

Boundary Control of Korteweg-de Vries and Kuramoto-Sivashinsky PDEs

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Abstract

The Korteweg-de Vries (KdV) and the Kuramoto-Sivashinsky (KS) partial differential equations are used to model nonlinear propagation of one-dimensional phenomena. The KdV equation is used in fluid mechanics to describe waves propagation in shallow water surfaces, while the KS equation models front propagation in reaction-diffusion systems. In this article, the boundary control of these equations is considered when they are posed on a bounded interval. Different choices of controls are studied for each equation.

Keywords and Phrases. Controllability, stabilizability, higher order partial differential equations, dispersive equations, parabolic equations

Introduction

The Korteweg-de Vries (KdV) and the Kuramoto-Sivashinsky (KS) equations have very different properties because they do not belong to the same class of partial differential equations (PDEs). The first one is a third-order nonlinear dispersive equation

$$(1) \quad y_t + y_x + y_{xxx} + yy_x = 0,$$

and the second one is a fourth-order nonlinear parabolic equation

$$(2) \quad u_t + u_{xxxx} + \lambda u_{xx} + uu_x = 0,$$

where $\lambda > 0$ is called the anti-diffusion parameter. However, they have one important characteristic in common. They are both used to model nonlinear propagation phenomena in the space x -direction when the variable t stands for time. The KdV equation serves as a model for waves propagation in shallow water surfaces [Korteweg & de Vries, 1895] and the KS equation models front propagation in reaction-diffusion phenomena including some instability effects [Kuramoto & Tsuzuki, 1975], [Sivashinsky, 1977].

From a control point of view, a new common characteristic arises. Because of the order of the spatial derivatives involved, when studying these equations

on a bounded interval $[0, L]$, two boundary conditions have to be imposed at the same point, for instance at $x = L$. Thus, we can consider control systems where we control one boundary condition but not all the boundary data at one end-point of the interval. This configuration is not possible for the classical Wave and Heat equations where at each extreme, only one boundary condition exists and therefore controlling one or all the boundary data at one point is the same.

Being the KdV equation of third order in space, three boundary conditions have to be imposed, one at the left end-point $x = 0$ and two at the right end-point $x = L$. For the KS equation, four boundary conditions are needed to get a well-posed system, two at each extreme. We will focus on the cases where Dirichlet and Neumann boundary conditions are considered because lack of controllability phenomena appear. This holds for some special values of the length of the interval for the KdV equation and depends on the anti-diffusion coefficient λ for the KS equation.

The particular cases where the lack of controllability occurs can be seen as isolated anomalies. However, those phenomena give us important information on the systems. In particular, any method insensible to the value of those constants cannot control or stabilize

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the system when acting from the corresponding control input where trouble appear. In all these cases, for both the KdV and the KS equations, the space of uncontrollable states is finite-dimensional and therefore some methods coming from the control of Ordinary Differential Equations can be applied.

General Definitions

Infinite-dimensional control systems described by PDEs have attracted a lot of attention since the 1970s. In this framework, the state of the control system is given by the solution of an evolution PDE. This solution can be seen as a trajectory in an infinite dimensional Hilbert space H , for instance the space of square integrable functions or some Sobolev space. Thus, for any time t , the state belongs to H . Concerning the control input, this is either an internal force distributed in the domain, or a punctual force localized within the domain, or some boundary data as considered in this article. For any time t , the control belongs to a control space U , which can be for instance the space of bounded functions. The main control properties to be mentioned in this article are controllability, stability and stabilization. A control system is said to be exactly controllable if the system can be driven from any initial state to another one in finite time. This kind of properties holds, for instance, for hyperbolic system as the Wave equation. The notion of null-controllability means that the system can be driven to the origin from any initial state. The main example for this property is the Heat equation, which presents regularizing effects. Even if the initial data is discontinuous, right after $t = 0$ the solution of the Heat equation becomes very smooth and therefore it is not possible to impose a discontinuous final state. A system is said to be asymptotically stable if the solutions of the system without any control converge as the time goes to infinity to a stationary solution of the PDE. When this convergence holds with a control depending at each time on the state of the system (feedback control), the system is said to be stabilizable by means of a feedback control law.

All these properties have local versions when a smallness condition for the initial and/or the final state is added. This local character is normally due to the nonlinearity of the system.

The KdV Equation

The classical approach to deal with nonlinearities is first to linearize the system around a given state or trajectory, then to study the linear system and finally to go back to the nonlinear one by means of an inversion argument or a fixed point theorem. Linearizing (1) around the origin, we get the equation

$$(3) \quad y_t + y_x + y_{xxx} = 0,$$

which can be studied on a finite interval $[0, L]$ under the following three boundary conditions

$$(4) \quad y(t, 0) = h_1(t), \quad y(t, L) = h_2(t), \\ \text{and} \quad y_x(t, L) = h_3(t).$$

Thus, viewing $h_1(t), h_2(t), h_3(t) \in \mathbb{R}$ as controls and the solution $y(t, \cdot) : [0, L] \rightarrow \mathbb{R}$ as the state, we can consider the linear control system (3)-(4) and the nonlinear one (1)-(4).

We will report on the role of each input control when the other two are off. The tools used are mainly the duality controllability-observability, Carleman estimates, the multiplier method, the compactness-uniqueness argument, the Backstepping method and fixed point theorems. Surprisingly, the control properties of the system depend strongly on the location of the controls.

Theorem 1 The linear KdV system (3)-(4) is:

1. Null-controllable when controlled from h_1 (i.e., $h_2 = h_3 = 0$) [Glass & Guerrero, 2008].
2. Exactly controllable when controlled from h_2 (i.e., $h_1 = h_3 = 0$) if and only if L does not belong to a set O of critical lengths defined in [Glass & Guerrero, 2010].
3. Exactly controllable when controlled from h_3 (i.e., $h_1 = h_2 = 0$) if and only if L does not belong to a set of critical lengths N defined in [Rosier, 1997].
4. Asymptotically stable to the origin if $L \notin N$ and no control is applied [Perla Menzala et al, 2002].
5. Stabilizable by means of a feedback law using h_1 only (i.e., $h_2 = h_3 = 0$) [Cerpa & Coron, 2013].

If $L \in N$ or $L \in O$, one says that L is a critical length since the linear control system (3)-(4) loses

controllability properties when only one control input is applied. In those cases, there exists a finite-dimensional subspace of $L^2(0, L)$ which is unreachable from 0 for the linear system. The sets N and O contain infinitely many critical lengths but they are countable sets.

When one is allowed to use more than one boundary control input, there is no critical spatial domain and the exact controllability holds for any $L > 0$. This is proved in [Zhang, 1999] when three boundary controls are used. The case of two control inputs is solved in [Rosier, 1997], [Glass & Guerrero, 2010] and [Cerpa et al, 2013].

Previous results concern the linearized control system. Considering the nonlinearity yy_x , we obtain the original KdV control system and the following results.

Theorem 2 The nonlinear KdV system (1)-(4) is:

1. Locally null-controllable when controlled from h_1 (i.e., $h_2 = h_3 = 0$) [Glass & Guerrero, 2008].
2. Locally exactly controllable when controlled from h_2 (i.e., $h_1 = h_3 = 0$) if L does not belong to the set O of critical lengths [Glass & Guerrero, 2010].
3. Locally exactly controllable when controlled from h_3 (i.e., $h_1 = h_2 = 0$). If L belongs to the set of critical lengths N , then a minimal time of control may be required (see [Cerpa, 2013]).
4. Asymptotically stable to the origin if $L \notin N$ and no control is applied [Perla Menzala et al, 2002].
5. Locally stabilizable by means of a feedback law using h_1 only (i.e., $h_2 = h_3 = 0$) [Cerpa & Coron, 2013].

Item 3 in Theorem 2 is a truly nonlinear result obtained by applying a power series method introduced in [Coron & Crépeau, 2004]. All other items are implied by perturbation arguments based on the linear control system. The related control system formed by (1) with boundary controls

$$(5) \quad y(t, 0) = h_1(t), \quad y_x(t, L) = h_2(t), \\ \text{and} \quad y_{xx}(t, L) = h_3(t),$$

is studied in [Cerpa et al, 2013] and the same phenomenon of critical lengths appears.

The KS Equation

Applying the same strategy than for KdV, we linearize (2) around the origin to get the equation

$$(6) \quad u_t + u_{xxxx} + \lambda u_{xx} = 0,$$

which can be studied on the finite interval $[0, 1]$ under the following four boundary conditions

$$(7) \quad u(t, 0) = v_1(t), \quad u_x(t, 0) = v_2(t), \\ u(t, 1) = v_3(t), \quad \text{and} \quad u_x(t, 1) = v_4(t).$$

Thus, viewing $v_1(t), v_2(t), v_3(t), v_4(t) \in \mathbb{R}$ as controls and the solution $u(t, \cdot) : [0, 1] \rightarrow \mathbb{R}$ as the state, we can consider the linear control system (6)-(7) and the nonlinear one (2)-(7). The role of the parameter λ is crucial. The KS equation is parabolic and the eigenvalues of system (6)-(7) with no control ($v_1 = v_2 = v_3 = v_4 = 0$) go to $-\infty$. If λ increases, then the eigenvalues move to the right. When $\lambda > 4\pi^2$ the system becomes unstable because there are a finite number of positive eigenvalues. In this unstable regime the system loses control properties for some values of λ .

Theorem 3 The linear KS control system (6)-(7) is:

1. Null-controllable when controlled from v_1 and v_2 (i.e., $v_3 = v_4 = 0$). The same is true when controlling v_3 and v_4 (i.e., $v_1 = v_2 = 0$) [Lin Guo, 2002], [Cerpa & Mercado, 2011].
2. Null-controllable when controlled from v_2 (i.e., $v_1 = v_3 = v_4 = 0$) if and only if λ does not belong to a countable set M defined in [Cerpa, 2010].
3. Asymptotically stable to the origin if $\lambda < 4\pi^2$ and no control is applied [Liu & Krstic, 2001].
4. Stabilizable by means of a feedback law using v_2 only (i.e., $v_1 = v_3 = v_4 = 0$) if and only if $\lambda \notin M$ [Cerpa, 2010].

In the critical case $\lambda \in M$, the linear system is not null-controllable anymore if we control v_2 only (item 2 in Theorem 3). The space of non-controllable states is finite-dimensional. To obtain the null controllability of the linear system in these cases, we have to add another control. Controlling with v_2 and v_4 does not improve the situation in the critical cases. Unlike that, the system becomes null-controllable if we can act on v_1 and v_2 . This result with two input controls

has been proved in [Lin Guo, 2002] for the case $\lambda = 0$ and in [Cerpa & Mercado, 2011] in the general case (item 1 in Theorem 3).

It is known from [Liu & Krstic, 2001] that if $\lambda < 4\pi^2$, then the system is exponentially stable in $L^2(0, 1)$. On the other hand, if $\lambda = 4\pi^2$, then zero becomes an eigenvalue of the system and therefore the asymptotic stability fails. When $\lambda > 4\pi^2$ the system has positive eigenvalues and becomes unstable. In order to stabilize this system, a finite-dimensional based feedback law can be designed by using the pole placement method (item 4 in Theorem 3).

Previous results concern the linearized control system. If we add the nonlinearity uu_x we obtain the original KS control system and the following results.

Theorem 4 The KS control system (2)-(7) is:

1. Locally null-controllable when controlled from v_1 and v_2 (i.e., $v_3 = v_4 = 0$). The same is true when controlling v_3 and v_4 (i.e., $v_1 = v_2 = 0$) [Cerpa & Mercado, 2011].
2. Asymptotically stable to the origin if $\lambda < 4\pi^2$ and no control is applied [Liu & Krstic, 2001].

There are less results for the nonlinear systems than for the linear one. This is due to the fact that the spectral techniques used to study the linear system with only one control input are not robust enough to deal with perturbations in order to address the nonlinear control system.

Summary and Future Directions

The KdV and the KS equations possess both non-control results when one boundary control input is applied. This is due to the fact that both are higher-order equations and therefore when posed on a bounded interval, more than one boundary condition should be imposed at the same point. The KdV equation is exact controllable when acting from the right and null-controllable when acting from the left. On the other hand, the KS equation, being parabolic as the Heat equation, is not exact controllable but null-controllable. Most of the results are implied by the behaviors of the corresponding linear system, which are very well understood.

For the KdV equation, the main directions to investigate at this moment are the controllability and

the stability for the nonlinear equation in critical domains. Among others, some questions concerning controllability, minimal time of control and decay rates for the stability are open. Regarding the KS equation, there are few results for the nonlinear system with one control input even if we are not in a critical value of the anti-diffusion parameter. In the critical cases, the controllability and stability issues are wide open.

In general, for PDEs there are few results about delay phenomena, output feedback laws, adaptive control and other classical questions in control theory. The existing results on these topics mainly concern the more popular Heat and Wave equations. As KdV and KS equations are one-dimensional in space, many mathematical tools are available to tackle those problems. For all that, to our opinion, the KdV and KS equations are excellent candidates to continue investigating these control properties in a PDE framework.

Cross References

- Controllability and Observability
- Stability: Lyapunov, Linear Systems
- Feedback stabilization of nonlinear systems
- Control of 1D hyperbolic systems
- Control of fluids and fluid-structure interactions

Recommended Reading

The book [Coron, 2007] is a very good reference to study the control of PDEs. In [Cerpa, 2013], a tutorial presentation of the KdV control system is given. Control system for PDEs with boundary conditions and internal controls are considered in [Rosier & Zhang, 2009] and the references therein for the KdV equation and in [Armaou & Christofides, 2000] and [Christofides & Armaou, 2000] for the KS equation. Control topics as delay and adaptive control are studied in the framework of PDEs in [Krstic, 2009] and [Smyshlyaev & Krstic, 2010], respectively.

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