delay. In this paper we consider

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = 0, \ (t,x) \in (0,\infty) \times (0,L), \\ z(t,0) = u(t-D), \ t \in (0,\infty), \\ z(t,L) = z_x(t,0) = z_x(t,L) = 0, \ t \in (0,\infty), \\ z(0,x) = z_0(x), \ x \in (0,L), \end{cases} \tag{3}$$

where $u : [-D,\infty) \to \mathbb{R}$ is the delayed boundary control, which we ask it to satisfy

$$u(t-D) = 0 \text{ when } t \in [0,D). \tag{4}$$

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Let us note that we are dealing with a unstable partial differential equation (PDE). Indeed, it is known from (Liu and Krstić, 2001, Remark 1) that uncontrolled (3) is unstable if $\lambda > 4\pi^2/L^2$.

Only in the undelayed case, which is when $D = 0$, there are boundary stabilization results for (1) or (2); as can be seen in Liu and Krstić (2001), Cerpa (2010), and Coron and Lü (2015). To our knowledge, this is the first paper including delay in the boundary stabilization of (2).

In Datko et al. (1986) and Datko (1988) there are examples of one-dimensional hyperbolic PDEs stabilized by means of boundary feedback controls that are unrobust with respect to delays, in the sense that the inclusion of delay in the boundary feedback control might cause the apparition of a unstable solution. This behavior has also been observed in Nicaise and Pignotti (2006) for the multi-dimensional wave equation and in Nicaise and Rebiai (2011) for the multi-dimensional Schrödinger equation. Nevertheless, let us note that the delay not always causes destabilization. Indeed, there are examples of boundary feedback controls that are robust with respect to delays, in the sense described above, as can be seen in Xu and Wang (2013) for the Timoshenko beam system, in Baudouin et al. (2018) for the Korteweg-de Vries equation, and in Prieur and Trélat (2018) for a reaction-diffusion equation.

The stabilization of abstract second order evolution equations by means of unbounded feedback controls with delay has been addressed in Fridman et al. (2010), thus in particular solving a large class of boundary stabilization problems for hyperbolic PDEs. Of course, (3) cannot be put in terms of their formalism.

The main contribution of this paper is the exponential stabilization of (3) in $H_0^2(0,L)$ by means of a feedback control that is designed from a finite-dimensional system with input delay $D \geq 0$.

The rest of this paper is organized as follows. Our main result is presented, along with some comments on its proof, in Section 2. Then, its proof is given in Section 3. Finally, in Section 4 we give some concluding remarks.

Notation. y_t (respectively y_x) stands for the partial derivative of the function y with respect to t (respectively x). The scalar product of the Hilbert space $L^2(0,L)$ is denoted by $(\cdot,\cdot)_{L^2(0,L)}$. The Sobolev space $H^2(0,L)$ is formed by the $y \in L^2(0,L)$ such that $y_x \in L^2(0,L)$ and $y_{xx} \in L^2(0,L)$ (the partial derivatives are taken in the sense of distributions). The Sobolev space $H_0^2(0,L)$ is the closure of $C_0^\infty(0,L)$ in $H^2(0,L)$ and it is equipped with the norm $\|y\|^2_{H_0^2(0,L)} = \int_0^L |y_{xx}|^2 dx$. The theory of Sobolev spaces may be found in Adams and Fournier (2003).

2. MAIN RESULT

In order to present our main result we need to introduce the anti-diffusion set of critical parameters

$$\mathcal{AD} = \left\{(j^2+k^2)\pi^2 L^{-2} \;/\; (j,k) \in \mathbb{N}^2 \text{ with the same parity and } j<k\right\} \cup \left\{4l^2\pi^2L^{-2} \;/\; l \in \mathbb{N}\right\},$$

which has been found in Cerpa et al. (2017) for the study of the null controllability of (3) when $D = 0$. Our main result is the following one.

Theorem 1. Let $\lambda \in (0,\infty)\backslash\mathcal{AD}$ and $z_0 \in H_0^2(0,L)$. Then, (3) is exponentially stabilizable with a feedback control that is designed from a finite-dimensional system with input delay $D \geq 0$. Furthermore, there exist $C \geq 1$ and $\omega > 0$ such that

$$|u(t-D)| + \|z(t,\cdot)\|_{H_0^2(0,L)} \leq Ce^{-\omega t}\|z_0\|_{H_0^2(0,L)}, \; t \geq 0.$$

To prove our main result we follow the approach developed in Prieur and Trélat (2018). Thus, we first consider the spectral decomposition of the spatial operator associated to (3) with the purpose to decompose the unstable infinite-dimensional system into two parts. The first part consists of a finite-dimensional system containing all the unstable eigenvalues, whereas the second part consists of a infinite-dimensional system containing the remaning eigenvalues, which are stable. Then, after applying the Artstein transform to deal with the delay phenomenon, we design a feedback law based on the pole-shifting theorem to exponential stabilize the unstable finite-dimensional system. Finally, thanks to the use of a Lyapunov function, we prove that the same feedback law exponential stabilize the original unstable infinite-dimensional system.

Remark 2. The assumption $\lambda \in (0,\infty)\backslash\mathcal{AD}$ is required to check the Kalman condition on the pair of matrices associated to the unstable finite-dimensional system mentioned above, so that later we can stabilize it by applying the pole-shifting theorem. Accordingly, another approach is needed to stabilize (3) if $\lambda \in \mathcal{AD}$.

3. FEEDBACK DESIGN

3.1 Spectral decomposition

In order to get rid of the non-homogeneous boundary conditions in (3), let us introduce the lifting function

$$y(t,x) = z(t,x) - u_D(t)d(x), \tag{5}$$

where $u_D(t) = u(t-D)$ and $d(x) = 2L^{-3}x^3 - 3L^{-2}x^2 + 1$. Since $d(0) = 1$ and $d(L) = d'(0) = d'(L) = 0$, and also $u_D(0) = 0$ by (4), we get from (3) that $y(t,x)$ satisfies

$$\begin{cases} y_t = -y_{xxxx} - \lambda y_{xx} - du_D' - \lambda d'' u_D, \\ y(t,0) = y(t,L) = y_x(t,0) = y_x(t,L) = 0, \\ y(0,x) = z_0(x), \end{cases} \tag{6}$$

Let us view (6) as an infinite-dimensional system. To this end, let us introduce the operator

$$\begin{cases} A : D(A) \subset L^2(0,L) \to L^2(0,L), \\ A\phi = -\phi'''' - \lambda\phi'', \\ D(A) = H^4(0,L) \cap H_0^2(0,L). \end{cases} \tag{7}$$

Therefore, setting $a(x) = -\lambda d''(x)$ and $b(x) = -d(x)$, we can view (6) as

$$y_t(t,\cdot) = Ay(t,\cdot) + a(\cdot)u_D(t) + b(\cdot)u_D'(t). \tag{8}$$

Since A is a self-adjoint operator with compact resolvent, we can consider a Hilbert basis $\{\phi_m\}_{m\in\mathbb{N}} \subset D(A)$ of $L^2(0,L)$ composed by eigenfunctions of A. Furthermore, the corresponding sequence of eigenvalues $\{\sigma_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$ satisfies

$$\begin{cases} \sigma_m \geq \sigma_{m+1} > -\infty \text{ for each } m \in \mathbb{N}, \\ \sigma_m \to -\infty \text{ as } m \to \infty. \end{cases} \tag{9}$$

From (Cerpa, 2010, Section 2) we know that $\sigma_m < \lambda^2/4$ for each $m \in \mathbb{N}$, which tells us, in view of (9), that only a finite number of eigenvalues are unstable. Then, let $n \in \mathbb{N}$ be the number of nonnegative eigenvalues. This yields that $\sigma_m \geq 0$ if $m \in \{1,\ldots,n\}$ and $\sigma_m < 0$ if $m \in \{n+1,\ldots\}$.

Let us construct a finite-dimensional system containing all the unstable eigenvalues. Any solution $y(t,\cdot)$ of (8) can be written in the form

$$y(t,x) = \sum_{m=1}^{\infty} y_m(t)\phi_m(x). \tag{10}$$

Let us set $a_m = (a,\phi_m)_{L^2(0,L)}$ and $b_m = (b,\phi_m)_{L^2(0,L)}$. From (8), (10) and the fact that $A\phi_m = \sigma_m\phi_m$ we deduce

$$\begin{cases} y_1'(t) = \sigma_1 y_1(t) + a_1 u_D(t) + b_1 u_D'(t), \\ \vdots \quad \vdots \qquad \vdots \\ y_n'(t) = \sigma_n y_n(t) + a_n u_D(t) + b_n u_D'(t), \end{cases} \tag{11}$$

Thus, (11) is a system of $n \in \mathbb{N}$ ordinary differential equations controlled by $u_D(t)$ and $u_D'(t)$. Let us set

$$\alpha_D(t) = \alpha(t-D) = u_D'(t), \tag{12}$$

and consider now $u_D(t)$ as a new state and $\alpha_D(t)$ as the control we aim to design. Then, with the aid of the matrices

$$X_1 = \begin{bmatrix} u_D \\ y_1 \\ \vdots \\ y_n \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ a_1 & \sigma_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \dots & \sigma_n \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad (13)$$

we can construct a unstable finite-dimensional system from (11) together with (12), namely

$$X_1'(t) = A_1 X_1(t) + B_1 \alpha_D(t). \quad (14)$$

Finally, recalling that $\alpha_D(t) = \alpha(t - D)$, we see that the control in (14) has a input delay $D \geq 0$.

3.2 Stabilization of the finite-dimensional system

In order to deal with the delay in (14) we proceed as in Bresch-Pietri et al. (2018) and Prieur and Trélat (2018). Thus, let us consider the Artstein transform

$$Z_1(t) = X_1(t) + \int_{t-D}^{t} e^{(t-s-D)A_1} B_1 \alpha(s)\, ds. \quad (15)$$

Using (15), which has been introduced in Artstein (1982), we can transform (14) into the unstable finite-dimensional system

$$Z_1'(t) = A_1 Z_1(t) + e^{-DA_1} B_1 \alpha(t). \quad (16)$$

This time we see that the control in (16) has no input delay. Let us check that the pair of matrices $\left(A_1, e^{-DA_1} B_1\right)$ satisfies the Kalman condition for any $D \geq 0$. To this end, we need the following result.

Lemma 3. The eigenfunctions of A satisfy $\phi_m'''(0) \neq 0$ for each $m \in \mathbb{N}$ if and only if $\lambda \in (0,\infty)\backslash\mathcal{AD}$. Furthermore, if $\lambda \in (0,\infty)\backslash\mathcal{AD}$, then the eigenvalues of A are simple.

Proof. Let us recall that A is defined in (7). We only show the second statement of this lemma, since the first one has already been shown in (Cerpa et al., 2017, Section 2.3). We proceed as in the proof of (Cerpa, 2010, Lemma 2.1). Thus, let ϕ_1 and ϕ_2 be two eigenfunctions associated to the same eigenvalue σ. Then, we have that $\Phi = \phi_1'''(0)\phi_2 - \phi_2'''(0)\phi_1$ satisfies $A\Phi = \sigma\Phi$ with $\Phi \in D(A)$. The extra information of $\Phi'''(0) = 0$ allows us to infer that $\Phi = 0$ in $[0, L]$. Accordingly, ϕ_1 and ϕ_2 are linearly dependent because $\phi_1'''(0)$ and $\phi_2'''(0)$ are different from zero by the first statement of this lemma. •

Lemma 4. Let $\lambda \in (0,\infty)\backslash\mathcal{AD}$. Then, for any $D \geq 0$ the pair of matrices $\left(A_1, e^{-DA_1} B_1\right)$ satisfies the Kalman condition.

Proof. Since the matrices A_1 and e^{-DA_1} commute, and the matrix e^{-DA_1} is invertible, it follows

$$\begin{aligned} &\operatorname{rank}\left(e^{-DA_1} B_1, A_1 e^{-DA_1} B_1, \dots, A_1^n e^{-DA_1} B_1\right) \\ &= \operatorname{rank}\left(B_1, A_1 B_1, \dots, A_1^n B_1\right), \end{aligned}$$

which means that we have to check that the pair of matrices (A_1, B_1) satisfies the Kalman condition. Some computations lead us to

$$\begin{aligned} &\det\left(B_1, A_1 B_1, \dots, A_1^n B_1\right) \\ &= VdM(\sigma_1, \dots, \sigma_n)\Pi_{m=1}^n (a_m + \sigma_m b_m), \end{aligned} \quad (17)$$

where

$$VdM(\sigma_1, \dots, \sigma_n) = \Pi_{1 \leq i < j \leq n}(\sigma_j - \sigma_i),$$

is a Van der Monde determinant, which is different from zero because the eigenvalues of A are simple in virtue of Lemma 3. Some integrations by parts and the fact that $-\phi_m'''' - \lambda\phi_m'' = \sigma_m \phi_m$ yield

$$\begin{aligned} a_m + \sigma_m b_m &= (-\lambda d'' - \sigma_m d, \phi_m)_{L^2(0,L)} \\ &= (d, \phi_m'''')_{L^2(0,L)} \\ &= d(L)\phi_m'''(L) - d(0)\phi_m'''(0). \end{aligned} \quad (18)$$

Finally, since $d(L) = 0$ and $d(0) = 1$, we conclude that (17) is different from zero due to (18) and Lemma 3. •

Thanks to Lemma 4 we have that (16) is controllable. Accordingly, we can apply the pole-shifting theorem to exponential stabilize it.

Corollary 5. Let $\lambda \in (0,\infty)\backslash\mathcal{AD}$. Then, for any $D \geq 0$ there exists a gain matrix $K_1(D) \in \mathbb{R}^{1\times(n+1)}$ such that the matrix $A_2(D) = A_1 + e^{-DA_1} B_1 K_1(D)$ admits -1 as an eigenvalue of order $(n+1)$. Furthermore, there exists a symmetric positive matrix $P(D) \in \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$P(D)A_2(D) + A_2(D)^\top P(D) = -I_{n+1}. \quad (19)$$

In virtue of Corollary 5 we can construct the Lyapunov function

$$V_1(t) = \frac{1}{2} Z_1(t)^\top P(D) Z_1(t), \quad (20)$$

to prove that the feedback control $\alpha(t) = K_1(D)Z_1(t)$ exponential stabilize (16). However, from (12) together with (4) we see that we actually have to select

$$\begin{cases} \alpha(t) = 0 \text{ when } t \in [-D, 0), \\ \alpha(t) = K_1(D)Z_1(t) \text{ when } t \in [0, \infty). \end{cases} \quad (21)$$

Lemma 6. Let $\lambda \in (0,\infty)\backslash\mathcal{AD}$. Then, for any $D \geq 0$ we have

$$\begin{aligned} &\|\alpha_D(t)\|_{\mathbb{R}^{n+1}} \\ &\leq e^{D\|A_2(D)\|}\|K_1(D)^\top\|_{\mathbb{R}^{n+1}}\|Z_1(t)\|_{\mathbb{R}^{n+1}},\ t \geq 0. \end{aligned} \quad (22)$$

Proof. Let us recall that $\alpha_D(t) = \alpha(t - D)$. By (21) it follows that (22) is true for $t \in [0, D)$. Thus, let us prove (22) only for $t \in [D, \infty)$. Plugging (21) into (16) we find that $Z_1(t) = \Phi(t)Z_1(0)$ for $t \geq 0$, where $\Phi(t) = \exp\{A_2(D)t\}$. We deduce (22) from (21), the fact that $\Phi(t - D) = \Phi(-D)\Phi(t)$ for $t \in [D, \infty)$, and the application of Cauchy-Schwarz inequality. •

If $\sigma_{min}(P(D)) > 0$ and $\sigma_{max}(P(D)) > 0$ respectively denote the lowest and greatest eigenvalue of the symmetric positive matrix $P(D) \in \mathbb{R}^{(n+1)\times(n+1)}$, then it holds

$$\begin{aligned}&\sigma_{min}(P(D))\,\|Z_1(t)\|^2_{\mathbb{R}^{n+1}}\\ &\leq Z_1(t)^\top P(D) Z_1(t) \leq \sigma_{max}(P(D))\,\|Z_1(t)\|^2_{\mathbb{R}^{n+1}}.\end{aligned} \tag{23}$$

3.3 Stabilization of the infinite-dimensional system

Plugging (21) into (15) we get

$$Z_1(t) = X_1(t) + \int_{I(t)} e^{(t-s-D)A_1} B_1 K_1(D) Z_1(s)\,ds, \tag{24}$$

where $I(t) = (t-D,t)\cap(0,\infty)$. So far the feedback control $\alpha(t)$ makes $X_1(t)$ to go exponentially to zero as $t \to \infty$. Indeed, this comes from (24) and the fact that $Z_1(t)$ goes exponentially to zero as $t \to \infty$. Let us prove that the same feedback control $\alpha(t)$ also exponential stabilize (8). To this end, let us introduce the Lyapunov function constructed in Prieur and Trélat (2018), which is

$$\begin{aligned}V_2(t) \;=&\; -\frac{1}{2}\,(y(t,\cdot), Ay(t,\cdot))_{L^2(0,L)}\\ &+M(D)V_1(t) + M(D)\int_{I(t)} V_1(s)\,ds,\end{aligned} \tag{25}$$

where $M(D) > 0$. The first two terms in (25) are needed to handle the unstable part of (8) produced by the finite number of nonnegative eigenvalues, while its third term is needed to tackle the delay phenomenon. With (7) and (20) in mind, we can rewrite (25) as

$$\begin{aligned}V_2(t) \;=&\; -\frac{1}{2}\sum_{m=1}^{\infty}\sigma_m y_m(t)^2 + \frac{M(D)}{2} Z_1(t)^\top P(D) Z_1(t)\\ &+\frac{M(D)}{2}\int_{I(t)} Z_1(s)^\top P(D) Z_1(s)\,ds.\end{aligned} \tag{26}$$

The next three lemmas, which are independent from each other, give the necessary tools to prove our main result, which is the one given by Theorem 1.

Lemma 7. Let $\lambda \in (0,\infty)\backslash\mathcal{AD}$. Let us set the positive constants

$$C_1(D) = \max\left\{2, 2e^{4D\|A_1\|}\|B_1K_1(D)\|\right\},$$

$$C_2(D) = \min\left\{\frac{\sigma_1}{2}, \frac{1}{2}\right\},\; C_3(D) = \max\left\{\lambda^2, 2 - \frac{\lambda^2}{\sigma_{n+1}}\right\},$$

$$C_4(D) = \frac{2\sigma_1 C_1(D)}{\sigma_{min}(P(D))}.$$

Also, let us assume that $M(D) \geq C_4(D)$. Then, for any $D \geq 0$ we have

$$\begin{aligned}&\frac{C_2(D)}{2}\|X_1(t)\|^2_{\mathbb{R}^{n+1}}\\ &+\frac{C_2(D)}{2C_3(D)}\|y(t,\cdot)\|^2_{H_0^2(0,L)} \leq V_2(t),\; t \geq 0.\end{aligned} \tag{27}$$

Proof. Firstly, by (24) we have

$$\begin{aligned}\|X_1(t)\|^2_{\mathbb{R}^{n+1}} &= X_1(t)^\top Z_1(t)\\ &-X_1(t)^\top \int_{I(t)} e^{(t-s-D)A_1} B_1 K_1(D) Z_1(s)\,ds,\end{aligned} \tag{28}$$

from which we get, by applying the Cauchy-Schwarz and Cauchy inequalities,

$$\begin{aligned}\|X_1(t)\|^2_{\mathbb{R}^{n+1}} \;\leq&\; C_1(D)\|Z_1(t)\|^2_{\mathbb{R}^{n+1}}\\ &+C_1(D)\int_{I(t)}\|Z_1(s)\|^2_{\mathbb{R}^{n+1}}\,ds.\end{aligned} \tag{29}$$

Then, combining (26) with (23) and (29) we find

$$\begin{aligned}V_2(t) \;\geq&\; M(D)\frac{\sigma_{min}(P(D))}{2C_1(D)}\|X_1(t)\|^2_{\mathbb{R}^{n+1}}\\ &-\frac{1}{2}\sum_{m=1}^{\infty}\sigma_m y_m(t)^2.\end{aligned} \tag{30}$$

Until the end of the proof we need to keep in mind that $\sigma_1 > \sigma_m \geq 0$ if $m \in \{2,\ldots,n\}$ and $0 > \sigma_{n+1} \geq \sigma_m$ if $m \in \{n+1,\ldots\}$. In (30) we use

$$\begin{aligned}&-\frac{1}{2}\sum_{m=1}^{\infty}\sigma_m y_m(t)^2\\ &\geq -\frac{\sigma_1}{2}\|X_1(t)\|^2_{\mathbb{R}^{n+1}} - \frac{1}{2}\sum_{m=n+1}^{\infty}\sigma_m y_m(t)^2,\end{aligned} \tag{31}$$

and then consider that $M(D) \geq C_4(D)$ to obtain

$$\begin{aligned}V_2(t) \;\geq&\; C_2(D)\|X_1(t)\|^2_{\mathbb{R}^{n+1}}\\ &-C_2(D)\sum_{m=n+1}^{\infty}\sigma_m y_m(t)^2.\end{aligned} \tag{32}$$

Secondly, thanks to (10) and some integrations by parts we have

$$\begin{aligned}\|y(t,\cdot)\|^2_{H_0^2(0,L)} &= \sum_{(i,j)\in\mathbb{N}^2} y_i(t)y_j(t)\left(\phi_i'', \phi_j''\right)_{L^2(0,L)}\\ &= \sum_{(i,j)\in\mathbb{N}^2} y_i(t)y_j(t)\left(\phi_i'''', \phi_j\right)_{L^2(0,L)}.\end{aligned} \tag{33}$$

Plugging $-\phi_i'''' - \lambda\phi_i'' = \sigma_i\phi_i$ into (33) and then applying Cauchy inequality it follows

$$\begin{aligned}&\|y(t,\cdot)\|^2_{H_0^2(0,L)} = -\lambda\,(y'', y)_{L^2(0,L)} - \sum_{m=1}^{\infty}\sigma_m y_m(t)^2\\ &\leq \lambda^2\sum_{m=1}^{\infty} y_m(t)^2 - 2\sum_{m=n+1}^{\infty}\sigma_m y_m(t)^2\\ &\leq \lambda^2\|X_1(t)\|^2_{\mathbb{R}^{n+1}} + \sum_{m=n+1}^{\infty}\left(\lambda^2 - 2\sigma_m\right) y_m(t)^2.\end{aligned} \tag{34}$$

Since $\lambda^2 - 2\sigma_m = -\sigma_m\left(2-\lambda^2/\sigma_m\right) \le -\sigma_m\left(2-\lambda^2/\sigma_{n+1}\right)$ for $m \in \{n+1,\ldots\}$ we find

$$\begin{aligned}\|y(t,\cdot)\|^2_{H_0^2(0,L)} \le\ & C_3(D)\|X_1(t)\|^2_{\mathbb{R}^{n+1}} \\ & -C_3(D)\sum_{m=n+1}^{\infty}\sigma_m y_m(t)^2.\end{aligned} \tag{35}$$

Finally, from the combination of (32) and (35) we arrive at (27). •

Lemma 8. Let $\lambda \in (0,\infty)\backslash\mathcal{AD}$. Let us set the positive constants

$$\begin{aligned}C_5(D) =\ & 4\|a\|^2_{L^2(0,L)}C_1(D) \\ & +4e^{2D\|A_2(D)\|}\|b\|^2_{L^2(0,L)}\|K_1(D)^\top\|^2_{\mathbb{R}^{n+1}}, \\ C_6(D) =\ & \max\{-1/\sigma_{n+1}, C_5(D), 2\sigma_{max}(P(D))\}.\end{aligned}$$

Also, let us assume that $M(D) \ge C_6(D)$. Then, for any $D \ge 0$ we have

$$V_2(t) \le \exp\left\{-\frac{1}{M(D)}t\right\}V_2(0),\ t \ge 0. \tag{36}$$

Proof. Let us note that (36) is true for $t = 0$. Thus, let us prove (36) only for $t > 0$ by computing the derivative of $V_2(t)$. By (20) and the use of (19) we get

$$\begin{aligned}&\frac{d}{dt}V_1(t) + \frac{d}{dt}\int_{I(t)} V_1(s)\,ds \\ &= -\frac{1}{2}\|Z_1(t)\|^2_{\mathbb{R}^{n+1}} - \frac{1}{2}\int_{I(t)}\|Z_1(s)\|^2_{\mathbb{R}^{n+1}}ds.\end{aligned} \tag{37}$$

Taking into account that A is a self-adjoint operator it follows

$$\frac{d}{dt}\left(y(t,\cdot), Ay(t,\cdot)\right)_{L^2(0,L)} = 2\left(Ay(t,\cdot), y_t(t,\cdot)\right)_{L^2(0,L)}.$$

Here we plug (8), then apply Cauchy inequality and finally consider (12), (13) and (22) to obtain

$$\begin{aligned}&-\frac{d}{dt}\left(y(t,\cdot), Ay(t,\cdot)\right)_{L^2(0,L)} \\ &\le -\|Ay(t,\cdot)\|^2_{L^2(0,L)} + 2\|a\|^2_{L^2(0,L)}\|X_1(t)\|^2_{\mathbb{R}^{n+1}} \\ &+2e^{2D\|A_2(D)\|}\|b\|^2_{L^2(0,L)}\|K_1(D)^\top\|^2_{\mathbb{R}^{n+1}}\|Z_1(t)\|^2_{\mathbb{R}^{n+1}}.\end{aligned} \tag{38}$$

Let us recall that $\sigma_m \ge 0$ if $m \in \{1,\ldots,n\}$ and $0 > \sigma_{n+1} \ge \sigma_m$ if $m \in \{n+1,\ldots\}$. Then, the choice of $M(D) \ge -1/\sigma_{n+1}$ allows us to infer

$$-\frac{1}{2}\|Ay(t,\cdot)\|^2_{L^2(0,L)} \le \frac{1}{2M(D)}\sum_{m=1}^{\infty}\sigma_m y_m(t)^2. \tag{39}$$

With (25) in mind, we combine (37), (38), (29), and (39) to obtain

$$\begin{aligned}V_2'(t) \le & -\left(\frac{M(D)}{2} - \|a\|^2_{L^2(0,L)}C_1(D)\right. \\ & \left. - e^{2D\|A_2(D)\|}\|b\|^2_{L^2(0,L)}\|K_1(D)^\top\|^2_{\mathbb{R}^{n+1}}\right)\|Z_1(t)\|^2_{\mathbb{R}^{n+1}} \\ & -\left(\frac{M(D)}{2} - \|a\|^2_{L^2(0,L)}C_1(D)\right)\int_{I(t)}\|Z_1(s)\|^2_{\mathbb{R}^{n+1}}ds \\ & +\frac{1}{2M(D)}\sum_{m=1}^{\infty}\sigma_m y_m(t)^2.\end{aligned} \tag{40}$$

Before going any further, let us note that the choice of $M(D) \ge C_5(D)$ implies

$$\begin{aligned}&\frac{M(D)}{2} - \|a\|^2_{L^2(0,L)}C_1(D) \\ &-e^{2D\|A_2(D)\|}\|b\|^2_{L^2(0,L)}\|K_1(D)^\top\|^2_{\mathbb{R}^{n+1}} \ge \frac{M(D)}{4}.\end{aligned} \tag{41}$$

Therefore, combining (40) with (41), and then applying (23), we get

$$\begin{aligned}V_2'(t) \le\ & \frac{1}{M(D)}\frac{1}{2}\sum_{m=1}^{\infty}\sigma_m y_m(t)^2 \\ & -\frac{1}{2\sigma_{max}(P(D))}\frac{M(D)}{2}Z_1(t)^\top P(D)Z_1(t) \\ & -\frac{1}{2\sigma_{max}(P(D))}\frac{M(D)}{2}\int_{I(t)} Z_1(s)^\top P(D)Z_1(s)\,ds,\end{aligned}$$

from which we deduce (36) in virtue of the choice of $M(D) \ge 2\sigma_{max}(P(D))$ and (26). •

Lemma 9. Let $\lambda \in (0,\infty)\backslash\mathcal{AD}$. Let us set the positive constant

$$C_7(D) = \frac{1}{2} + \frac{M(D)}{2}\sigma_{max}(P(D))\frac{L^4}{\pi^4}.$$

Then, for any $D \ge 0$ we have

$$V_2(0) \le C_7(D)\|y_0\|^2_{H_0^2(0,L)}. \tag{42}$$

Proof. In view of $I(0) = \emptyset$ it follows from (24) that $Z_1(0) = X_1(0)$. Accordingly, by (26) together with (34) we find

$$\begin{aligned}V_2(0) =\ & \frac{1}{2}\|y_0\|^2_{H_0^2(0,L)} + \frac{\lambda}{2}\left(y_0'', y_0\right)_{L^2(0,L)} \\ & +\frac{M(D)}{2}X_1(0)^\top P(D)X_1(0).\end{aligned}$$

By (4) and (5) we deduce that $y_0 \in H_0^2(0,L)$. Then, one integration by parts, the application of (23) and the fact that $\|X_1(0)\|^2_{\mathbb{R}^{n+1}} \le \|y_0\|^2_{L^2(0,L)}$ give

$$V_2(0) \le \frac{1}{2}\|y_0\|^2_{H_0^2(0,L)} + \frac{M(D)}{2}\sigma_{max}(P(D))\|y_0\|^2_{L^2(0,L)},$$

from which we arrive at (42) thanks to Poincaré inequality $\|y_0\|^2_{L^2(0,L)} \le (L^4/\pi^4)\|y_0\|^2_{H_0^2(0,L)}$. •

Let us give a proof of Theorem 1.

Proof. In order to apply the previous three lemmas we need to choose $M(D) = \max\{C_4(D), C_6(D)\}$. Then, by (27), (36), and (42) it follows

$$\frac{C_2(D)}{2}\|X_1(t)\|^2_{\mathbb{R}^{n+1}} + \frac{C_2(D)}{2C_3(D)}\|y(t,\cdot)\|^2_{H_0^2(0,L)}$$
$$\leq C_7(D)\exp\left\{-\frac{1}{M(D)}t\right\}\|y_0\|^2_{H_0^2(0,L)},\ t \geq 0.$$

from which we deduce the desired result thanks to (4), (5) and (13). •

3.4 Inversion of the Artstein transform

We would like to express $\alpha(t)$ in terms of $X_1(t)$. To this end, we present the inversion of the Artstein transform realized in Bresch-Pietri et al. (2018). Thus, taking into account that $(t-D,t) = (t-D, \max\{t-D,0\}) \cup J(t)$, where $J(t) = (\max\{t-D,0\},t)$, we plug (15) into (21) and then use the fact that $\alpha(t) = 0$ when $t \in [-D,0)$ to get

$$\begin{aligned}&\alpha(t) = K_1(D)X_1(t)\\ &+K_1(D)\int\limits_{J(t)} e^{(t-s-D)A_1}B_1\alpha(s)\,ds,\ t \geq 0.\end{aligned} \tag{43}$$

Then, for a function F let us define

$$\begin{aligned}&(T_D F)(t)\\ &= K_1(D)\int\limits_{J(t)} e^{(t-s-D)A_1}B_1F(s)\,ds,\ t \geq 0.\end{aligned} \tag{44}$$

Let us consider $t \geq 0$. In virtue of (44) we can rewrite (43) as $\alpha(t) = K_1(D)X_1(t) + (T_D\alpha)(t)$, which is equivalent to $(I - T_D)\alpha(t) = K_1(D)X_1(t)$. The convergence of the Neumann series associated to T_D has been shown in (Bresch-Pietri et al., 2018, Section 4.1), which implies the existence of $(I-T_D)^{-1}$. Accordingly, we have

$$\alpha(t) = \sum_{m=0}^{\infty}\left(T_D^m K_1(D)X_1\right)(t),\ t \in [0,\infty). \tag{45}$$

Remark 10. The series in (45) converges for any $D \geq 0$. Let us note that the value of $\alpha(t)$ depends on $X_1(s)$ over $0 < s < t$. Further details may be found in Bresch-Pietri et al. (2018).

4. CONCLUSION

In this paper we have exponentially stabilized in $H_0^2(0,L)$ the linear Kuramoto-Sivashinsky equation by means of a feedback control designed from a finite-dimensional system with input delay $D \geq 0$. As in the null controllability, our main assumption was $\lambda \in (0,\infty)\backslash\mathcal{AD}$. To prove our main result we have followed the approach developed in Prieur and Trélat (2018), which relies on the Artstein transform, a careful spectral analysis and the pole-shifting theorem. Our next step will be to exponential stabilize in $H_0^2(0,L)$ the Kuramoto-Sivashinsky equation with a delayed boundary control.

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