On the controllability of the Boussinesq equation in low regularity*

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Abstract

In this paper we consider the internal control problem for the Boussinesq equation posed on the torus $\mathbb{T}$. Previous results had dealt with this problem when the state space is $H^2(\mathbb{T}) \times L^2(\mathbb{T})$. The main goal of this work is to improve the regularity until $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ for $s > -1/2$. The exact controllability of the linearized equation is proved by using the moment method and spectral analysis. In order to get the same result for the nonlinear equation, we use a fixed point argument in Bourgain spaces.

1 Introduction

The Boussinesq equation

$$u_{tt} + u_{xxxx} - u_{xx} + (u^2)_{xx} = 0 \tag{1.1}$$

is a nonlinear dispersive mathematical model appearing in Physics to study nonlinear strings [1]. The equation (1.1) is also known as the good Boussinesq equation, due to the fact that well-posedness property can be shown, by looking at the highest order terms when we consider only the main part of equation (1.1) this one can be split in two Schrödinger equations traveling in opposite directions.

Other Boussinesq-like equation appearing in the literature is obtained by changing the sign of the high order term u_{xxxx} in (1.1), that is,

$$u_{tt} - u_{xxxx} - u_{xx} + (u^2)_{xx} = 0. \tag{1.2}$$

The nice well-posedness property is lost and the equation becomes ill-possed. This can be seen by realizing that the highest order terms of equation (1.2) can be split in two heat-like equations where one of them has the wrong sign in the diffusion, for more details, see [10]. The bad Boussinesq equation (1.2) is relevant in applications since it describes the propagation of water waves of small amplitude in shallow waters with flat bottom.

Another model that approximates the Boussinesq equation (1.2) was introduced in [12] and looks like

$$u_{tt} + u_{xxtt} + u_{xx} = (u^2)_{xx}. \tag{1.3}$$

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The mix of derivatives helps to improve the well-posedness results for (1.3), which justify the fact that this equation is called the improved Boussinesq equation. See [18].

The well-posedness of the equation (1.1) has been studied in different domain, for instance in the whole real line $\mathbb{R}$ by [1, 10, 5, 6, 9], in the half line $\mathbb{R}^+$ by [16], in the Torus $\mathbb{T}$ in [4, 14, 7] and in a bounded interval by [15]. All these results have been proven in a local setting since the lack of good energy estimates. Concerning controllability properties for Boussinesq equations, different type of problems have been treated. The control of (1.1) has been studied on an interval with boundary inputs in [11, 3] and on the torus $\mathbb{T}$ with distributed inputs in [17]. Regarding control properties of the improved Boussinesq equation (1.3), see the article [2].

In this paper, we will concentrate in the equation (1.1) on the torus $\mathbb{T}$, that is the bounded domain $(-\pi, \pi)$ with periodic boundary conditions. Initially, in [4] the local well-posedness of (1.1) on $\mathbb{T}$ was shown assuming initial conditions $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ provided that $0 \leq s \leq 1$. Later, in [7] the local well-posedness of (1.1) was shown for initial data in $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ for $s > -\frac{1}{4}$. This result was improved in [9] where the sharp regularity $H^{-\frac{1}{2}}(\mathbb{T}) \times H^{-\frac{5}{2}}(\mathbb{T})$ was reached.

In [17] the author proves the exact controllability in the state space $H^2(\mathbb{T}) \times L^2(\mathbb{T})$. The main goal of this work is to study the case with regularity $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ for $s \geq -\frac{1}{2}$. We want to answer the following question: Can we find a control $h = h(x,t)$ such that the solution of system

$$\begin{cases} u_{tt} + u_{xxxx} - u_{xx} + (N(u))_{xx} = g(x)h(x,t), & x \in \mathbb{T},\, t \in [0,T], \\ u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x), & x \in \mathbb{T}, \end{cases} \tag{1.4}$$

satisfies

$$u(x,T) = u_{0T}(x), \quad \text{and} \quad u_t(x,T) = u_{1T}(x)? \tag{1.5}$$

The solution $u = u(x,t)$ of (1.4) is a complex-valued function depending on space and time, the nonlinearity N is any linear combination of u^2, $u\bar{u}$ and $\bar{u}^2$, and the function $g(x)$ is a given nonzero real-valued function. This function $g(x)$ can have a support strictly contained on the torus, thus it can represent a localization of the control $h(x,t)$, which would be only able to act on a part of the domain.

Our main result, giving a positive answer to the exact controllability in a local sense, is the following.

Theorem 1 *Let $T > 0$, $s \geq -\frac{1}{2}$ and $g \in H^s(\mathbb{T}) \cap L^2(\mathbb{T}) \setminus \{0\}$ be given. There exists $r > 0$ such that for any $(u_0, u_1), (u_{0T}, u_{1T}) \in H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ satisfying*

$$\int_{\mathbb{T}} u_0(x)dx \;=\; \int_{\mathbb{T}} u_{0T}(x)dx \;=\; 0$$

and

$$\|u_0\|_s \leq r, \quad \|u_1\|_{s-2} \leq r, \quad \|u_{0T}\|_s \leq r, \quad \|u_{1T}\|_{s-2} \leq r,$$

there exists a control $h \in L^2(0,T; H^{s-2}(\mathbb{T}))$, such that the solution $u \in C([0,T]; H^s(\mathbb{T}))$ of (1.4) satisfies (1.5).

The main idea for this article is based on [13] where it is shown that the Schrödinger equation with a cubic non linear term posed on an interval with periodic boundary conditions is locally exactly controllable in $H^s(\mathbb{T})$ for $s \geq 0$. In [13] the authors perform a spectral analysis of the linear operator and by solving a moment problem, they found the characterization of the internal control for the linear problem. The non-linear problem is treated as a perturbation by fixed point theory. In this paper, in order to achieve lower regularity, we study the controllability initially in $H^s(\mathbb{T})$ for $s \geq -1/4$, and then, for $-1/2 \leq s \leq -1/4$.

In order to prove our main theorem we need first some spectral properties and a well-posedness framework. This is provided in Section 2. Then, by using the moment method we prove in Section 3 the controllability of the linear system. Finally, Section 4 is devoted to apply a fixed point argument in Bourgain spaces in order to deal with the nonlinear equation and obtain the local exact controllability stated in Theorem 1.

2 Spectral Analysis and Well-posedness

From now on, we denote by Z any space $Z(\mathbb{T})$ defined on the tours. For $s \in \mathbb{R}$ and $T > 0$ fixed, we use the notation $L^2(H^s) := L^2([0,T]; H^s(\mathbb{T}))$ and $\mathcal{X}_s := C([0,T]; \mathcal{H}_s)$, where $\mathcal{H}_s$ is the product Hilbert space

$$\mathcal{H}_s := H^s_o(\mathbb{T}) \times H^{s-2}(\mathbb{T}),$$

equipped with the norm

$$\|\vec{w}\|_{\mathcal{H}^s} := \left\{\|w_1\|_s^2 + \|w_2\|_{s-2}^2\right\}^{1/2}$$

for $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. In this paper, we use the notation

$$H^s_o(\mathbb{T}) = \left\{w \in H^s(\mathbb{T}) \Big/ \int_{\mathbb{T}} w(x)\, dx = 0\right\}.$$

For any $v(x) = \sum_{n\in\mathbb{Z}} v_n e^{inx}$, the $H^s(\mathbb{T})$ Sobolev norm can be defined by

$$\|v\|_s := \left(\sum_{n\in\mathbb{Z}} |v_n|^2 < n >^{2s}\right)^{\frac{1}{2}}$$

with $< n >= (1 + |n|^2)^{1/2}$.

From now on, we will denote by $C > 0$ a general constant which may varie from line to line.

2.1 Spectral Analysis

We analyse the linear operator related to the Boussinesq equation and study its spectral structure, eigenvalues and eigenfunctions, which will play an important role in the characterization of the solutions of (1.4).

Let us consider the linear periodic Boussinesq problem

$$\begin{cases} u_{tt} + u_{xxxx} - u_{xx} = 0, \quad x \in \mathbb{T}, t \in \mathbb{R}, \\ \partial_x^j u(-\pi, t) = \partial_x^j u(\pi, t), \quad \text{for} \quad j = 0, 1, 2, 3, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \end{cases} \tag{2.6}$$

Let $A : D(A) \subset \mathcal{H}_s \to \mathcal{H}_s$ be the linear operator associated to (2.6) defined as

$$A := \begin{pmatrix} 0 & I \\ \partial_x^2 - \partial_x^4 & 0 \end{pmatrix}, \tag{2.7}$$

with a domain $\mathcal{D}(A) := H_o^{s+4}(\mathbb{T}) \times H^s(\mathbb{T})$. We can see that $A^* = -A$. In an abstract setting, by considering $h \in L^2(0,T; H^{s-2}(\mathbb{T}))$, $\vec{y} = \begin{pmatrix} u \\ u_t \end{pmatrix}$, $\vec{y_0} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$, and $Bh = \begin{pmatrix} 0 \\ g(x)h(x,t) \end{pmatrix}$, the linear system can be written as

$$\begin{cases} \dfrac{d\vec{y}}{dt} = A\vec{y} + Bh, \\ \vec{y}(0) = \vec{y}_0. \end{cases} \tag{2.8}$$

The discrete spectrum of the operator A is

$$\sigma(A) := \{\lambda_n = i\,\mathrm{sgn}(n)\sqrt{n^2(n^2+1)} \quad / \quad n = 0, \pm 1, \pm 2, ...\}. \tag{2.9}$$

The normalized eigenfunction associated with $\lambda_0 = 0$ is $\vec{\phi}_0 = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and in the other cases the normalized eigenfunctions associated with λ_n for $n \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ are

$$\vec{\phi}_{n\pm} = \frac{\vec{\eta}_{n\pm}}{\|\vec{\eta}_{n\pm}\|_{\mathcal{H}_s}}, \qquad \text{for} \quad n = \pm 1, \pm 2, \ldots$$

where $\{\vec{\eta}_0, \vec{\eta}_{n\pm}\}_{n\in\mathbb{Z}} \in \mathcal{H}_s$ is the Riesz basis formed by $\vec{\eta}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$\vec{\eta}_{n+} = \begin{pmatrix} \frac{1}{\lambda_n} e^{inx} \\ e^{inx} \end{pmatrix}, \quad \vec{\eta}_{n-} = \begin{pmatrix} \frac{1}{\lambda_n} e^{-inx} \\ e^{-inx} \end{pmatrix}, \qquad \text{for } n = \pm 1, \pm 2, \ldots$$

We have that $\{\vec{\phi}_0,\ \vec{\phi}_{n\pm}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{H}_s$, that is, for any $\vec{y}_0 \in \mathcal{H}_s$, there exists unique coefficients β_0 and $\beta_{n\pm}$ such that

$$\vec{y}_0 := \beta_0 \vec{\phi}_0 + \sum_{n\in\mathbb{Z}^*} \left(\beta_{n+}\vec{\phi}_{n+} + \beta_{n-}\vec{\phi}_{n-} \right). \tag{2.10}$$

Remark that the expansion given in (2.10) is a Fourier series. Moreover, the solution $\vec{y}$ of (2.8) is

$$\begin{aligned} \vec{y}(x,t) = {} & \beta_0 e^{\lambda_0 t} \vec{\phi}_0(x) + \sum_{n\in\mathbb{Z}^*} e^{\lambda_n t} (\beta_{n-}\vec{\phi}_{n-}(x) + \beta_{n+}\vec{\phi}_{n+}(x)) \\ & + \int_0^t \left\{ \alpha_0(t') e^{\lambda_0(t-t')} \vec{\phi}_0(x) + \sum_{n\in\mathbb{Z}^*} \int_0^t e^{\lambda_n(t-t')} \left(\alpha_{n+}(t')\vec{\phi}_{n+}(x) + \alpha_{n-}(t')\vec{\phi}_{n-}(x) \right) \right\} dt' \end{aligned} \tag{2.11}$$

where

$$\alpha_0(t) = \langle Bh(x,t), \vec{\phi}_0(x) \rangle_s$$

and

$$\alpha_{n\pm}(t) = \langle Bh(x,t), \vec{\phi}_{n\pm}(x) \rangle_s, \qquad \text{for } n \in \mathbb{Z}^*.$$

The coefficients $\beta_0, \beta_\pm$ are associated to the initial data $\vec{y}_0$ as in the Fourier decomposition (2.10).

2.2 Well-posedness

The well-posedness results for the linear system are a direct consequence of the spectral analysis obtained in Subsection 2.1. The system (2.8) is well-posed in $\mathcal{H}_s$ for any $s \in \mathbb{R}$, because the operator A, defined in (2.7), generates a group of isometries on the space $\mathcal{H}_s$. Therefore, if $h \in L^2(H^{s-2})$, by semigroup theory there exists a unique solution $\vec{y} \in \mathcal{X}_s$ of (2.8). Additionally, by Duhamel's principle, the solution of (2.8) can be written in integral form as

$$\vec{y}(t) = e^{At}\vec{y}_0 + \int_0^t e^{A(t-t')}Bh(t')dt' \tag{2.12}$$

while the solution of the nonlinear system (1.4) can be written as

$$\vec{y}(t) = e^{At}\vec{y}_0 + \int_0^t e^{A(t-t')}Bh(t')dt' + \int_0^t e^{A(t-t')} \begin{pmatrix} 0 \\ N(u)_{xx} \end{pmatrix} dt'. \tag{2.13}$$

However, if we are in a low regularity framework, the nonlinear term cannot be estimated if we only use Sobolev spaces. Since, we aim to achieve $s \geq -\frac{1}{2}$ with a quadratic nonlinear term we consider two cases taking advantage of the bilinear estimates done initially in [7] for $s > -\frac{1}{4}$, and after in [9] for $-\frac{1}{2} \leq s \leq -\frac{1}{4}$.

2.2.1 Case $s > -\frac{1}{4}$

The integral form (2.13) of the solution of (2.8) is not going to be enough to achieve the desired regularity. We need to take advantage of the structure of the semigroup and consider that the solution of (1.4) is

$$u(t) = W_0(t)u_0 + W_1(t)u_1 + \int_0^t W_1(t-t')gh(t')dt' + \int_0^t W_1(t-t')N(u)_{xx}(t')dt' \tag{2.14}$$

where W_0, W_1 are the two vector components of the semigroup generated by the operator A. They are defined by $W_0(t) := \sum_{n\in\mathbb{Z}} W_0^n(t)$ and $W_1(t) := \sum_{n\in\mathbb{Z}} W_1^n(t)$, with

$$W_0^n(t)u_0 = \left(\frac{e^{t\lambda_n} + e^{-t\lambda_n}}{2}\hat{u}_0\right)^{\widehat{}-1} = \left(\frac{e^{i\,\mathrm{sgn}(n)t\sqrt{n^2(n^2+1)}} + e^{-i\,\mathrm{sgn}(n)t\sqrt{n^2(n^2+1)}}}{2}\hat{u}_0\right)^{\widehat{}-1}, \tag{2.15}$$

$$W_1^n(t)u_1 = \left(\frac{e^{t\lambda_n} - e^{-t\lambda_n}}{2\lambda_n}\hat{u}_1\right)^{\widehat{}-1} = \left(\frac{e^{i\,\mathrm{sgn}(n)t\sqrt{n^2(n^2+1)}} - e^{-i\,\mathrm{sgn}(n)t\sqrt{n^2(n^2+1)}}}{2i\,\mathrm{sgn}(n)\sqrt{n^2(n^2+1)}}\hat{u}_1\right)^{\widehat{}-1}, \tag{2.16}$$

where we denote $\widehat{}$ and $\widehat{}^{-1}$ the Fourier transform in space and its inverse transform, respectively. Formulas (2.15)-(2.16) come from the similarity of the matrices

$$\begin{pmatrix} 0 & 1 \\ -(n^2+n^4) & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} i\,\mathrm{sgn}(n)\sqrt{n^2(n^2+1)} & 0 \\ 0 & -i\,\mathrm{sgn}(n)\sqrt{n^2(n^2+1)} \end{pmatrix}.$$

This structure appears when the operator A is seen by using sub-blocks.

Thus, the solution written as (2.14) allows to consider Bourgain spaces and use some bilinear estimates already proven in the literature. We recall that for $s, b \in \mathbb{R}$ fixed, the Bourgain space $X_{s,b}$ is the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|w\|_{X_{s,b}} := \| < |\tau| - \sqrt{n^2(n^2+1)} >^b < n >^s \tilde{w}\|_{\ell^2_n L^2_\tau}$$

where $\tilde{}$ and $\tilde{}^{-1}$ are the Fourier transform in time and space and its inverse transform, respectively. The restriction in time of the norm $\|w\|_{X_{s,b}}$, for $T>0$, defines the space $X^T_{s,b}$ with the norm

$$\|u\|_{X^T_{s,b}} := \inf_{w\in X_{s,b}} \left\{\|w\|_{X_{s,b}} \Big/ w(t,.) = u(t,.) \text{ on } [0,T]\right\}.$$

Additionally, not only for the well-posedness problem but also for the control problem we require some estimates for the linear and nonlinear problem.

As usual when dealing with dispersive equations in Bourgain spaces, we consider a cutoff function $\theta \in C_0^\infty(\mathbb{R})$ with $0\le\theta\le 1$, $\theta\equiv 1$ in $[-1,1]$, and $\text{supp}(\theta)\subset[-2,2]$. For $0<T<1$, one defines $\theta_T(t)=\theta(\frac{t}{T})$. Thus, the solution of the linear system (2.6) is written as

$$u(t) = \theta_T(t)\left(W_0(t)u_0 + W_1(t)u_1 + \int_0^t W_1(t-t')gh(t')dt'\right), \tag{2.17}$$

while the solution of the nonlinear system (1.4) as

$$u(t) = \theta_T(t)\left(W_0(t)u_0 + W_1(t)u_1 + \int_0^t W_1(t-t')gh(t')dt' + \int_0^t W_1(t-t')N(u)_{xx}(t')dt'\right). \tag{2.18}$$

The following estimates will be useful in the sequel in order to deal with the linear and nonlinear part of (2.18).

Lemma 2 *[7, Lemma 2.1] Let $u(t)$ be the solution* (2.17) *of* (2.6)*. Then,*

$$\|u\|_{X_{s,b}} \le C\left(\|u_0\|_s + \|u_1\|_{s-2}\right). \tag{2.19}$$

Lemma 3 *[7, Lemma 2.2] Let $s>-1/4$, $-\frac{1}{2}<b'\le 0\le b\le b'+1$ and $0<T\le 1$. Then,*

$$\|\theta_T\int_0^t W_1(t-t')N(u)_{xx}(t')dt'\|_{X_{s,b}} \le T^{1-(b-b')}\Big\|\left(\frac{[N(u)_{xx}]\tilde{}(n,\tau)}{2\lambda_n}\right)^{\tilde{}-1}\Big\|_{X_{s,b'}}$$

where τ is the frequency variable corresponding to time and n the frequency variable corresponding to space.

It is worth to mention that Lemma 3 played a key role in the proof of the local well-posedness for (1.4) when $s>-\frac{1}{4}$, as stated in the following result.

Theorem 4 *[7, Theorem 1.3] Let $s>-\frac{1}{4}$ and $T>0$. Then, there exists $r>0$ such that for any $u_0\in H^s(\mathbb{T})$ and $u_1\in H^{s-2}(\mathbb{T})$ with*

$$\|u_0\|_s\le r, \qquad \textit{and} \qquad \|u_1\|_{s-2}\le r$$

and any $h\in L^2(H^{s-2})$, there exists a unique solution u of (1.4) *such that*

$$u\in C([0,T],H^s(\mathbb{T}))\cap X_{s,b}.$$

Remark 5 *In fact, Farah and Scialom stated in [7] this result with a smallness condition on the time of existence. Looking at their proof, we see that the same result can be obtained, for any fixed time T but with a smallness condition on the initial data. Concerning the term h, it does not appear in [7] but can be easily added as a source term by classical semigroup theory.*

2.2.2 Case $-\frac{1}{2} \leq s \leq -\frac{1}{4}$

For lower regularity, instead of using the characterization (2.14)-(2.15), we use the ideas of Kishimoto in [9]. Considering the change of variable

$$v := u + i(1-\partial_x^2)^{-1}\partial_t u$$

which transforms the Cauchy problem associated to (1.4) into the Schrödinger equation

$$\begin{cases} iv_t + v_{xx} = \frac{1}{2}(u-\bar{u}) - \frac{1}{4}\omega^2(v+\bar{v})^2 + Gh, \\ v(0,x) = v_0(x), \end{cases} \tag{2.20}$$

for v a complex-valued function, the operator $\omega^2 = \frac{-\partial_x^2}{1-\partial_x^2}$ and the initial data $v_0 = u_0 + i(1-\partial_x^2)^{-1}u_1$.

The solution of (1.4) is recovered by considering

$$u := Re(v) \quad \text{and} \quad (u_0, u_1) := (Re(v_0), (1-\partial_x^2)Im(v_0)).$$

To consider the control systems (1.4) with a control $Gh(x,t) = g(x)h(x,t)$, it can be deduced from the change of variable that Gh should be real and the bilinear estimate to be used will be in this cases given by the following.

Proposition 6 *[9, Proposition 2.5] Let $\lambda \geq 1$ and $-\frac{1}{4} \geq s \geq -\frac{1}{2}$. Then, we have*

$$\|\Lambda^{-1}\omega_\lambda^2(u^\lambda\overline{v^\lambda})\|_{W^s} \lesssim C_s(\lambda)\|u^\lambda\|_{W^s}\|v^\lambda\|_{W^s}$$

with $C_s(\lambda) = \begin{cases} \lambda^{-2s-\frac{1}{2}}, & -\frac{1}{4} > s \geq -\frac{1}{2}, \\ (log(1+\lambda))^{1/2} & s = -\frac{1}{4}. \end{cases}$

Here, the parameter λ with $\lambda \geq 1$ is the scaling of the solution

$$u^\lambda(x,t) := \lambda^{-2}u(\lambda^{-2}t, \lambda^{-1}x),$$

and the pseudo-differential operators are

$$\omega_\lambda^2 = \mathcal{F}_n^{-1}\frac{\lambda^2 n^2}{1+\lambda^2 n^2}\mathcal{F}_x \quad \text{and} \quad \Lambda^\sigma = \mathcal{F}_{\tau,n}^{-1} < \tau + n^2 >^\sigma \mathcal{F}_{t,x},$$

for $\sigma \in \mathbb{R}$.

It is necessary to introduce the spaces W^s and Y^s uniquely defined for $-\frac{1}{4} \geq s \geq -\frac{1}{2}$, through the norms of the projection over the frequency size

$$\|v\|_{W^s} := \|P_{\{<\tau+n^2>\lesssim<n>\}}v\|_{X_{s,1}} + \|P_{\{<\tau+n^2>\gtrsim<n>\}}v\|_{X_{s+1,1}} + \|P_{\{<\tau+n^2>\gg<n>^2\}}v\|_{Y_s},$$

and

$$\|u\|_{Y^s} := \| < n >^s \tilde{u}\|_{\ell^2_n L^1_\tau},$$

where, as before, $< n > := (1+|n|^2)^{1/2}$. Let us notice that for for $-\frac{1}{4} \geq s \geq -\frac{1}{2}$ and $0 < \theta < 1$ with $(s,\theta) \neq (-1/2, 1)$ we have the embeddings

$$X_{s,1}, X_{s+\theta,1-\theta} \cap Y^s \hookrightarrow W^s \hookrightarrow X_{s,0} \cap Y^s.$$

This approach improves the well-posedness property for the Cauchy problem (1.4) until the space $H^{-\frac{1}{2}}(\mathbb{T}) \times H^{-\frac{5}{2}}(\mathbb{T})$.

Theorem 7 *[9, Theorem 1.1] Let* $-\frac{1}{2} \leq s \leq -\frac{1}{4}$ *and* $T > 0$*. Then, there exists* $r > 0$ *such that for any* $u_0 \in H^s(\mathbb{T})$ *and* $u_1 \in H^{s-2}(\mathbb{T})$ *with*

$$\|u_0\|_s \leq r, \quad \text{and} \quad \|u_1\|_{s-2} \leq r$$

and any $h \in L^2(H^{s-2})$*, there exists a unique solution* u *of* (1.4) *such that*

$$u \in C([0,T], H^s(\mathbb{T})) \cap X_{s,b}.$$

3 Linear control system

Once the well-posedness of the system is established, the study of the linear control system is our next step. In this section, it is showed that a linear operator can define a control driving system (1.4)-(1.5) from the initial state to the final state. The proof strongly uses the spectral decomposition studied in Subsection 2.1, which allows to manage the terms in a simpler way.

Proposition 8 *Let* $s \in \mathbb{R}$ *and* $T > 0$*. There exists a bounded linear operator*

$$\Theta : (H_o^s \times H^{s-2})^2 \to L^2(H^{s-2})$$

such that for any $\vec{y}_0 := (u_0, u_1) \in H_o^s \times H^{s-2}$ *and* $\vec{y}_T := (u_{0T}, u_{1T}) \in H_o^s \times H^{s-2}$*, the control defined by* $h := \Theta(\vec{y}_0, \vec{y}_T)$ *drives the solution of*

$$\frac{d\vec{y}}{dt} = A\vec{y} + Bh, \quad \vec{y}(0) = \vec{y}_0, \tag{3.21}$$

to $\vec{y}(T) = \vec{y}_T$*. Moreover, there exists* $C > 0$ *such that for any* $\vec{y}_0, \vec{y}_T \in H_o^s \times H^{s-2}$*, we have*

$$\|\Theta(\vec{y}_0, \vec{y}_T)\|_{L^2(H^{s-2})} \leq C\|(\vec{y}_0, \vec{y}_T)\|_{(H^s \times H^{s-2})^2}. \tag{3.22}$$

Proof. Let $\vec{y}_0, \vec{y}_T \in H_o^s \times H^{s-2}$ and write them using the decompositions

$$\vec{y}_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \beta_0 \vec{\phi}_0 + \sum_{n \in \mathbb{Z}^*} \left(\beta_{n+}\vec{\phi}_{n+} + \beta_{n-}\vec{\phi}_{n-}\right),$$

$$\vec{y}_T = \begin{pmatrix} u_{0T} \\ u_{1T} \end{pmatrix} = \gamma_0 \vec{\phi}_0 + \sum_{n \in \mathbb{Z}^*} \left(\gamma_{n+}\vec{\phi}_{n+} + \gamma_{n-}\vec{\phi}_{n-}\right).$$

By previous computations, we can write the solution of (3.21) at time T as

$$\begin{aligned}\vec{y}(x,T) = \beta_0 e^{\lambda_0 T}\vec{\phi}_0(x) + \sum_{n \in \mathbb{Z}^*} e^{\lambda_n T}(\beta_{n+}\vec{\phi}_{n+}(x) + \beta_{n-}\vec{\phi}_{n-}(x)) + \int_0^T \alpha_0(t) e^{\lambda_0(T-t)} dt \vec{\phi}_0(x) \\ + \sum_{n \in \mathbb{Z}^*} \int_0^T e^{\lambda_n(T-t)} \alpha_{n+}(t) dt\, \vec{\phi}_{n+}(x) + \sum_{n \in \mathbb{Z}^*} \int_0^T e^{\lambda_n(T-t)} \alpha_{n-}(t) dt\, \vec{\phi}_{n-}(x)\end{aligned} \tag{3.23}$$

with

$$\alpha_0(t) = \int_{\mathbb{T}} g(x)h(x,t)\,dx, \quad \alpha_{n\pm}(t) = \int_{\mathbb{T}} g(x)h(x,t)\overline{\phi_{n\pm}^{(2)}}(x)\,dx$$

where for a vector $\vec{q}$, we denote $q^{(2)}$ its second component.

Looking at each component, we see that the problem we want to solve is to find functions $\alpha_0(t), \alpha_{n\pm}(t)$ such that

$$\beta_0 + \int_0^T \alpha_0(t)dt = \gamma_0 e^{-\lambda_0 T}, \tag{3.24}$$

$$\beta_{n-} + \int_0^T e^{-\lambda_n t}\alpha_{n-}(t)dt = \gamma_{n-} e^{-\lambda_{n-} T}, \tag{3.25}$$

$$\beta_{n+} + \int_0^T e^{\lambda_n t}\alpha_{n+}(t)dt = \gamma_{n+} e^{-\lambda_{n+} T}. \tag{3.26}$$

In order to solve this moment problem, we need some special basis for $L^2(0,T)$. By Subsection 2.1, we have that, up to normalization,

$$\phi^{(2)}_{n-} = e^{-inx} \quad \text{and} \quad \phi^{(2)}_{n+} = e^{inx}.$$

If we define $p_n(t) = e^{\lambda_n t}$, then $\mathcal{P} \equiv \{p_n\}_{n\in\mathbb{Z}}$ forms a Riesz basis for its closed span, $\mathcal{P}_T$ in $L^2(0,T)$. By [8] there exists $\mathcal{Q} \equiv \{q_n\}_{n\in\mathbb{Z}}$, the unique Riesz basis dual to $\mathcal{P}$ in $\mathcal{P}_T$. Thus, we have

$$\int_0^T q_j(t)\overline{p_n(t)}dt = \delta_{nj}, \quad -\infty < j, n < \infty. \tag{3.27}$$

Using the basis $\mathcal{Q}$, we look for the control h, driving system (3.21) from $\vec{y}_0$ to $\vec{y}_T$, in the form

$$h(x,t) = g(x)h_0 q_0(t) + \sum_{j\in\mathbb{Z}^*} q_j(t)g(x)\left(h_{j-}\phi^{(2)}_{j-} + h_{j+}\phi^{(2)}_{j+}\right). \tag{3.28}$$

We are led to find the coefficients $h_0, h_{n\pm}$, for $n\in\mathbb{Z}$. Plugging (3.28) into (3.24), we obtain initially for $n = 0$,

$$\begin{aligned}\gamma_0 e^{-\lambda_0 T} - \beta_0 &= \int_0^T \alpha_0(t)dt = \int_0^T \int_{\mathbb{T}} g(x)h(x,t)dxdt \\ &= \int_0^T \int_{\mathbb{T}} g^2(x)h_0 dxdt = h_0 T \int_{\mathbb{T}} g^2(x)dx,\end{aligned}$$

and then for $n\in\mathbb{Z}^*$

$$\begin{aligned}\gamma_{n-}e^{-\lambda_n T} - \beta_{n-} &= \int_0^T \int_{\mathbb{T}} h(x,t)g(x)\bar{\phi}^{(2)}_{n-}(x)dx\, e^{-\lambda_n t}dt \\ &= \int_{\mathbb{T}} g^2(x)h_0 \int_0^T e^{-\lambda_n t}q_0(t)\bar{\phi}^{(2)}_{n-}(x)dx \\ &\quad + \int_{\mathbb{T}} g^2(x)\sum_{j\in\mathbb{Z}}\int_0^T e^{-\lambda_n t}q_j(t)dt\left\{h_{j+}\phi^{(2)}_{j+}(x) + h_{j-}\phi^{(2)}_{j-}(x)dx\right\}\bar{\phi}^{(2)}_{j-}(x)dx \\ &= h_{n+}\int_{\mathbb{T}} g^2(x)\phi^{(2)}_{n+}(x)\bar{\phi}^{(2)}_{n-}(x)dx + h_{n-}\int_{\mathbb{T}} g^2(x)\phi^{(2)}_{n-}(x)\bar{\phi}^{(2)}_{n-}(x)dx \\ &= h_{n+}b_{n+} + h_{n-}a\end{aligned}$$

and

$$
\begin{aligned}
\gamma_{n+}e^{-\lambda_n T}-\beta_{n+} &= \int_0^T\int_{\mathbb{T}} h(x,t)g(x)\bar{\phi}^{(2)}_{n+}(x)dxe^{-\lambda_n t}dt\\
&= \int_{\mathbb{T}} g^2(x)h_0\int_0^T e^{-\lambda_n t}q_0(t)\bar{\phi}^{(2)}_{n+}(x)\\
&\quad + \int_{\mathbb{T}} g^2(x)\sum_{j\in\mathbb{Z}}\int_0^T e^{-\lambda_n t}q_j(t)dt\left\{h_{j+}\phi^{(2)}_{j+}(x)+h_{j-}\phi^{(2)}_{j-}(x)dx\right\}\bar{\phi}^{(2)}_{n+}(x)dx\\
&= h_{n+}\int_{\mathbb{T}} g^2(x)\phi^{(2)}_{n+}(x)\bar{\phi}^{(2)}_{n+}(x)dx+h_{n-}\int_{\mathbb{T}} g^2(x)\bar{\phi}^{(2)}_{n-}(x)\bar{\phi}^{(2)}_{n+}(x)dx\\
&= h_{n+}a+h_{n-}b_{n-}
\end{aligned}
$$

where we are considering

$$
\begin{aligned}
a &:= \int_{\mathbb{T}} g^2(x)\phi^{(2)}_{n+}(x)\bar{\phi}^{(2)}_{n+}(x)dx=\int_{\mathbb{T}} g^2(x)\phi^{(2)}_{n-}(x)\bar{\phi}^{(2)}_{n-}(x)dx=\frac{1}{2\pi}\int_{\mathbb{T}} g^2(x)dx,\\
b_{n+} &:= \int_{\mathbb{T}} g^2(x)\phi^{(2)}_{n+}(x)\bar{\phi}^{(2)}_{n-}(x)dx=\frac{1}{2\pi}\int_{\mathbb{T}} g^2(x)e^{2inx}dx,\\
b_{n-} &:= \int_{\mathbb{T}} g^2(x)\phi^{(2)}_{n-}(x)\bar{\phi}^{(2)}_{n+}(x)dx=\frac{1}{2\pi}\int_{\mathbb{T}} g^2(x)e^{-2inx}dx.
\end{aligned}
$$

We define

$$
\rho_0=\gamma_0 e^{-\lambda_0 T}-\beta_0,\quad \rho_{n-}=\gamma_{n-}e^{-\lambda_n T}-\beta_{n-},\quad \rho_{n+}=\gamma_{n+}e^{-\lambda_n T}-\beta_{n+}
$$

to write the systems

$$
\begin{aligned}
\rho_0 &= Th_0 a,\\
\rho_{n+} &= h_{n+}a+h_{n-}b_{n-},\\
\rho_{n-} &= h_{n+}b_{n+}+h_{n-}a,
\end{aligned}
\tag{3.29}
$$

that we should solve in order to find the control h. The determinant of each system (3.29) is $(b_{n+}b_{n-}-a^2)$, which is not zero and then it allows to compute the solution

$$
h_0=\frac{\rho_0}{Ta},\quad h_{n+}=\frac{\rho_{n-}b_{n+}-\rho_{n+}a}{b_{n+}b_{n-}-a^2},\quad \text{and}\quad h_{n-}=\frac{\rho_{n+}b_{n-}-\rho_{n-}a}{b_{n+}b_{n-}-a^2}.
$$

Now, we have to prove that the control h is well-defined as an element in the space $L^2(0,T;H^{s-2}(\mathbb{T}))$. To do that, we estimate its norm as done in [13]. Let us consider the expansion of each $g(x)\phi^{(2)}_{n\pm}$ as follows

$$
g(x)\phi^{(2)}_{n\pm}=a_0^{n\pm}+\sum_{j\in\mathbb{Z}^*}\left(a^{n\pm}_{j+}\phi^{(2)}_{j+}+a^{n\pm}_{j-}\phi^{(2)}_{j-}\right)
$$

where the coefficients are given by

$$
\begin{aligned}
a_0^{n\pm} &= \int_{-\pi}^{\pi} g(x)\phi^{(2)}_{n\pm}(x)dx,\\
a^{n\pm}_{j+} &= \int_{-\pi}^{\pi} g(x)\phi^{(2)}_{n\pm}(x)\overline{\phi_{j+}(x)^{(2)}}dx,\\
a^{n\pm}_{j-} &= \int_{-\pi}^{\pi} g(x)\phi^{(2)}_{n\pm}(x)\overline{\phi_{j-}(x)^{(2)}}dx.
\end{aligned}
$$

Plugging this decomposition in (3.28), we obtain that the control can be written as

$$\begin{aligned} h(x,t) = \phi_0^{(2)}(x) &\left(a_0^0 h_0 q_0(t) + \sum_{n\in\mathbb{Z}^*} a_0^{n+} h_{n+} q_n(t) + \sum_{n\in\mathbb{Z}^*} a_0^{n-} h_{n-} q_n(t) \right) \\ &+ \sum_{j\in\mathbb{Z}^*} \phi_{j+}^{(2)}(x) \left(a_{j+}^0 h_0 q_0(t) + \sum_{n\in\mathbb{Z}^*} a_{j+}^{n+} h_{n+} q_n(t) + \sum_{n\in\mathbb{Z}^*} a_{j+}^{n-} h_{n-} q_n(t) \right) \\ &\quad + \sum_{j\in\mathbb{Z}^*} \phi_{j-}^{(2)}(x) \left(a_{j-}^0 h_0 q_0(t) + \sum_{n\in\mathbb{Z}^*} a_{j-}^{n+} h_{n+} q_n(t) + \sum_{n\in\mathbb{Z}^*} a_{j-}^{n-} h_{n-} q_n(t) \right) \end{aligned}$$

and taking the norm in $L^2(0,T;H^p(\mathbb{T}))$ for some p, we obtain

$$\begin{aligned} \|h(x,t)\|^2_{L^2(H^p)} = \int_0^T &\left\{ \left| a_0^0 h_0 q_0(t) + \sum_{n\in\mathbb{Z}^*} a_0^{n+} h_{n+} q_n(t) + \sum_{n\in\mathbb{Z}^*} a_0^{n-} h_{n-} q_n(t) \right|^2 \right. \\ &+ \sum_{j\in\mathbb{Z}^*} (1+j^2)^p \left| a_{j+}^0 h_0 q_0(t) + \sum_{n\in\mathbb{Z}^*} a_{j+}^{n+} h_{n+} q_n(t) + \sum_{n\in\mathbb{Z}^*} a_{j+}^{n-} h_{n-} q_n(t) \right|^2 \\ &\left. + \sum_{j\in\mathbb{Z}^*} (1+j^2)^p \left| a_{j-}^0 h_0 q_0(t) + \sum_{n\in\mathbb{Z}^*} a_{j-}^{n+} h_{n+} q_n(t) + \sum_{n\in\mathbb{Z}^*} a_{j-}^{n-} h_{n-} q_n(t) \right|^2 \right\} dt. \end{aligned}$$

By applying Cauchy Schwarz inequality, we get

$$\begin{aligned} \|h(x,t)\|^2_{L^2(H^p)} \le C &\left\{ (a_0^0 h_0)^2 + \sum_{n\in\mathbb{Z}^*} (a_0^{n+} h_{n+})^2 + \sum_{n\in\mathbb{Z}^*} (a_0^{n-} h_{n-})^2 \right. \\ &+ \sum_{j\in\mathbb{Z}^*} (1+j^2)^p \left[(a_{j+}^0 h_0)^2 + \sum_{n\in\mathbb{Z}^*} (a_{j+}^{n+} h_{n+})^2 + \sum_{n\in\mathbb{Z}^*} (a_{j+}^{n-} h_{n-})^2 \right] \\ &\left. + \sum_{j\in\mathbb{Z}^*} (1+j^2)^p \left[(a_{j-}^0 h_0)^2 + \sum_{n\in\mathbb{Z}^*} (a_{j-}^{n+} h_{n+})^2 + \sum_{n\in\mathbb{Z}^*} (a_{j-}^{n-} h_{n-})^2 \right] \right\} \end{aligned}$$

or equivalently

$$\begin{aligned} \|h(x,t)\|^2_{L^2(H^p)} \le C &\left\{ (h_0)^2 \left[c(a_0^0)^2 + \sum_{j\in\mathbb{Z}^*} (1+j^2)^p \left((a_{j+}^0)^2 + (a_{j-}^0)^2 \right) \right] \right. \\ &+ \sum_{n\in\mathbb{Z}^*} (h_{n+})^2 \left[\sum_{j\in\mathbb{Z}^*} (1+j^2)^p \left((a_{j+}^{n+})^2 + (a_{j-}^{n+})^2 \right) + (a_0^{n+})^2 \right] \\ &\left. + \sum_{n\in\mathbb{Z}^*} (h_{n-})^2 \left[\sum_{j\in\mathbb{Z}^*} (1+j^2)^p \left((a_{j+}^{n-})^2 + (a_{j-}^{n-})^2 \right) + (a_0^{n-})^2 \right] \right\} \end{aligned} \tag{3.30}$$

where $C > 0$ bounds uniformly the L^2-norms of functions $\{q_j\}_{j\in\mathbb{Z}}$. Notice that due to the form of $\{\phi_{j\pm}^{(2)}\}_{j\in\mathbb{Z}}$, we obtain, for generic indices $n, j \in \mathbb{Z}$, that

$$|a_n^j| = \left| \int_{\mathbb{T}} g(x) \phi_j^{(2)}(x) \bar{\phi}_n^{(2)}(x)\, dx \right| \simeq |a_{n-j}^0|.$$

Using this, we obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}}(1+|j|)^{2p}|a_n^j|^2 &= \sum_{j\in\mathbb{Z}}(1+|j|)^{2p}|a_{n-j}^0|^2\\ &= \sum_{j\in\mathbb{Z}}(1+|n+j|)^{2p}|a_j^0|^2\\ &\leq C(1+|n|)^{2p}\sum_{j\in\mathbb{Z}}(1+|j|)^{2p}|a_j^0|^2\\ &\leq C(1+|n|)^{2p}\|g\|_p^2.\end{aligned}$$

With the latter estimate, we can go back to (3.30) to get

$$\begin{aligned}\|h\|_{L^2(H^p)}^2 &\leq C\|g\|_p^2\left((h_0)^2+\sum_{n\in\mathbb{Z}^*}(1+|n|)^{2p}[(h_{n+})^2+(h_{n-})^2]\right)\\ &\leq C\|g\|_p^2\left(\left(\frac{\rho_0}{Ta}\right)^2+\sum_{n\in\mathbb{Z}^*}(1+|n|)^{2p}\left[\left|\frac{\rho_{n-}b_{n+}-\rho_{n+}a}{b_{n+}b_{n-}-a^2}\right|^2+\left|\frac{\rho_{n+}b_{n-}-\rho_{n-}a}{b_{n+}b_{n-}-a^2}\right|^2\right]\right)\\ &\leq C\|g\|_p^2\left((\beta_0)^2+(\gamma_0)^2+\sum_{n\in\mathbb{Z}^*}(1+|n|)^{2p}[(\beta_{n+})^2+(\beta_{n-})^2+(\gamma_{n+})^2+(\gamma_{n-})^2]\right)\\ &\leq C\|g\|_p^2(\|u_0\|_p+\|u_1\|_p), \qquad (3.31)\end{aligned}$$

where the constant C varies line to line. In particular, the constant C bounds uniformly the quantities $\frac{1}{(b_{n+}b_{n-})-a^2}$. This is possible because the terms $(b_{n+}b_{n-})$ converge to zero, as they are the coefficients in the Fourier decomposition of $g = g(x)$.

If we take $p = s$, since $u_0 \in H^s$ and $u_1 \in H^{s-2}$, then (3.31) makes sense. The existence of the control h in the space $L^2(0,T;H^s(\mathbb{T}))$ and estimate (3.31) end the proof of Proposition 8. ■

4 Nonlinear control system

As a final step the controllability for the nonlinear system is proved. The integral equation form for the nonlinear Boussinesq equation (1.4)-(1.5) can be written as

$$u(t)=\theta_T(t)\left(W_0(t)u_0+W_1(t)u_1+\int_0^t W_1(t-t')gh(t')dt'+\int_0^t W_1(t-t')N(u)_{xx}(t')dt'\right). \qquad (4.32)$$

Thus, we see that we have to prove the existence of u satisfying (4.32) and

$$u(x,0)=u_0, \quad u(x,T)=u_{0T}, \quad u_t(x,0)=u_1, \quad u_t(x,T)=u_{1T}.$$

For any u, we define

$$w(t,u):=\int_0^t W_1(t-t')N(u)_{xx}(t')dt', \qquad (4.33)$$

and using Proposition 8 for final state $(u_{0T}, u_{1T}) \in H_o^s \times H^{s-2}$, we choose

$$h_u = \Theta_T\Big(0, 0, u_{0T} - w(T,u), u_{1T} - w_t(T,u)\Big)$$

From (4.32), we define the map $\Gamma : \mathcal{X}_s \to \mathcal{X}_s$ as

$$\Gamma(u) = \theta_T(t)\left(W_0(t)u_0 + W_1(t)u_1 + \int_0^t W_1(t-t')g(x)h_u(t')dt' \right. \\ \left. + \int_0^t W_1(t-t')N(u)_{xx}(t')dt'\right). \quad (4.34)$$

The reader should notice that it is enough to show that Γ is a contraction in a space $X_{b,s}^T$ (and consequently has a fixed point) to obtain the exact controllability of the system (1.4)-(1.5).

From Lemma 2 and Lemma 3 (when the hypothesis are satisfied), we obtain

$$\|\Gamma(u)\|_{X_{s,b}^T} \le C\Big(\|u_0\|_s + \|u_1\|_{s-2} + \|gh_u\|_{X_{s,b'}^T} + \|N(u)\|_{X_{s,b'}^T}\Big)$$

for $-\frac{1}{2} \le b' \le 0 \le b \le b'+1$. Additionally, from (3.22), we get

$$\|gh_u\|_{X_{s,b'}^T} \;\le\; C\Big(\|u_0\|_s + \|u_1\|_{s-2} + \|u_{0T}\|_s + \|u_{1T}\|_{s-2} + \|w(T,u)\|_s + \|w_t(T,u)\|_{s-2}\Big).$$

For the linear terms one has $\|f\|_{\mathcal{X}^s} \le C\|f\|_{X_{s,b}}$ for any $f \in X_{s,b}$ (see [7, Lemma 2.1]), and together with Lemma 2 and Lemma 3, we have

$$\begin{aligned} \|w(T,u)\|_s &\le \Big\|\int_0^T W_1(T-t')(N(u))_{xx}(t')dt'\Big\|_s \\ &\le \sup_{[0,T]}\Big\|\int_0^t W_1(t-t')(N(u))_{xx}(t')dt'\Big\|_s \\ &\le \Big\|\int_0^T W_1(t-t')(N(u))_{xx}(t')\Big\|_{X_{s,b}} \\ &\le C\|(N(u))_{xx}(t')\|_{X_{s,b'}} \\ &\le C\|u\|^2_{X_{s,b'}^T} \le C\|u\|^2_{X_{s,b}^T} \end{aligned} \quad (4.35)$$

for the last line, we are taking into account that $X_{s,b}$ is continuously embedded in $X_{s,b'}$ for $b' < b$. Taking the derivative with respect to time in (4.33), when $\chi_{[0,t]}$ is the characteristic function in $[0,t]$ and δ_t is the Delta Dirac function centered in t, we obtain

$$\begin{aligned} w_t(t,u) &= \frac{d}{dt}\left[\int_0^T \chi_{[0,t]}W_1(t-t')\left(N(u(t'))\right)_{xx}(t')dt'\right] \\ &= \int_0^T \delta_t(t')W_1(t-t')\left(N(u(t'))\right)_{xx}(t')dt' + \int_0^t W_0(t-t')\left(N(u(t'))\right)_{xx}(t')dt' \\ &= \int_0^t W_0(t-t')\left(N(u(t'))\right)_{xx}(t')dt'. \end{aligned} \quad (4.36)$$

Following the previous computation and taking into account that X_{s,a_1} is continuously embedded in X_{s,a_2} for $a_2 < a_1$, we obtain

$$\begin{aligned}
\|w_t(T,u)\|_{s-2} &\leq \sup_{[0,T]} \| \int_0^t W_0(t-t') \left(N(u(t'))\right)_{xx} dt' \|_{s-2} \\
&\leq \| \int_0^T W_0(t-t')(N(u(t')))_{xx} dt' \|_{X_{s-2,b}} \\
&\leq C \|(N(u))_{xx}\|_{X_{s-2,b'}} \\
&\leq C \|u\|^2_{X^T_{s-2,b'}} \leq C\|u\|^2_{X^T_{s,b}}. \qquad (4.37)
\end{aligned}$$

Therefore, from (4.35),(4.36) and (4.37), the norm of (4.34) can be bounded as

$$\|\Gamma(u)\|_{X^T_{s,b}} \leq C\left(\|u_0\|_s + \|u_1\|_{s-2} + \|u_{0T}\|_s + \|u_{1T}\|_{s-2} + \|u\|^2_{X^T_{b,s}}\right).$$

For $R > 0$, we denote B_R the ball of radius R and center 0 in $X_{s,b}$, i.e.,

$$B_R = \{u \in X^T_{s,b}, \quad \|u\|_{X^T_{s,b}} < R\}$$

and we obtain

$$\|\Gamma(u)\|_{X^T_{s,b}} \leq C\|u_0\|_s + C\|u_1\|_{s-2} + C\|u_{0T}\|_s + c\|u_{1T}\|_{s-2} + cR^2.$$

If $\delta > 0$ and $R > 0$ are selected such that

$$4C\delta + CR^2 \leq R \quad \text{and} \quad CR < \frac{1}{2}$$

we obtain that the image of the map (4.34) stays in the ball B_R, i.e.

$$\|\Gamma(u)\|_{X^T_{s,b}} \leq R$$

for $\|u_0\|_s \leq \delta$, $\|u_1\|_{s-2} \leq \delta$, $\|u_{0T}\|_s \leq \delta$ and $\|u_{1T}\|_{s-2} \leq \delta$.

Let us show that Γ in (4.34) is a contraction. Taking $u, v \in B_R$

$$\begin{aligned}
\|\Gamma(u) - \Gamma(v)\|_{X^T_{s,b}} &\leq \| \int_0^t W_1(t-t') g(x) \Theta(0,0,-w(T,u)+w(T,v),-w_t(T,u)+w_t(T,v)))(t')dt' \|_{X^T_{s,b}} \\
&\quad + \| \int_0^t W_1(t-t') \left(N(u)-N(v)\right)_{xx} (t')dt' \|_{X^T_{s,b}}.
\end{aligned}$$

In a similar way as before, by Lemma 3,

$$\begin{aligned}
\|\Gamma(u) - \Gamma(v)\|_{X^T_{s,b}} &\leq \|g(x)\Theta(0,0,-w(T,u)+w(T,v),-w_t(T,u)+w_t(T,v))\|_{X^T_{s,b'}} \\
&\quad + \|(N(u)-N(v))_{xx}\|_{X^T_{s,b'}} \\
&\leq \|w(T,u)-w(T,v)\|_{X^T_{s,b'}} + \|w_t(T,u)-w_t(T,v)\|_{X^T_{s,b'}} \\
&\quad + \|(N(u)-N(v))_{xx}\|_{X^T_{s,b'}} \\
&\leq \|u-v\|_{X^T_{s,b}} + \|(N(u)-N(v))_{xx}\|_{X^T_{s,b'}}. \qquad (4.38)
\end{aligned}$$

Since this estimate (4.38) depends on the nonlinear term, we analyze the three possible cases. If $N(u) = u^2$,

$$\begin{aligned} \|u^2 - v^2\|_{X^T_{s,b}} &\leq (\|u\|_{X^T_{s,b}} + \|v\|_{X^T_{s,b}})\|u - v\|_{X^T_{s,b}} \\ &\leq CR\|u - v\|_{X^T_{s,b}} \\ &\leq \frac{1}{2}\|u - v\|_{X^T_{s,b}}. \end{aligned} \tag{4.39}$$

For $N(u) = \overline{u^2}$ the estimate follows directly from (4.39). Finally, if $N(u) = u\bar{u}$,

$$\begin{aligned} \|u\bar{u} - v\bar{v}\|_{X^T_{s,b}} &= \|u(\bar{u} - \bar{v}) - \bar{v}(v - u)\|_{X^T_{s,b}} \\ &= (\|u\|_{X^T_{s,b}} + \|v\|_{X^T_{s,b}})\|u - v\|_{X^T_{s,b}} \\ &\leq \frac{1}{2}\|u - v\|_{X^T_{s,b}}. \end{aligned}$$

Therefore, Γ is a contraction on B_R. Thus, we proved the existence of a unique fixed point $u \in B_R$. This fixed point u is the controlled solution of the integral equation (4.32), which ends the proof of Theorem 1.

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