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BOUNDARY CONTROLLABILITY OF THE KORTEWEG–DE VRIES EQUATION ON A BOUNDED DOMAIN*

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Abstract. This paper studies boundary controllability of the Korteweg–de Vries equation posed on a finite interval, in which, because of the third-order character of the equation, three boundary conditions are required to secure the well-posedness of the system. We consider the cases where one, two, or all three of those boundary data are employed as boundary control inputs. The system is first linearized around the origin and the corresponding linear system is proved to be exactly boundary controllable if using two or three boundary control inputs. In the case where only one control input is allowed to be used, the linearized system is known to be only *null* controllable if the single control input acts on the left end of the spatial domain. By contrast, if the single control input acts on the right end of the spatial domain, the linearized system is shown to be exactly controllable if and only if the length of the spatial domain does not belong to a set of *critical values*. Moreover, the nonlinear system is shown to be locally exactly boundary controllable via the *contraction mapping principle* if the associated linearized system is exactly controllable.

Key words. boundary control, exact controllability, Korteweg–de Vries equation, nonlinear systems

AMS subject classifications. 93B05, 35Q53

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1. Introduction.

In this paper we study a class of distributed parameter control systems described by the Korteweg–de Vries (KdV) equation posed on a finite domain with nonhomogeneous boundary conditions:

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \quad (x,t) \in (0,L) \times (0,T), \\ y(0,t) = h_1(t),\, y_x(L,t) = h_2(t),\, y_{xx}(L,t) = h_3(t), \quad t \in (0,T). \end{cases} \tag{1.1}$$

This system can be considered as a model for propagation of surface water waves in the situation where a wave maker is putting energy in a finite-length channel from the left ($x = 0$) while the right end ($x = L$) of the channel is free (corresponding to the case $h_2 = h_3 = 0$) (see [5]). Since the work of Colin and Ghidaglia in the late 1990s [5, 6, 7], the system (1.1) has been mainly studied for its well-posedness in the classical Sobolev space $H^s(0,L)$ [13, 19]. So far, the system is known to be locally well-posed in the space $H^s(0,L)$ for any $s > -\frac{3}{4}$, as stated in the following theorem.

THEOREM A (see [14]). *Let $s > -\frac{3}{4}$, $T > 0$, and $r > 0$ be given with*

$$s \neq \frac{2j-1}{2}, \quad j = 1,2,3,\ldots.$$

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Then there exists $T^* > 0$ such that for given s-compatible[1] *data*

$$y_0 \in H^s(0,L), \quad h_1 \in H^{\frac{s+1}{3}}(0,T), \quad h_2 \in H^{\frac{s}{3}}(0,T), \quad h_3 \in H^{\frac{s-1}{3}}(0,T)$$

satisfying

$$\|y_0\|_{H^s(0,L)} + \|h_1\|_{H^{\frac{s+1}{3}}(0,T)} + \|h_2\|_{H^{\frac{s}{3}}(0,T)} + \|h_3\|_{H^{\frac{s-1}{3}}(0,T)} \le r,$$

the system (1.1) *admits a unique solution*

$$y \in C([0,T^*]; H^s(0,L)) \cap L^2(0,T^*; H^{s+1}(0,L))$$

satisfying the initial condition

$$y|_{t=0} = y_0.$$

Moreover, the solution y depends Lipschitz continuously on y_0 and $h_j, j = 1,2,3,$ in the corresponding spaces.

In this paper we are interested in studying the IBVP (1.1) from a control point of view: How can solutions of the system (1.1) be influenced by choosing appropriate control inputs $h_j,\ j = 1,2,3$?

In particular, we are concerned with the following exact boundary control problem.

Exact control problem. Given $T > 0$ and $y_0,\ y_T \in L^2(0,L)$, can one find appropriate control inputs $h_j,\ j = 1,2,3$, such that the corresponding solution y of (1.1) satisfies

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T?$$

Boundary control problems for the KdV equation on a finite domain have been extensively studied in the past (see [25, 20, 26, 21, 8, 3, 9, 4, 23, 27] and the references therein). Most of those works have focused on the system

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0,\ (x,t) \in (0,L) \times (0,T), \\ u(0,t) = g_1(t),\ u(L,t) = g_2(t),\ u_x(L,t) = g_3(t), \quad t \in (0,T), \end{cases} \tag{1.2}$$

which possess a different set of boundary conditions than those of the system (1.1). Controllability of this system was first studied by Rosier [20] in 1997 using only one control input g_3:

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0,\ (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0,\ u(L,t) = 0,\ u_x(L,t) = g_3(t), \quad t \in (0,T). \end{cases} \tag{1.3}$$

It was discovered rather surprisingly that whether the associated linear system

$$\begin{cases} u_t + u_x + u_{xxx} = 0,\ (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0,\ u(L,t) = 0,\ u_x(L,t) = g_3(t), \quad t \in (0,T), \end{cases} \tag{1.4}$$

[1] When $\frac{5}{2} < s < \frac{7}{2}$, the given initial and boundary data, $y_0,\ h_j, j = 1,2,3$, are said to be s-compatible if $y_0(0) = h_1(0), \quad y_0'(L) = h_2(0), \quad y_0''(L) = h_3(0)$. The reader is referred to Definition 1.3 in [14] for the precise definition of s-compatibility for the IBVP (1.1) for other values of s.

is exactly controllable depends on the length L of the spatial domain $(0, L)$. More precisely, Rosier [20] showed that the linear system is exactly controllable in the space $L^2(0, L)$ if and only if

$$L \notin \mathcal{S} := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}};\ k, l \in \mathbb{N}^* \right\}. \tag{1.5}$$

With the linear result in hand and using the *contraction mapping principle*, Rosier [20] showed further that the nonlinear system (1.3) is locally exactly controllable in the space $L^2(0, L)$ so long as $L \notin \mathcal{S}$.

THEOREM B (Rosier [20]). *Let $T > 0$ be given and assume $L \notin \mathcal{S}$. There exists $r > 0$ such that for any $u_0,\ u_T \in L^2(0, L)$ with*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \le r,$$

there exists $g_3 \in L^2(0, T)$ such that the system (1.3) admits a unique solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u|_{t=0} = u_0, \qquad u|_{t=T} = u_T.$$

The system (1.2) was later studied by Glass and Guerrero [10] for its boundary controllability using only g_2 as a control input:

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0,\ (x, t) \in (0, L) \times (0, T), \\ u(0, t) = 0,\ u(L, t) = g_2(t),\ u_x(L, t) = 0, \quad t \in (0, T). \end{cases} \tag{1.6}$$

They showed that the corresponding linear system

$$\begin{cases} u_t + u_x + u_{xxx} = 0,\ (x, t) \in (0, L) \times (0, T), \\ u(0, t) = 0,\ u(L, t) = g_2(t),\ u_x(L, t) = 0, \quad t \in (0, T), \end{cases} \tag{1.7}$$

is exactly controllable in the space $L^2(0, L)$ if and only if $L \notin \mathcal{N}^2$, where

$$\begin{aligned} \mathcal{N} = \Big\{ L \in \mathbb{R}^+ : L^2 = -(a^2+ab+b^2) \text{ with } a, b \in \mathbb{C} \\ \text{satisfying} \quad ae^a = be^b = -(a+b)e^{-(a+b)} \Big\}. \end{aligned} \tag{1.8}$$

Then the nonlinear system (1.6) was shown to be locally exactly controllable in the space $L^2(0, L)$ if $L \notin \mathcal{N}$.

THEOREM C (Glass and Guerrero [10]). *Let $T > 0$ and $L \notin \mathcal{N}$ be given. There exists $r > 0$ such that for any $u_0,\ u_T \in L^2(0, L)$ with*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \le r,$$

one can find $g_2 \in H^{\frac{1}{6}-\epsilon}(0, T)$ for any $\epsilon > 0$ such that the system (1.6) admits a solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

[2]Note that, as pointed out in [10], the set $\mathcal{S}$ defined by (1.5) can be similarly described as $\mathcal{S} = \big\{ L \in \mathbb{R}^+ : L^2 = -(a^2+ab+b^2) \text{ with } a, b \in \mathbb{C} \quad \text{satisfying} \quad e^a = e^b = e^{-(a+b)} \big\}$.

satisfying

$$u|_{t=0} = u_0, \qquad u|_{t=T} = u_T.$$

While the critical length phenomenon occurs when a single control input (either g_2 or g_3) is used, it will not happen, however, if more than one control input is allowed to be used. It was already pointed out in [20] that the linear system associated to (1.2) is exactly controllable for any $L > 0$ if both g_2 and g_3 are allowed to be used as control inputs. Moreover, the nonlinear system

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \ (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0, \ u(L,t) = g_2(t), \ u_x(L,t) = g_3(t), \quad t \in (0,T), \end{cases} \tag{1.9}$$

was shown in [20] to be locally exactly controllable in $L^2(0,L)$ without any interval length restriction.

THEOREM D (Rosier [20]). *Let $T > 0$ and $L > 0$ be given and $k > 0$ be an integer. There exists $\delta > 0$ such that for any $u_0,\ u_T \in L^2(0,L)$ with*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \le \delta,$$

there exist $g_2 \in H_0^k(0,T)$ and $g_3 \in L^2(0,T)$ such that the system (1.9) *admits a solution*

$$u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$u|_{t=0} = u_0, \qquad u|_{t=T} = u_T.$$

In [9] Glass and Guerrero considered the system (1.2) using g_1 and g_2 as control inputs:

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \ (x,t) \in (0,L) \times (0,T), \\ u(0,t) = g_1(t), \ u(L,t) = g_2(t), \ u_x(L,t) = 0, \quad t \in (0,T), \end{cases} \tag{1.10}$$

and showed that the system is also locally exactly controllable for any $L > 0$.

THEOREM E (Glass and Guerrero [9]). *Let $T > 0$ and $L > 0$ be given. There exists $r > 0$ such that for any $u_0,\ u_T \in L^2(0,L)$ with*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \le r,$$

there exist g_1 and $g_2 \in L^2(0,T)$ such that the system (1.10) *admits a solution*

$$u \in C([0,T]; H^{-1}(0,L)) \cap L^2(0,T; L^2(0,L))$$

satisfying

$$u|_{t=0} = u_0, \qquad u|_{t=T} = u_T.$$

Another interesting controllability result regarding the system (1.2) is that it is *null controllable* only if the control acts from the left side of the spatial domain $(0,L)$, which was proved by Rosier [21] and Glass and Guerrero [9].

THEOREM F (see [21, 9]). *Let $T > 0$ and $L > 0$ be given. Let*

$$v \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfy

$$\begin{cases} v_t + v_x + vv_x + v_{xxx} = 0,\ v(x,0) = v_0(x), & (x,t) \in (0,L) \times (0,T), \\ v(0,t) = v(L,t) = v_x(L,t) = 0, \quad t \in (0,T). \end{cases} \tag{1.11}$$

Then, there exists $\delta > 0$ such that for any $u_0 \in L^2(0,L)$ with

$$\|u_0 - v_0\|_{L^2(0,L)} \leq \delta$$

there exists $g_1 \in H^{\frac{1}{2}-\epsilon}(0,T)$ for any $\epsilon > 0$ such that the system (1.2) *admits a solution*

$$u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$u|_{t=0} = u_0, \qquad u|_{t=T} = v|_{t=T}.$$

While boundary controllability of the system (1.2) has been well studied, there are very few results for the system (1.1). To our knowledge, the only known result is due to Guilleron [11]. He has considered the system (1.1) with $h_2 = h_3 = 0$ and shown that the corresponding linear system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, & (x,t) \in (0,L) \times (0,T), \\ y(0,t) = h_1(t), & y_x(L,t) = y_{xx}(L,t) = 0 \end{cases}$$

is null controllable by applying a Carleman estimates approach to obtain the needed observability inequality.

The purpose of this paper is to fill the gap and to determine if the system (1.1) possesses controllability results similar to those established for system (1.2). Naturally one would like to try the same approaches that have worked effectively for system (1.2). However, one will encounter some difficulties that demand special attention and some new tools will be needed. In particular, when we use only h_2 as a control input, the linear system associated to (1.1) is

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \quad y(x,0) = y_0(x), \quad (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0,\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = 0, \quad t \in (0,T), \end{cases} \tag{1.12}$$

and its adjoint system is given by

$$\begin{cases} \varphi_t + \varphi_x + \varphi_{xxx} = 0, \quad \varphi(x,T) = \varphi_T(x), \quad (x,t) \in (0,L) \times (0,T), \\ \varphi(L,t) + \varphi_{xx}(L,t) = 0,\ \varphi_x(0,t) = 0,\ \varphi(0,t) = 0, \quad t \in (0,T). \end{cases} \tag{1.13}$$

It is well known that the exact controllability of system (1.12) is equivalent to the following observability inequality for the adjoint system (1.13):

$$\|\varphi_T\|_{L^2(0,L)} \leq C\|\varphi_x(L,\cdot)\|_{L^2(0,T)}. \tag{1.14}$$

However, the usual multiplier method and compactness arguments as those used in dealing with the control of system (1.4) only lead to

$$\|\varphi_T\|_{L^2(0,L)} \leq C_1\|\varphi_x(L,\cdot)\|_{L^2(0,T)} + C_2\|\varphi(L,\cdot)\|_{L^2(0,T)}. \tag{1.15}$$

How to remove the extra term in (1.15) presents a challenge and demands a new tool. This new tool turns out to be the hidden regularity (or the sharp Kato smoothing property [12]) for solutions of the KdV equation. As we will demonstrate later in this paper, for solutions of the system (1.13), the inequality

$$\sup_{0<x<L} \|\varphi(x,\cdot)\|_{H^{\frac{1}{3}}(0,T)} \leq C\|\varphi_T\|_{L^2(0,L)} \tag{1.16}$$

holds and will play a crucial role in validating the observability estimate (1.14).

In this paper, we will first consider the case that only h_2 is employed as a control input and show that the system

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, (x,t) \in (0,L)\times(0,T), \\ y(0,t) = 0,\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = 0, \quad t \in (0,T), \end{cases} \tag{1.17}$$

is locally exactly controllable as long as $L \notin \mathbb{F}$, where

$$\mathbb{F} = \left\{ L \in \mathbb{R}^+ : L^2 = -(a^2+ab+b^2) \text{ with } a,b \in \mathbb{C} \text{ satisfying } \frac{e^a}{a^2} = \frac{e^b}{b^2} = \frac{e^{-(a+b)}}{(a+b)^2} \right\}. \tag{1.18}$$

THEOREM 1.1. *Let $T > 0$ and $L \notin \mathbb{F}$ be given. There exists $\delta > 0$ such that for any $y_0,\ y_T \in L^2(0,L)$ with*

$$\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} \leq \delta,$$

one can find $h_2 \in L^2(0,T)$ such that the system (1.17) *admits a unique solution*

$$y \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$$

satisfying

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T.$$

Remark 1.2. One can expect that the set $\mathbb{F}$ is nonempty and countable as suggested by [10], where the authors proved that the set $\mathcal{N}$ (see (1.8)) is nonempty and countable.

Instead of employing control input h_2, one can just use the control input h_3:

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, (x,t) \in (0,L)\times(0,T), \\ y(0,t) = 0,\ y_x(L,t) = 0,\ y_{xx}(L,t) = h_3(t), \quad t \in (0,T). \end{cases} \tag{1.19}$$

The corresponding system is also locally exactly controllable if the length L of the interval $(0,L)$ does not belong to the set $\mathcal{N}$ as defined in (1.8).

THEOREM 1.3. *Let $T > 0$ and $L \notin \mathcal{N}$ be given. There exists $\delta > 0$ such that for any $y_0,\ y_T \in L^2(0,L)$ with*

$$\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} \leq \delta,$$

there exists $h_3 \in H^{-\frac{1}{3}}(0,T)$ *such that the system* (1.19) *admits a unique solution*

$$y \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T.$$

Similar to the system (1.2), the critical length phenomenon will not occur if more than one control input is employed. For the system where h_1 and h_2 are used as control inputs,

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, (x,t) \in (0,L) \times (0,T), \\ y(0,t) = h_1(t),\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = 0, \quad t \in (0,T), \end{cases} \tag{1.20}$$

we have the following local exact controllability result.

THEOREM 1.4. *Let* $T > 0$ *and* $L > 0$ *be given. There exists* $\delta > 0$ *such that for any* $y_0,\ y_T \in L^2(0,L)$ *with*

$$\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} \leq \delta,$$

one can find $h_1 \in H^{\frac{1}{3}}(0,T)$ *and* $h_2 \in L^2(0,T)$ *such that the system* (1.20) *admits a unique solution*

$$y \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T.$$

For the system using both h_2 and h_3 as control inputs,

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0,\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = h_3(t), \quad t \in (0,T), \end{cases} \tag{1.21}$$

we have the following local exact controllability result.

THEOREM 1.5. *Let* $T > 0$ *and* $L > 0$ *be given. There exists* $\delta > 0$ *such that for any* $y_0,\ y_T \in L^2(0,L)$ *with*

$$\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} \leq \delta,$$

one can find $h_3 \in H^{-\frac{1}{3}}(0,T)$ *and* $h_2 \in L^2(0,T)$ *such that the system* (1.21) *admits a unique solution*

$$y \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T.$$

For the system using h_1 and h_3 as control inputs,

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, (x,t) \in (0,L) \times (0,T), \\ y(0,t) = h_1(t),\ y_x(L,t) = 0,\ y_{xx}(L,t) = h_3(t), \quad t \in (0,T), \end{cases} \tag{1.22}$$

we have the next theorem.

THEOREM 1.6. *Let $T > 0$ and $L > 0$ be given. There exists $\delta > 0$ such that for any $y_0,\ y_T \in L^2(0, L)$ with*

$$\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} \le \delta,$$

one can find $h_1 \in H^{\frac{1}{3}}(0, T)$ and $h_3 \in H^{-\frac{1}{3}}(0, T)$ such that the system (1.22) admits a unique solution

$$y \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T.$$

If all three boundary control inputs are allowed to be used, then we can show that system (1.1) is locally exactly controllable around any smooth solution of the KdV equation.

THEOREM 1.7. *Let $T > 0$ and $L > 0$ be given. Assume that $u \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}))$ satisfies*

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}.$$

Then there exists $\delta > 0$ such that for any $y_0, y_T \in L^2(0, L)$, satisfying

$$\|y_0 - u(\cdot, 0)\|_{L^2(0,L)} + \|y_T - u(\cdot, T)\|_{L^2(0,L)} \le \delta$$

one can find control inputs

$$h_1 \in H^{\frac{1}{3}}(0, T), \quad h_2 \in L^2(0, T), \quad h_3 \in H^{-\frac{1}{3}}(0, T)$$

such that (1.1) admits a unique solution

$$y \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T.$$

Finally, with the help of some hidden regularity properties for solutions of the KdV equation we can improve some controllability results for the system (1.2). Concerning system

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \ (x, t) \in (0, L) \times (0, T), \\ u(0, t) = 0, \ u(L, t) = g_2(t), \ u_x(L, t) = 0, \quad t \in (0, T), \end{cases} \tag{1.23}$$

we can show that the control input g_2, used in Theorem C, belongs in fact to the space $H^{\frac{1}{3}}(0, T)$.

THEOREM 1.8. *Let $T > 0$ and $L \notin \mathcal{N}$ be given. There exists $r > 0$ such that for any $u_0,\ u_T \in L^2(0, L)$ with*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \le r,$$

there exists $g_2 \in H^{\frac{1}{3}}(0, T)$ such that the system (1.23) admits a solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u|_{t=0} = u_0, \qquad u|_{t=T} = u_T.$$

Regarding system

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \ (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0, \ u(L,t) = g_2(t), \ u_x(L,t) = g_3(t), \quad t \in (0,T), \end{cases} \tag{1.24}$$

Theorem D can be improved as follows.

THEOREM 1.9. *Let $T > 0$ and $L > 0$ be given. There exists $\delta > 0$ such that for any $u_0, \ u_T \in L^2(0,L)$ with*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \leq \delta,$$

there exist $g_2 \in H^{\frac{1}{3}}(0,T)$ and $g_3 \in L^2(0,T)$ such that system (1.24) *admits a solution*

$$u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$u|_{t=0} = u_0, \qquad u|_{t=T} = u_T.$$

Moreover, we also consider system (1.2) using only two control inputs g_1 and g_3

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \ (x,t) \in (0,L) \times (0,T), \\ u(0,t) = g_1(t), \ u(L,t) = 0, \ u_x(L,t) = g_3(t), \quad t \in (0,T), \end{cases} \tag{1.25}$$

which has not been studied in the literature before. We can show that the critical length phenomenon will not occur for this system either.

THEOREM 1.10. *Let $T > 0$ and $L > 0$ be given. There exists $\delta > 0$ such that for any $u_0, \ u_T \in L^2(0,L)$ with*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \leq \delta,$$

one can find $g_1 \in H^{\frac{1}{3}}(0,T)$ and $g_3 \in L^2(0,T)$ such that the system (1.25) *admits a solution*

$$u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$u|_{t=0} = u_0, \qquad u|_{t=T} = u_T.$$

The paper is organized as follows. In section 2, we present various linear estimates including hidden regularities for solutions of the linear systems associated to (1.1) and (1.2) which will play important roles in establishing our exact controllability results in this paper. The associated linear systems are shown to be exactly controllable in section 3, while the nonlinear systems are shown to be locally exactly controllable using the standard contraction mapping principle in section 4. Finally, in section 5 we provide some concluing remarks together with some open problems for further studies. The paper ends with an appendix, where the proofs of some technical lemmas used in the paper are furnished.

2. Linear estimates.

2.1. The forward linear system.

In this subsection, we consider the following linear problem associated to the nonlinear system (1.1):

$$\begin{cases} y_t + y_x + y_{xxx} = f, \quad y(x,0) = y_0(x), \quad x \in (0,L),\ t > 0, \\ y(0,t) = h_1(t),\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = h_3(t), \quad t > 0. \end{cases} \tag{2.1}$$

In the case $h_1 = h_2 = h_3 = 0$ and $f = 0$, the solution y can be written as

$$y(t) = W_0(t)y_0.$$

Here $W_0(t)$ is the C_0-semigroup in the space $L^2(0,L)$ (see [16, Cor. 4.4]) generated by the dissipative linear operator

$$A\psi = -\psi''' - \psi',$$

whose domain is

$$\mathcal{D}(A) = \{\psi \in H^3(0,L) : \psi(0) = \psi'(L) = \psi''(L) = 0\}.$$

The solution y of (2.1), when $h_1 = h_2 = h_3 = 0$ and $y_0 = 0$, is given by

$$y(t) = \int_0^t W_0(t-\tau)f(\tau)d\tau.$$

In the case $y_0 = 0$ and $f = 0$, the solution of (2.1) is denoted by

$$y(t) = W_{bdr}(t)\vec{h},$$

where $\vec{h} = (h_1, h_2, h_3)$. The operator $W_{bdr}(t)$ is called the boundary integral operator associated to (2.1), whose explicit representation can be found in [13, 14].

As demonstrated in [13, 14], the linear system (2.1) is well-posed in the space $H^s(0,L)$ for any $0 \le s \le 3$ with

$$y_0 \in H^s(0,L),\ f \in W^{\frac{s}{3},1}(0,T;L^2(0,L))$$

and

$$\vec{h} = (h_1,h_2,h_3) \in \mathcal{H}^s_{loc}(\mathbb{R}^+) := H_{loc}^{\frac{s+1}{3}}(\mathbb{R}^+) \times H_{loc}^{\frac{s}{3}}(\mathbb{R}^+) \times H_{loc}^{\frac{s-1}{3}}(\mathbb{R}^+).$$

In particular, in the case $s = 0$, the result can be stated as follows.

PROPOSITION 2.1. *Let $T > 0$ be given. For any $y_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$, and*

$$(h_1,h_2,h_3) \in \mathcal{H}_T := H^{\frac{1}{3}}(0,T) \times L^2(0,T) \times H^{-\frac{1}{3}}(0,T),$$

the IBVP (2.1) *admits a unique solution*

$$y \in X_T := C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L)).$$

Moreover, there exists $C > 0$ such that

$$\|y\|_{X_T} \le C\left(\|f\|_{L^1(0,T;L^2(0,L))} + \|y_0\|_{L^2(0,L)} + \|(h_1,h_2,h_3)\|_{\mathcal{H}_T}\right).$$

In addition, the solution y of (2.1) possesses the following hidden (or sharp trace) regularities.

PROPOSITION 2.2. *Let $T > 0$ be given. For any $y_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$ and $(h_1,h_2,h_3) \in \mathcal{H}_T$, the solution y of system (2.1) satisfies*

$$\sup_{0<x<L} \|\partial_x^j y(x,\cdot)\|_{H^{\frac{1-j}{3}}(0,T)} \leq C_j \left(\|f\|_{L^1(0,T;L^2(0,L))} + \|y_0\|_{L^2(0,L)} + \|(h_1,h_2,h_3)\|_{\mathcal{H}_T}\right) \tag{2.2}$$

for $j = 0,1,2$.

The next proposition states similar hidden (or sharp trace) regularity results for the linear system

$$\begin{cases} u_t + u_x + u_{xxx} = f, \quad u(x,0) = u_0(x), \quad x \in (0,L),\ t > 0, \\ u(0,t) = g_1(t),\ u(L,t) = g_2(t),\ u_x(L,t) = g_3(t), \quad t > 0, \end{cases} \tag{2.3}$$

associated to (1.2).

PROPOSITION 2.3. *Let $T > 0$ be given. For any $u_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$, and*

$$(g_1,g_2,g_3) \in \mathcal{G}_T := H^{\frac{1}{3}}(0,T) \times H^{\frac{1}{3}}(0,T) \times L^2(0,T),$$

the IBVP (2.3) admits a unique solution $u \in X_T$. Moreover, there exists $C > 0$ such that

$$\|u\|_{X_T} \leq C \left(\|f\|_{L^1(0,T;L^2(0,L))} + \|y_0\|_{L^2(0,L)} + \|(g_1,g_2,g_3)\|_{\mathcal{G}_T}\right).$$

In addition, the solution u possesses the sharp trace estimates

$$\sup_{0<x<L} \|\partial_x^j u(x,\cdot)\|_{H^{\frac{1-j}{3}}(0,T)} \leq C_j \left(\|f\|_{L^1(0,T;L^2(0,L))} + \|u_0\|_{L^2(0,L)} + \|(g_1,g_2,g_3)\|_{\mathcal{G}_T}\right) \tag{2.4}$$

for $j = 0,1,2$.

The proofs of Propositions 2.2 and 2.3 can be found in [28] (cf. also [1, 2, 14]).

Remark 2.4. Systems (2.1) and (2.3) are equivalent in the following sense: for given $\{y_0, f, h_1, h_2, h_3\}$ one can find $\{u_0, f, g_1, g_2, g_3\}$ such that the corresponding solution y of (2.1) is exactly the same as the corresponding u for system (2.3) and vice versa. Indeed, for given $y_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$, and $\vec{h} \in \mathcal{H}_T$, system (2.1) admits a unique solution $y \in X_T$. Let $u_0 = y_0$, and set

$$g_1(t) = h_1(t), \quad g_2(t) = y(L,t), \quad g_3(t) = h_2(t).$$

Then, according to (2.2), we have $\vec{g} \in \mathcal{G}_T$. Because of the uniqueness of the IBVP (2.3), with such selected (u_0, f, g_1, g_2, g_3), the corresponding solution $u \in X_T$ of (2.3) must be equal to y since y also solves (2.3) with the given auxiliary data (u_0, f, g_1, g_2, g_3). On the other hand, for any given $u_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$, and $\vec{g} \in \mathcal{G}_T$, let $u \in X_T$ be the corresponding solution of the system (2.3). By (2.4), we have $u_{xx}(L,\cdot) \in H^{-\frac{1}{3}}(0,T)$. Thus, if set $y_0 = u_0$ and

$$h_1(t) = g_1(t), \quad h_2(t) = g_3(t), \quad h_3(t) = u_{xx}(L,t),$$

then $\vec{h} \in \mathcal{H}_T$ and the corresponding solution $y \in X_T$ of (2.1) must be equal to u, which also solves (2.1) with the auxiliary data $(y_0, f, \vec{h})$.

2.2. The backward adjoint linear system. In this subsection, we consider the backward adjoint system of (2.1),

$$\begin{cases}\psi_t+\psi_x+\psi_{xxx}=0, \quad \psi(x,T)=\psi_T(x), \quad (x,t)\in(0,L)\times(0,T),\\ \psi(0,t)=0, \quad \psi_x(0,t)=0, \quad \psi(L,t)+\psi_{xx}(L,t)=0, \quad t\in(0,T),\end{cases} \tag{2.5}$$

which (by transformation $x'=L-x,\ t'=T-t$) is equivalent to the following forward system:

$$\begin{cases}\varphi_t+\varphi_x+\varphi_{xxx}=0, \quad \varphi(x,0)=\varphi_0(x), \quad (x,t)\in(0,L)\times(0,T),\\ \varphi(L,t)=0, \quad \varphi_x(L,t)=0, \quad \varphi(0,t)+\varphi_{xx}(0,t)=0, \quad t\in(0,T).\end{cases} \tag{2.6}$$

The solution of (2.6) can be written as

$$\varphi(x,t)=S(t)\varphi_0,$$

where $S(t)$ is the C_0 semigroup in the space $L^2(0,L)$ generated by the operator

$$A_1f=-f'-f'''$$

with the domain

$$\mathcal{D}(A_1)=\{f\in H^3(0,L):\ f(0)+f''(0)=0,\ f(L)=f'(L)=0\}.$$

Notice that the existence of $S(t)$ is due to dissipativity of the operator A_1 (see [16, Cor. 4.4]).

PROPOSITION 2.5. *For any $\varphi_0\in L^2(0,L)$ the IBVP* (2.6) *admits a unique solution $\varphi\in X_T$. Moreover, there exists $C>0$ such that*

$$\|\varphi\|_{X_T}\le C\|\varphi_0\|_{L^2(0,L)}$$

and

$$\int_0^T\left(|\varphi(0,t)|^2+|\varphi_x(0,t)|^2\right)dt\le C\|\varphi_0\|^2_{L^2(0,L)}.$$

Proof. The proof is very similar to that of Propositions 3.1 and 3.2 in [20] and is therefore omitted. ∎

Thus, when $\varphi_0\in L^2(0,L)$, the corresponding solution φ has the trace $\varphi(0,\cdot)\in L^2(0,T)$. The next theorem reveals that φ has a stronger trace regularity: $\varphi(0,\cdot)\in H^{\frac{1}{3}}(0,T)$. It will play an important role in establishing exact controllability of the system (1.1), as shown in the next section.

THEOREM 2.6 (hidden regularities). *For any $\varphi_0\in L^2(0,L)$, the solution $\varphi\in X_T$ of IBVP* (2.6) *possesses the sharp trace properties*

$$\sup_{0<x<L}\|\partial_x^j\varphi(x,\cdot)\|_{H^{\frac{1-j}{3}}(0,T)}\le C_j\|\varphi_0\|_{L^2(0,L)}$$

for $j=0,1,2$.

Remark 2.7. Equivalently, the solution of the system (2.5) has the sharp trace estimates

$$\sup_{0<x<L}\|\partial_x^j\psi(x,\cdot)\|_{H^{\frac{1-j}{3}}(0,T)}\le C_j\|\psi_T\|_{L^2(0,L)}$$

for $j=0,1,2$.

To prove Theorem 2.6, we first consider the following linear system:

$$\begin{cases} w_t + w_{xxx} = f, \quad (x,t) \in (0,L) \times (0,T), \\ w_{xx}(0,t) = k_1(t), \quad w(L,t) = k_2(t), \quad w_x(L,t) = k_3(t), \quad t \in (0,T), \\ w(x,0) = w_0(x), \quad x \in (0,L). \end{cases} \tag{2.7}$$

PROPOSITION 2.8. *If $w_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$, and $\vec{k} := (k_1,k_2,k_3) \in \mathcal{K}_T$ with $\mathcal{K}_T = H^{-\frac{1}{3}}(0,T) \times H^{\frac{1}{3}}(0,T) \times L^2(0,T)$, then system* (2.7) *admits a unique solution $w \in X_T$ which, in addition, has the hidden (or sharp trace) regularities*

$$\partial_x^j w \in L_x^\infty(0,L,H^{\frac{1-j}{3}}(0,T)) \text{ for } j = 0,1,2.$$

Moreover, there exist constants $C > 0,\ C_j > 0,\ j = 0,1,2,$ such that

$$\|w\|_{X_T} \le C\left(\|w_0\|_{L^2(0,L)} + \|\vec{k}\|_{\mathcal{K}_T} + \|f\|_{L^1(0,T;L^2(0,L))}\right),$$

where

$$\|\vec{k}\|_{\mathcal{K}_T} := \left(\|k_1\|^2_{H^{-\frac{1}{3}}(0,T)} + \|k_2\|^2_{H^{\frac{1}{3}}(0,T)} + \|k_3\|^2_{L^2(0,T)}\right)^{1/2}$$

and

$$\sup_{0<x<L} \|\partial_x^j w(x,\cdot)\|_{H^{\frac{1-j}{3}}(0,T)} \le C_j\left(\|w_0\|_{L^2(0,L)} + \|\vec{k}\|_{\mathcal{K}_T} + \|f\|_{L^1(0,T;L^2(0,L))}\right)$$

for $j = 0,1,2$.

Proof. The proof is similar to the one in [28]. Its sketch will be presented in the appendix for the convenience of interested readers. □

Now we turn to proving Theorem 2.6.

Proof of Theorem 2.6. Let

$$\mathcal{X}_T := \left\{u \in X_T; \quad \partial_x^j u \in L_x^\infty(0,L;H^{\frac{1-j}{3}}(0,T)),\ j = 0,1,2\right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{\mathcal{X}_T} := \|u\|_{X_T} + \sum_{j=0}^{2} \|\partial_x^j u\|_{L_x^\infty(0,L;H^{\frac{1-j}{3}}(0,T))}.$$

According to Proposition 2.8, for any $v \in \mathcal{X}_\beta$, where $0 < \beta \le T$, and for any $\varphi_0 \in L^2(0,L)$, the system

$$\begin{cases} w_t + w_{xxx} = -v_x, \quad (x,t) \in (0,L) \times (0,\beta), \\ w_{xx}(0,t) = -v(0,t), \quad w(L,t) = 0, \quad w_x(L,t) = 0, \quad t \in (0,\beta), \\ w(x,0) = \varphi_0(x), \quad x \in (0,L), \end{cases} \tag{2.8}$$

admits a unique solution $w \in \mathcal{X}_\beta$ and, moreover,

$$\|w\|_{\mathcal{X}_\beta} \le C\left(\|\varphi_0\|_{L^2(0,L)} + \|v(0,\cdot)\|_{H^{-\frac{1}{3}}(0,\beta)} + \|v_x\|_{L^1(0,\beta;L^2(0,L))}\right),$$

where the constant C depends only on T. As we have

$$\|v_x\|_{L^1(0,\beta;L^2(0,L))} \le \beta^{\frac{1}{2}} \|v\|_{\mathcal{X}_\beta}$$

and

$$\begin{aligned}\|v(0,\cdot)\|_{H^{-\frac{1}{3}}(0,\beta)} &\le \|v(0,\cdot)\|_{L^2(0,\beta)} \le \beta^{\frac{2}{3}}\|v(0,\cdot)\|_{L^6(0,\beta)}\\ &\le C\beta^{\frac{2}{3}}\|v(0,\cdot)\|_{H^{\frac{1}{3}}(0,\beta)} \le C\beta^{\frac{2}{3}}\|v\|_{\mathcal{X}_\beta},\end{aligned}$$

the system (2.8) defines a map Γ from the space $\mathcal{X}_\beta$ to $\mathcal{X}_\beta$ for any $0<\beta\le\max\{1,T\}$,

$$w=\Gamma(v) \text{ for any } v\in\mathcal{X}_\beta,$$

where $w\in\mathcal{X}_\beta$ is the corresponding solution of (2.8) and

$$\|\Gamma(v)\|_{\mathcal{X}_\beta}\le C_1\|\varphi_0\|_{L^2(0,L)}+C_2\beta^{\frac{1}{2}}\|v\|_{\mathcal{X}_\beta},$$

where C_1 and C_2 are two constants depending only on T. Choose $r>0$ and $0<\beta\le\max\{1,T\}$ such that

$$r=2C_1\|\varphi_0\|_{L^2(0,L)},\quad 2C_2\beta^{\frac{1}{2}}\le\frac{1}{2}.$$

Then, for any

$$v\in B_{\beta,r}=\left\{v\in\mathcal{X}_\beta;\quad \|v\|_{\mathcal{X}_\beta}\le r\right\},$$

we have

$$\|\Gamma(v)\|_{\mathcal{X}_\beta}\le r.$$

Moreover, for any $v_1,\ v_2\in B_{\beta,r}$, we get

$$\|\Gamma(v_1)-\Gamma(v_2)\|_{\mathcal{X}_\beta}\le 2C_2\beta^{\frac{1}{2}}\|v_1-v_2\|_{\mathcal{X}_\beta}\le\frac{1}{2}\|v_1-v_2\|_{\mathcal{X}_\beta}.$$

Therefore the map Γ is a contraction mapping on $B_{\beta,r}$. Its fixed point $w=\Gamma(w)\in\mathcal{X}_\beta$ is the desired solution for $t\in(0,\beta)$. As the chosen β is independent of φ_0, the standard continuation extension argument yields that the solution w belongs to $\mathcal{X}_T$. The proof is complete. ∎

We conclude this section with an elementary estimate for solutions of system (2.6).

PROPOSITION 2.9. *Any solution φ of the adjoint problem* (2.6) *with initial data $\varphi_0\in L^2(0,L)$ satisfies*

$$\|\varphi_0\|^2_{L^2(0,L)}\le\frac{1}{T}\|\varphi\|^2_{L^2((0,T)\times(0,L))}+\|\varphi_x(0,\cdot)\|^2_{L^2(0,T)}+\|\varphi(0,\cdot)\|^2_{L^2(0,T)}.$$

Proof. Multiplying both sides of the equation in (2.6) by $(T-t)\varphi$ and integrating by parts over $(0,L)\times(0,T)$, we get

$$\int_0^L(T-t)\varphi^2\big|_0^T dx+\int_0^T(T-t)\left(\varphi^2(0,t)+\varphi_x^2(0,t)\right)dt+\int_0^T\int_0^L\varphi^2dxdt=0.$$

Consequently,

$$\int_0^L\varphi_0^2dx\le\frac{1}{T}\int_0^L\int_0^T\varphi^2dtdx+\int_0^T\varphi^2(0,t)dt+\int_0^T\varphi_x^2(0,t)dt.\qquad ∎$$

Equivalently, the following estimate holds for solutions ψ of the system (2.5):

$$\|\psi_T\|^2_{L^2(0,L)} \leq \frac{1}{T}\|\psi\|^2_{L^2((0,T)\times(0,L))} + \|\psi_x(L,\cdot)\|^2_{L^2(0,T)} + \|\psi(L,\cdot)\|^2_{L^2(0,T)}. \tag{2.9}$$

As a comparison, it is worth pointing out that for the adjoint system of (2.3), which is given by

$$\begin{cases} \nu_t + \nu_x + \nu_{xxx} = 0, \quad \nu(x,T) = \nu_T(x), \quad (x,t) \in (0,L)\times(0,T), \\ \nu(L,t) = 0, \quad \nu_x(0,t) = 0, \quad \nu(0,t) = 0, \end{cases} \tag{2.10}$$

the following inequality holds:

$$\|\nu_T\|^2_{L^2(0,L)} \leq \frac{1}{T}\|\nu\|^2_{L^2((0,T)\times(0,L))} + \|\nu_x(L,\cdot)\|^2_{L^2(0,T)}. \tag{2.11}$$

The extra term $\|\psi(L,\cdot)\|^2_{L^2(0,T)}$ in (2.9) brings new challenges in establishing the observability of the adjoint system (2.5).

3. Linear control systems.

3.1. A single control input. In this subsection, we consider boundary controllability of the linear system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \quad (x,t) \in (0,L)\times(0,T), \\ y(0,t) = 0,\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = 0, \quad t \in (0,T), \end{cases} \tag{3.1}$$

which employs only one control input $h_2 \in L^2(0,T)$.

Proposition 3.1. *Let $L \notin \mathbb{F}$ (see (1.18)) and $T > 0$ be given. There exists a bounded linear operator*

$$\Psi : L^2(0,L) \times L^2(0,L) \to L^2(0,T)$$

such that for any $y_0,\ y_T \in L^2(0,L)$, if one chooses $h_2 = \Psi(y_0, y_T)$, then system (3.1) admits a solution $y \in X_T$ satisfying

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T.$$

As is well known, the exact controllability of the system (3.1) is related to the observability of its adjoint system

$$\begin{cases} \psi_t + \psi_x + \psi_{xxx} = 0, \quad (x,t) \in (0,L)\times(0,T), \\ \psi(0,t) = 0, \quad \psi_x(0,t) = 0, \quad \psi(L,t) + \psi_{xx}(L,t) = 0, \quad t \in (0,T), \\ \psi(x,T) = \psi_T(x), \quad x \in (0,L). \end{cases} \tag{3.2}$$

Lemma 3.2. *For all $T > 0$ and all $L \notin \mathbb{F}$ there exists $C = C(L,T) > 0$ such that for any $\psi_T \in L^2(0,L)$, the solution ψ of (3.2) satisfies*

$$\|\psi_T\|_{L^2(0,L)} \leq C\|\psi_x(L,t)\|_{L^2(0,T)}. \tag{3.3}$$

Proof. Proceeding as in [20], if (3.3) is false, then there exists a sequence $\{\psi_T^n\}_{n\in\mathbb{N}} \in L^2(0,L)$ with $\|\psi_T^n\|_{L^2(0,L)} = 1$ such that the corresponding solutions of (3.2) satisfy

$$1 = \|\psi_T^n\|_{L^2(0,L)} > n\|\psi_x^n(L,\cdot)\|_{L^2(0,T)}.$$

Thus $\|\psi_x^n(L,\cdot)\|_{L^2(0,T)} \to 0$ as $n \to \infty$. By Proposition 2.5 and Theorem 2.6, the sequences $\{\psi^n\}_{n\in\mathbb{N}}$ and $\{\psi^n(L,t)\}_{n\in\mathbb{N}}$ are bounded in $L^2(0,T;H^1(0,L))$ and $H^{\frac{1}{3}}(0,T)$, respectively. In addition, according to Proposition 2.9

$$(3.4)\ \|\psi_T^n\|^2_{L^2(0,L)} \leq \frac{1}{T}\|\psi^n\|^2_{L^2(0,T;L^2(0,L))} + \|\psi_x^n(L,\cdot)\|^2_{L^2(0,T)} + \|\psi^n(L,\cdot)\|^2_{L^2(0,T)}.$$

Since $\psi_t^n = -(\psi_x^n + \psi_{xxx}^n)$ is bounded in $L^2(0,T;H^{-2}(0,L))$ and by the embedding

$$H^1(0,L) \hookrightarrow L^2(0,L) \hookrightarrow H^{-2}(0,L),$$

the sequence $\{\psi^n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;L^2(0,L))$ (see [24]). Furthermore, the second term on the right in (3.4) converges to zero in $L^2(0,T)$, and by the compact embedding

$$H^{\frac{1}{3}}(0,T) \hookrightarrow L^2(0,T)$$

the sequence $\{\psi^n(L,\cdot)\}_{n\in\mathbb{N}}$ has a convergent subsequence on $L^2(0,T)$. Therefore $\{\psi_T^n\}_{n\in\mathbb{N}}$ is an $L^2(0,L)$-Cauchy sequence. Let us denote $\psi_T = \lim_{n\to\infty}\psi_T^n$ and let ψ be the corresponding solution of (3.2). Since $\psi_x^n(L,t) \to \psi_x(L,t)$ as $n \to \infty$ in $L^2(0,T)$ and $\|\psi_T^n\|_{L^2(0,L)} = 1$ for any n, we have $\|\psi_T\|_{L^2(0,L)} = 1$ and $\psi_x(L,t) = 0$. By Lemma 3.3, one can conclude that $\psi \equiv 0$, therefore $\psi_T(x) \equiv 0$, which contradicts the fact that $\|\psi_T\|_{L^2(0,L)} = 1$. ∎

LEMMA 3.3. *For given $T > 0$, let us define*

$$N_T = \{\psi_T \in L^2(0,L) : \\ \psi \in X_T \ \text{ is the mild solution of } (3.2) \text{ satisfying } \psi_x(L,\cdot) = 0 \text{ in } L^2(0,T)\}.$$

Then, $N_T = \{0\}$ if and only if $L \notin \mathbb{F}$.

Proof. The proof uses the same arguments as that given in [20, 10] and will be presented in the appendix for the convenience of interested readers. ∎

Now we turn to prove Proposition 3.1.

Proof of Proposition 3.1. Without loss of generality, we assume that $y_0 = 0$. Let ψ be a solution of the system (3.2) and multiply both sides of the equation in (3.1) by ψ and integrate over the domain $(0,L)\times(0,T)$. Integration by parts leads to

$$\int_0^L y(x,T)\psi_t(x)dx = \int_0^T h_2(t)\psi_x(L,t)dt.$$

Let us denote by Υ the linear and bounded map from $L^2(0,L) \to L^2(0,L)$ defined by

$$\Upsilon : \psi_T(\cdot) \to y(\cdot,T)$$

with y being the solution of (3.1) when $h_2(t) = \psi_x(L,t)$, where ψ is the solution of the system (3.2). According to Lemma 3.2,

$$(\Upsilon(\psi_T),\psi_T)_{L^2(0,L)} = \|\psi_x(L,\cdot)\|^2_{L^2(0,T)} \geq C^{-2}\|\psi_T\|^2_{L^2(0,L)}. \tag{3.5}$$

Thus Υ is invertible by the Lax–Milgram theorem. Consequently, for given $y_T \in L^2(0,L)$, we can define $\psi_T = \Upsilon^{-1}y_T$. We solve system (3.2) and get $\psi \in X_T$. Then, we set $h_2(t) = \psi_x(L,t)$ in system (3.1) and see that the corresponding solution $y \in X_T$ satisfies

$$y|_{t=0} = 0, \quad y|_{t=T} = y_T.$$

The proof is complete. ∎

3.2. Double control inputs. In this subsection, we first consider boundary controllability of the linear system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \quad (x,t) \in (0,L) \times (0,T), \\ y(0,t) = h_1(t), \quad y_x(L,t) = h_2(t), \quad y_{xx}(L,t) = 0, \quad t \in (0,T), \end{cases} \tag{3.6}$$

with two control inputs $h_1 \in H^{\frac{1}{3}}(0,T)$ and $h_2 \in L^2(0,T)$.

PROPOSITION 3.4. *Let $T > 0$ be given. There exists a bounded linear operator*

$$\digamma : L^2(0,L) \times L^2(0,L) \to H^{\frac{1}{3}}(0,T) \times L^2(0,T)$$

such that for any $y_0,\ y_T \in L^2(0,L)$, if one chooses

$$(h_1, h_2) = \digamma(y_0, y_T),$$

then the system (3.6) *admits a solution $y \in X_T$ satisfying*

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T.$$

As before, we first establish the following observability estimate for the corresponding adjoint system (3.2).

LEMMA 3.5. *Let $T > 0$ be given. There exists a constant $C > 0$ such that for any $\psi_T \in L^2(0,L)$, the corresponding solution ψ of* (3.2) *satisfies*

$$\|\psi_T\|_{L^2(0,L)} \le C \left(\int_0^T (|\Delta_t^{-\frac{1}{3}} \psi_{xx}(0,t)|^2 + |\psi_x(L,t)|^2) dt \right), \tag{3.7}$$

where $\Delta_t := (I - \partial_t^2)^{\frac{1}{2}}$.

Proof. If the estimate (3.7) is false, then there exists a sequence $\{\psi_T^n\}_{n\in\mathbb{N}} \in L^2(0,L)$ with $\|\psi_T^n\|_{L^2(0,L)} = 1$ such that the corresponding solution ψ^n of (3.2) satisfies

$$1 = \|\psi_T^n\|_{L^2(0,L)} > n \left(\int_0^T (|\Delta_t^{-\frac{1}{3}} \psi_{xx}^n(0,t)|^2 + |\psi_x^n(L,t)|^2) dt \right)$$

for any n. Thus

$$\|\Delta_t^{-\frac{1}{3}} \psi_{xx}^n(0,\cdot)\|_{L^2(0,T)} \to 0 \quad \text{and} \quad \|\psi_x^n(L,\cdot)\|_{L^2(0,T)} \to 0 \tag{3.8}$$

when $n \to \infty$. Arguing as in the proof of Lemma 3.2 we can conclude that $\{\psi_T^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(0,L)$ converging to some $\psi_T \in L^2(0,L)$. The corresponding solution ψ of (3.2) satisfies $\psi_{xx}(0,t) = 0$ and $\psi_x(L,t) = 0$, i.e.,

$$\begin{cases} \psi_t + \psi_x + \psi_{xxx} = 0, \quad (x,t) \in (0,L) \times (0,T), \\ \psi(0,t) = 0,\ \psi_x(0,t) = 0,\ \psi_{xx}(0,t) = 0, \quad t \in (0,T), \\ \psi(L,t) + \psi_{xx}(L,t) = 0,\ \psi_x(L,t) = 0, \quad t \in (0,T), \\ \psi(x,T) = \psi_T(x), \quad x \in (0,L), \end{cases} \tag{3.9}$$

from which we have $\psi \equiv 0$ because of the unique continuation property ($\psi(0,t) = \psi_x(0,t) = \psi_{xx}(0,t) = 0$ for any $t \in (0,T)$). In particular, $\psi_T \equiv 0$, which contradicts the fact that $\|\psi_T\|_{L^2(0,L)} = 1$. ∎

Proof of Proposition 3.4. Without loss of generality, we assume that $y_0 = 0$. Let ψ be a solution of the system (3.2) and multiply both sides of the equation in (3.6) by ψ and integrate over the domain $(0, L) \times (0, T)$. Integration by parts leads to

$$\int_0^L y(x,T)\psi_T(x)dx = \int_0^T (h_1(t)\psi_{xx}(0,t) + h_2(t)\psi_x(L,t))\,dt.$$

Let us denote by Υ the linear and bounded map from $L^2(0, L) \to L^2(0, L)$ defined by

$$\Upsilon : \psi_T(\cdot) \to y(\cdot, T)$$

with y being the solution of (3.6) when

$$h_1(t) = \Delta_t^{-\frac{2}{3}}\psi_{xx}(0,t) \quad \text{and} \quad h_2(t) = \psi_x(L,t),$$

where ψ is the solution of system (3.2). Thus

$$\begin{aligned}(\Upsilon(\psi_T), \psi_T)_{L^2(0,L)} &= (\Delta_t^{-\frac{1}{3}}\psi_{xx}(0,\cdot), \Delta_t^{-\frac{1}{3}}\psi_{xx}(0,\cdot))_{L^2(0,T)} + \|\psi_x(L,\cdot)\|^2_{L^2(0,T)} \\ &\geq C^{-2}\|\psi_T\|^2_{L^2(0,L)}.\end{aligned}$$

The proof is then completed by using the Lax–Milgram theorem. ∎

We now consider boundary controllability of the linear system

$$\text{(3.10)} \qquad \begin{cases} y_t + y_x + y_{xxx} = 0, \quad (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0, \quad y_x(L,t) = h_2(t), \quad y_{xx}(L,t) = h_3(t), \quad t \in (0,T), \end{cases}$$

with two control inputs $h_2 \in L^2(0, T)$ and $h_3 \in H^{-\frac{1}{3}}(0, T)$.

PROPOSITION 3.6. *Let $T > 0$ be given. There exists a bounded linear operator*

$$\digamma_1 : L^2(0,L) \times L^2(0,L) \to L^2(0,T) \times H^{-\frac{1}{3}}(0,T)$$

such that for any $y_0,\ y_T \in L^2(0, L)$, if one chooses

$$(h_2, h_3) = \digamma_1(y_0, y_T),$$

then the system (3.6) *admits a solution $y \in X_T$ satisfying*

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T.$$

As before, Proposition 3.6 follows from the following observability estimates for the corresponding adjoint system (3.2).

LEMMA 3.7. *Let $T > 0$ be given. There exists a constant $C > 0$ such that for any $\psi_T \in L^2(0, L)$, the corresponding solution ψ of* (3.2) *satisfies*

$$\text{(3.11)} \qquad \|\psi_T\|_{L^2(0,L)} \leq C\left(\int_0^T (|\Delta_t^{\frac{1}{3}}\psi(L,t)|^2 + |\psi_x(L,t)|^2)dt\right).$$

Proof. The proof is similar to that of Lemma 3.5 and is therefore omitted. ∎

Finally we consider the linear system associated to (1.2) using only $g_1 \in H^{\frac{1}{3}}(0, T)$ and $g_3 \in L^2(0, T)$ as control inputs, i.e.,

$$\text{(3.12)} \qquad \begin{cases} u_t + u_x + u_{xxx} = 0,\ (x,t) \in (0,L) \times (0,T), \\ u(0,t) = g_1(t),\ u(L,t) = 0,\ u_x(L,t) = g_3(t), \quad t \in (0,T). \end{cases}$$

The critical length phenomenon will not occur and system (3.12) is exactly controllable for any $L > 0$, as stated in the following result.

PROPOSITION 3.8. *Let $T > 0$ be given. There exists a bounded linear operator*

$$\digamma_2 : L^2(0,L) \times L^2(0,L) \to H^{\frac{1}{3}}(0,T) \times L^2(0,T)$$

such that for any $u_0,\ u_T \in L^2(0,L)$, if one chooses

$$(g_1, g_3) = \digamma_2(u_0, u_T),$$

then the system (3.12) *admits a solution $u \in X_T$ satisfying*

$$u|_{t=0} = u_0, \qquad u|_{t=T} = u_T.$$

Proof. The proof is similar to that of Proposition 3.6 and is therefore skipped. ∎

4. Nonlinear control systems. In this section we first consider the nonlinear system

$$(4.1) \qquad \begin{cases} y_x + y_{xxx} + y_x + yy_x = 0, \quad x \in (0,L),\ t \in (0,T), \\ y(0,t) = 0,\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = 0, \quad t \in (0,T), \\ y(x,0) = y_0(x), \quad x \in (0,L), \end{cases}$$

and present the proof of Theorem 1.1.

Proof of Theorem 1.1. Rewrite system (4.1) in its integral form

$$(4.2) \qquad y(t) = W_0(t)y_0 + W_{bdr}(t)h_2 - \int_0^t W_0(t-\tau)(yy_x)(\tau)d\tau.$$

Here we have written $W_{bdr}(t)(0, h_2, 0)$ as $W_{bdr}(t)h_2$ for simplicity. For any $v \in X_T$, let us set

$$\nu(T, v) := \int_0^T W_0(T-\tau)(vv_x)(\tau)d\tau.$$

For any $y_0,\ y_T \in L^2(0,L)$, we use Proposition 3.1 to define

$$h_2 = \Psi(y_0, y_T + \nu(T, v)).$$

Then

$$v(t) = W_0(t)y_0 + W_{bdr}(t)\Psi(y_0, y_T + \nu(T,v)) - \int_0^t W_0(t-\tau)(vv_x)(\tau)d\tau$$

satisfies

$$v|_{t=0} = y_0, \qquad v|_{t=T} = y_T + \nu(T,v) - \nu(T,v) = y_T.$$

This leads us to consider the map

$$\Gamma(v) = W_0(t)y_0 + W_{bdr}(t)\Psi(y_0, y_T + \nu(T,v)) - \int_0^t W_0(t-\tau)(vv_x)(\tau, x)d\tau.$$

If we can show that the map Γ is a contraction in an appropriate metric space, then its fixed point v is a solution of (4.1) with $h_2 = \Psi(y_0, y_T + \nu(T; v))$ which satisfies

$$v|_{t=0} = y_0, \qquad v|_{t=T} = y_T.$$

Next we show that this is indeed the case; the map Γ is a contraction map in the ball

$$B_r = \{z \in X_T;\ \|z\|_{X_T} \leq r\}$$

for an appropriately chosen r. According to Proposition 2.1, there exists a constant $C_1 > 0$ such that

$$\|\Gamma(v)\|_{X_T} \leq C_1 \left(\|y_0\|_{L^2(0,L)} + \|\Psi(y_0, y_T + \nu(T,v))\|_{L^2(0,L)} + \int_0^T \|vv_x\|_{L^2(0,L)}(t)dt \right).$$

Since

$$\|\Psi(y_0, y_T + \nu(T,v))\|_{L^2(0,L)} \leq C_2 \left(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} + \|\nu(T,v)\|_{L^2(0,L)} \right),$$
$$\|\nu(T,v)\|_{L^2(0,L)} \leq \int_0^T \|W_0(T-\tau)vv_x\|_{L^2(0,L)}d\tau \leq \int_0^T \|vv_x\|_{L^2(0,L)}(t)dt$$

and the bilinear estimate

$$\int_0^T \|vv_x\|_{L^2(0,L)}(t)dt \leq C_3\|v\|^2_{X_T},$$

we arrive at

$$\|\Gamma(v)\|_{X_T} \leq C_3(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)}) + C_4\|v\|^2_{X_T}$$

for any $v \in X_T$, where C_3 and C_4 are constants depending only on T. By choosing r, δ such that

$$r = 2C_3\delta, \qquad 4C_3C_4\delta < \frac{1}{2} \tag{4.3}$$

we get

$$\|\Gamma(v)\|_{X_T} \leq C_3\delta + 4C_4C_3\delta C_3\delta \leq 2C_3\delta \leq r$$

for any $v \in B_r$. In addition, for $v_1,\ v_2 \in B_r$,

$$\begin{aligned}\Gamma(v_1) - \Gamma(v_2) = {} & W_{bdr}(t)\Psi\left(0, \nu(T,v_1) - \nu(T,v_2)\right) \\ & + \frac{1}{2}\int_0^t W_0(t-\tau)\left[(v_1+v_2)(v_1-v_2)_x\right](\tau)d\tau\end{aligned}$$

and

$$\begin{aligned}\|\Gamma(v_1) - \Gamma(v_2)\|_{X_T} & \leq \{C_4(\|v_1\|_{X_T} + \|v_2\|_{X_T}) + C_4(\|v_1\|_{X_T} + \|v_2\|_{X_T})\}\,\|v_1 - v_2\|_{X_T} \\ & \leq 8C_3C_4\delta\|v_1 - v_2\|_{X_T} \\ & \leq \alpha\|v_1 - v_2\|_{X_T}\end{aligned}$$

with $\alpha = 8C_3C_4\delta < 1$. The proof is completed. $\square$

As Theorems 1.4, 1.5, and 1.6 can be proved using the same arguments in the proof of Theorem 1.1, their proofs will be skipped.

Now we consider the system

$$\begin{cases} y_x + y_{xxx} + y_x + yy_x = 0, \quad x \in (0, L),\ t \in (0, T), \\ y(0, t) = 0,\ y_x(L, t) = 0,\ y_{xx}(L, t) = h_3(t), \quad t \in (0, T), \end{cases} \tag{4.4}$$

to prove Theorem 1.3.

Proof of Theorem 1.3. The proof is based on Theorem 1.8, whose proof will be given later. For given $y_0,\ y_T \in L^2(0, L)$, let us set

$$u_0 = y_0, \quad u_T = y_T.$$

Then by Theorem 1.8, there exists $g_2 \in H^{\frac{1}{3}}(0, T)$ such that the system

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0,\ (x, t) \in (0, L) \times (0, T), \\ u(0, t) = 0,\ u(L, t) = g_2(t),\ u_x(L, t) = 0, \quad t \in (0, T), \end{cases} \tag{4.5}$$

admits a unique solution $u \in X_T$ satisfying

$$u|_{t=0} = y_0, \qquad u|_{t=T} = y_T.$$

Thus $y(x, t) := u(x, t)$ will be a desired solution of (4.4) with $h_3(t) = u_{xx}(L, t)$ satisfying

$$y|_{t=0} = y_0, \quad y|_{t=T} = y_T.$$

As $u \in X_T$ solves

$$\begin{cases} u_t + u_x + u_{xxx} = f, \quad u(x, 0) = y_0, \quad (x, y) \in (0, L) \times (0, T), \\ u(0, t) = 0, \quad u(L, t) = g_2(t), \quad u_x(L, t) = 0, \quad t \in (0, T), \end{cases}$$

with $g_2 \in H^{\frac{1}{3}}(0, T)$ and $f = -uu_x \in L^1(0, T; L^2(0, L))$, it follows from Proposition 2.3 that $u_{xx}(L, \cdot) \in H^{-\frac{1}{3}}(0, T)$. The proof of Theorem 1.3 is thus complete as long as Theorem 1.8 is proved. ∎

Proof of Theorem 1.8. According to Theorem C, we get a control $g_2 \in H^{\frac{1}{6}-\epsilon}(0, T)$. We just need to prove that this g_2 given by Theorem C belongs, in fact, to the space $H^{\frac{1}{3}}(0, T)$. Indeed, the solution $u \in X_T$ given in Theorem C can be written as

$$u = \kappa + \mu,$$

where κ solves

$$\begin{cases} \kappa_t + \kappa_{xxx} = f, \quad \kappa(x, 0) = u_0,\ (x, t) \in (0, L) \times (0, T), \\ \kappa(0, t) = \kappa(L, t) = \kappa_x(L, t) = 0, \quad t \in (0, T), \end{cases}$$

with $f = -u_x - uu_x$, and μ solves

$$\begin{cases} \mu_t + \mu_{xxx} = 0, \quad \mu(x, 0) = 0,\ (x, t) \in (0, L) \times (0, T), \\ \mu(0, t) = 0, \quad \mu(L, t) = g_2(t), \quad \mu_x(L, t) = 0, \quad t \in (0, T). \end{cases} \tag{4.6}$$

As $u_0 \in L^2(0, L)$ and $f = -u_x - uu_x \in L^1(0, T; L^2(0, L))$ (because $u \in X_T$), we have $\kappa \in X_T$ by Proposition 2.3. In addition, the following lemma (whose proof will be presented in the appendix) holds for system (4.6).

LEMMA 4.1. *The solution* μ *of* (4.6) *belongs to* X_T *if and only if* g_2 *belongs to* $H^{\frac{1}{3}}(0,T)$.

Using this lemma, we get $u \in X_T$ if and only if $g_2 \in H^{\frac{1}{3}}(0,T)$. The proof of Theorem 1.8 is complete. ▯

Next we consider the system

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0, \quad x \in (0,L),\ t \in (0,T), \\ y(0,t) = h_1(t),\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = h_3(t), \quad t \in (0,T), \\ y(x,0) = y_0(x), \quad x \in (0,L), \end{cases} \tag{4.7}$$

and present the proof of Theorem 1.7.

Proof of Theorem 1.7. Consider first the following initial value control problem for the KdV equation posed on the whole line $\mathbb{R}$:

$$\begin{cases} z_t + z_x + zz_x + z_{xxx} = 0, \quad x,\ t \in \mathbb{R}, \\ z(x,0) = h(x), \end{cases} \tag{4.8}$$

where the initial value $h(x)$ is considered as a control input. The following result is due to Zhang [26].

THEOREM G. *Let* $s \geq 0$ *and* $T > 0$ *be given and suppose* $w \in C(\mathbb{R}; H^\infty(\mathbb{R}))$ *is a given solution of*

$$w_t + w_x + ww_x + w_{xxx} = 0, \quad x, t \in \mathbb{R}.$$

There exists $\delta > 0$ *such that for any* $y_0,\ y_T \in H^s(0,L)$ *with*

$$\|y_0 - w(\cdot,0)\|_{H^s(0,L)} \leq \delta, \qquad \|y_T - w(\cdot,T)\|_{H^s(0,L)} \leq \delta, \tag{4.9}$$

one can find a control input $h \in H^s(\mathbb{R})$ *which is an external modification of* y_0 *such that* (4.8) *admits a solution* $z \in C(\mathbb{R}; H^s(\mathbb{R}))$ *satisfying*

$$z(x,0) = y_0(x), \qquad z(x,T) = y_T(x) \quad \mathit{for\ any}\ x \in (0,L).$$

Let u be as in Theorem 1.7. Applying Theorem G with $w = u$ and $s = 0$, we get the existence of $\delta > 0$ such that for any $y_0,\ y_T \in L^2(0,L)$ with

$$\|y_0 - u(\cdot,0)\|_{L^2(0,L)} \leq \delta, \qquad \|y_T - u(\cdot,T)\|_{L^2(0,L)} \leq \delta, \tag{4.10}$$

there exists $h \in L^2(\mathbb{R})$ and the corresponding $z \in C(\mathbb{R}; L^2(\mathbb{R})) \cap L^2(\mathbb{R}; H^1_{loc}(\mathbb{R}))$ solution of (4.8). Let y be the restriction of z to the domain $(0,L) \times (0,T)$, and

$$h_1(t) = z(0,t), \quad h_2(t) = z_x(L,t), \quad h_3(t) = z_{xx}(L,t)$$

for $0 < t < T$. Then, according to [28], we have that $y \in X_T$ solves (4.7) with $h_1 \in H^{\frac{1}{3}}(0,T),\ h_2 \in L^2(0,T)$, and $h_3 \in H^{-\frac{1}{3}}(0,T)$. Moreover,

$$y|_{t=0} = y_0, \qquad y|_{t=T} = y_T.$$

The proof is complete. ▯

Finally we consider systems

$$\begin{cases} u_t + u_{xxx} + u_x + uu_x = 0, \quad x \in (0,L),\ t \in (0,T), \\ u(0,t) = 0,\ u(L,t) = g_2(t),\ u_x(L,t) = g_3(t), \quad t \in (0,T), \end{cases} \tag{4.11}$$

and

$$\begin{cases} u_t + u_{xxx} + u_x + uu_x = 0, \quad x \in (0,L),\ t \in (0,T), \\ u(0,t) = g_1(t),\ u(L,t) = g_2(t),\ u_x(L,t) = 0, \quad t \in (0,T), \end{cases} \tag{4.12}$$

and present the proofs of Theorems 1.9 and 1.10.

Proof of Theorem 1.9. We first consider the associated linear system

$$\begin{cases} u_t + u_{xxx} + u_x = 0, \quad x \in (0,L),\ t \in (0,T), \\ u(0,t) = 0,\ u(L,t) = g_2(t),\ u_x(L,t) = g_3(t), \quad t \in (0,T), \end{cases} \tag{4.13}$$

and show that it is exactly controllable in the space $L^2(0,L)$.

PROPOSITION 4.2. *Let $T > 0$ be given. For any $u_0,\ u_T \in L^2(0,L)$, there exist $g_2 \in H^{\frac{1}{3}}(0,T)$ and $g_3 \in L^2(0,T)$ such that the system* (4.13) *admits a unique solution*

$$u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$u|_{t=0} = u_0, \qquad u|_{t=T} = u_T.$$

Moreover, there exists a constant $C > 0$ depending only on T such that

$$\|g_2\|_{H^{\frac{1}{3}}(0,T)} + \|g_3\|_{L^2(0,T)} \le C \left(\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \right).$$

Then the proof of Theorem 1.9 is completed by using the fixed point argument as in the proof of Theorem 1.1.

To see that Proposition 4.2 is true, note that by Proposition 3.6, for given u_0 and u_T in $L^2(0,L)$, there exist $h_2 \in L^2(0,T)$ and $h_3 \in H^{-\frac{1}{3}}(0,T)$ such that the system

$$\begin{cases} y_t + y_{xxx} + y_x = 0, \quad x \in (0,L),\ t \in (0,T), \\ y(0,t) = 0,\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = h_3(t), \quad t \in (0,T), \end{cases}$$

admits a unique solution $y \in X_T$ satisfying

$$y|_{t=0} = u_0, \qquad y|_{t=T} = u_T.$$

Moreover, according to Proposition 2.2, $y(L,\cdot) \in H^{\frac{1}{3}}(0,T)$. Thus, we can just choose $g_2(t) = y(L,t)$ and $g_3(t) = h_2(t)$, and then $u(x,t) \equiv y(x,t)$ is the desired solution of system (4.13) in Proposition 4.2. ∎

Proof of Theorem 1.10. The proof is similar to the proof of Theorem 1.1 but uses Proposition 3.8 instead of Proposition 3.1. ∎

5. Concluding remarks. Our discussion has been focused on the boundary controllability of two classes of boundary control systems described by the KdV equation posed on a finite domain $(0,L)$, namely,

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad (x,t) \in (0,L) \times (0,T), \\ u(0,t) = g_1(t), \quad u(L,t) = g_2(t), \quad u_x(L,t) = g_3(t), \quad t \in (0,T), \end{cases} \tag{5.1}$$

and

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, \quad (x,t) \in (0,L) \times (0,T), \\ y(0,t) = h_1(t), \quad y_x(L,t) = h_2(t), \quad y_{xx}(L,t) = h_3(t) \quad t \in (0,T). \end{cases} \tag{5.2}$$

The linear systems associated to these equations are obtained by dropping the nonlinear terms uu_x and yy_x, respectively. The system (5.1) has been intensively studied and various controllability results have been established in the past. However, there have been few results for the second system (5.2) because of some difficulties in applying directly the methods that work effectively for system (5.1). In this paper, aided by the newly established hidden regularities of solutions of the KdV equation, we have succeeded in overcoming those difficulties and established various boundary controllability results for system (5.2) similar to those known for system (5.1) in the literature. Furthermore, with the new tool in hand, we have also been able to improve some known controllability results for system (5.1). Our results can be summarized as follows:

(i) The linear system associated to (5.2) is exactly controllable with two or three boundary controls in action. In any of those cases, the nonlinear system (5.2) is locally exactly controllable.

(ii) With only a single control h_2 in action ($h_1 = h_3 = 0$), the linear system associated to (5.2) is exactly controllable if and only if L does not belong to

$$\mathbb{F} = \Big\{ L \in \mathbb{R}^+ : L^2 = -(a^2 + ab + b^2) \text{ with } a, b \in \mathbb{C} \text{ satisfying } \\ \frac{e^a}{a^2} = \frac{e^b}{b^2} = \frac{e^{-(a+b)}}{(a+b)^2} \Big\}.$$

The nonlinear system (5.2) is also locally exactly controllable if $L \notin \mathbb{F}$.

(iii) When only control input h_3 is employed ($h_1 = h_2 = 0$), the linear system associated to (5.2) is exactly controllable if and only if L does not belong to

$$\mathcal{N} = \Big\{ L \in \mathbb{R}^+ : L^2 = -(a^2 + ab + b^2) \text{ with } a, b \in \mathbb{C} \text{ satisfying } \\ ae^a = be^b = -(a+b)e^{-(a+b)} \Big\}.$$

Moreover, if $L \notin \mathcal{N}$, then the nonlinear system (5.2) is locally exactly controllable.

(iv) The linear system associated to (5.1) is exactly controllable with control inputs g_1 and g_3 in action (put $g_2 = 0$). In this case, the nonlinear system (5.1) is locally exactly controllable.

(v) We have improved the regularity of the control g_2 in (5.1). In previous work [10], the control g_2 is known to belong to the space $H^{\frac{1}{6}-\epsilon}(0,T)$ for any $\epsilon > 0$. In this paper we are able to prove that the control input g_2 belongs in fact to the space $H^{\frac{1}{3}}(0,T)$.

While some significant progress has been made in the study of boundary controllability of the KdV equation on a bounded domain, there are still a lot of interesting questions left open for further investigations. One of them is the so-called critical length problem. As is well known now, the linear systems associated to (5.1) and (5.2) are not always exactly controllable if only a single control input is allowed to act on the right end of the spatial domain $(0,L)$. In general, if the associated linear system is not exactly controllable, one would tend to believe the nonlinear system is also not exactly controllable. However, for system (5.1) with only control input g_3 in action, though its associated linear system is not exactly controllable when $L \in S$ (see (1.5) for the definition of S), the nonlinear system (5.1) has been shown by Coron and Crépeau [8], Cerpa [3], and Cerpa and Crépeau [4] to be locally (large time) exactly controllable. The questions remain open for other critical length problems.

OPEN PROBLEM 5.1 (critical length problems).

(a) *Is the nonlinear system* (5.2) *with only control input* h_2 *in action exactly controllable when the length* L *of the spatial domain* $(0, L)$ *belongs to the set* $\mathbb{F}$*?*

(b) *Is the nonlinear system* (5.2) *with only control input* h_3 *in action exactly controllable when the length* L *of the spatial domain* $(0, L)$ *belongs to the set* $\mathcal{N}$*?*

(c) *Is the nonlinear system* (5.1) *with only control input* g_2 *in action exactly controllable when the length* L *of the spatial domain* $(0, L)$ *belongs to the set* $\mathcal{N}$*?*

Most controllability results that have been established so far for both systems (5.1) and (5.2) are local: one can only guide a small amplitude initial state to a small amplitude terminal state by choosing appropriate boundary control inputs. The following question arises naturally.

OPEN PROBLEM 5.2 (global controllability problem). *Are the nonlinear systems* (5.1) *and* (5.2) *globally exactly boundary controllable?*

The global interior stabilization result for the KdV equation on a finite interval,

$$(5.3)\quad \begin{cases} u_t + u_x + uu_x + u_{xxx} + a(x)u = 0, \quad u(x,0) = u_0(x), \quad x \in (0, L),\ t > 0, \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \end{cases}$$

is well known in the literature (see [17, 18, 22]).

THEOREM H. *Assume the function* $a \in L^\infty(0, L)$ *with* $a(x) \geq 0$ *and such that the support of the function* a *is a nonempty open subset of* $(0, L)$*. There exists* $\gamma > 0$ *such that for any* $u_0 \in L^2(0, L)$*, the corresponding solution* u *of* (5.3) *belongs to the space* $C([0, \infty); L^2(0, L))$ *and, moreover,*

$$\|u(\cdot, t)\|_{L^2(0,L)} \leq \alpha(\|u_0\|_{L^2(0,L)})e^{-\gamma t} \qquad \forall\ t \geq 0,$$

where $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ *is a nondecreasing continuous function.*

Combining Theorem H and Theorems 1.4, 1.5, 1.6 and 1.7, we have the following partial answers to Open Problem 5.2 for the nonlinear system (5.2).

THEOREM 5.1. *There exists* $\delta > 0$ *such that for any* $N > 0$ *the following holds. There exists a* $T > 0$ *depending only on* N *and* δ *such that for any* $\phi,\ \psi \in L^2(0, L)$ *with*

$$\|\phi\|_{L^2(0,L)} \leq N, \qquad \|\psi\|_{L^2(0,L)} \leq \delta,$$

one can find either

$$h_1 \in H^{\frac{1}{3}}(0, T), \quad h_2 \in L^2(0, T), \quad h_3 \in H^{-\frac{1}{3}}(0, T),$$

or

$$h_1 \in H^{\frac{1}{3}}(0, T), \quad h_2 = 0, \quad h_3 \in H^{-\frac{1}{3}}(0, T),$$

or

$$h_1 = 0, \quad h_2 \in L^2(0,T), \quad h_3 \in H^{-\frac{1}{3}}(0,T),$$

such that the nonlinear system (5.2) *admits a solution* $u \in C([0,T];L^2(0,L))$ *satisfying*

$$u|_{t=0} = \phi, \qquad u|_{t=T} = \psi.$$

Proof. Note first that for any $\phi \in L^2(0,L)$, system (5.3) admits a unique solution $u \in C([0,\infty);L^2(0,L))$ which also possesses the hidden regularity

$$\partial_x^k u(x,t) \in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T)), \quad k = 0,1,2.$$

Then, Theorem 5.1 follows from Theorem G and Theorems 1.4, 1.5, 1.6 and 1.7 using the same argument as that used in the proof of Theorem 3.22 in [23]. ∎

Remark 5.2. Theorem 5.1 provides only a partial answer to Open Problem 5.2 since the amplitude of the terminal state is still required to be small. A question remains: Can a small amplitude restriction on the terminal state be removed?

If one is allowed to use all three boundary control inputs, then the small amplitude restriction can be removed.

THEOREM 5.3. *Let* $N > 0$ *be given. There exists* $T > 0$ *such that for any* $\phi,\ \psi \in L^2(0,L)$ *with*

$$\|\phi\|_{L^2(0,L)} \le N, \qquad \|\psi\|_{L^2(0,L)} \le N,$$

the nonlinear equation

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad x \in (0,L)\times(0,T)$$

admits a solution $u \in C([0,T];L^2(0,L))$ *satisfying*

$$u|_{t=0} = \phi, \qquad u|_{t=T} = \psi.$$

Proof. For given $\phi,\ \psi \in L^2(0,L)$, let $\tilde\phi$ and $\tilde\psi$ be their extension from $(0,L)$ to $(0,2L)$ such that

$$\tilde\phi \in L^2(0,2L), \qquad \tilde\psi \in L^2(0,2L), \qquad \int_0^{2L} \tilde\phi(x)dx = \int_0^{2L} \tilde\psi(x)dx$$

and consider the following internal control problem of the KdV equation posed on the interval $(0,2L)$ with periodic boundary condition:

$$\begin{cases} v_t + v_x + vv_x + v_{xxx} + a(x)v = 0, \quad v(x,0) = \phi(x), \qquad x \in (0,2L), \\ v(0,t) = v(2L,t), \qquad v_x(0,t) = v_x(2L,t), \qquad v_{xx}(0,t) = v_{xx}(2L,t), \end{cases}$$

where $a \in L^\infty(0,2L)$ with its support contained in $(L,2L)$. The proof is completed by invoking Theorem 1.1 in [15]. ∎

Consequently, if one chooses

$$h_1(t) = u(0,t), \qquad h_2(t) = u_x(L,t), \qquad h_3(t) = u_{xx}(L,t),$$

then the system (5.2) will be guided from the given initial state ϕ to the given terminal state ψ. The only drawback is that we do not know exactly the regularities of the boundary inputs h_j, $j = 1, 2, 3$.

In Theorems 5.1 and 5.3 the time interval $(0, T)$ used to conduct control depends on the size of the initial state and terminal state. The larger the initial state, the longer the time interval $(0, T)$. Such controllability is usually called large time controllability. As is well known, the KdV equation possesses infinite propagation speed. Thus one may wonder the following.

OPEN PROBLEM 5.3. *Can the time interval* $(0, T)$ *in Theorems* 5.1 *and* 5.3 *be chosen arbitrarily small?*

6. Appendix.

6.1. Proofs of Proposition 2.8 and Lemma 4.1.

Proof of Proposition 2.8. The solution of the system

$$\begin{cases} w_t + w_{xxx} = f, \quad w(x,0) = w_0(x), \quad (x,t) \in (0,L) \times (0,T), \\ w_{xx}(0,t) = k_1(t), \quad w(L,t) = k_2(t), \quad w_x(L,t) = k_3(t), \quad t \in (0,T), \end{cases} \tag{6.1}$$

can be written as

$$w(t) = W_0(t)w_0 + W_{bdr}(t)\vec{k} + \int_0^t W_0(t-\tau)f(\tau)d\tau$$

with $\vec{k} = (k_1, k_2, k_3)$, where $W_0(t)$ is the C_0 semigroup in $L^2(0, L)$ generated by the operator

$$Bf = -f'''$$

with the domain

$$\mathcal{D}(B) = \{f \in H^3(0,L);\ f''(0) = f(L) = f'(L) = 0\}.$$

Then, $u(t) = W_0(t)w_0$ solves

$$\begin{cases} u_t + u_{xxx} = 0, \quad u(x,0) = w_0(x), \quad (x,t) \in (0,L) \times (0,T), \\ u_{xx}(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0, \quad t \in (0,T), \end{cases} \tag{6.2}$$

$v(t) = W_{bdr}(t)\vec{k}$ solves

$$\begin{cases} v_t + v_{xxx} = 0, \quad v(x,0) = 0, \quad (x,t) \in (0,L) \times (0,T), \\ v_{xx}(0,t) = k_1(t), \quad v(L,t) = k_2(t), \quad v_x(L,t) = k_3(t), \quad t \in (0,T), \end{cases} \tag{6.3}$$

and $z(t) = \int_0^t W_0(t-\tau)f(\tau)d\tau$ solves

$$\begin{cases} z_t + z_{xxx} = f, \quad z(x,0) = 0, \quad (x,t) \in (0,L) \times (0,T), \\ z_{xx}(0,t) = 0, \quad z(L,t) = 0, \quad z_x(L,t) = 0, \quad t \in (0,T). \end{cases} \tag{6.4}$$

As in the proof of Proposition 2.1, it is easy to see that for any $f \in L^1(0, T; L^2(0, L))$ and $w_0 \in L^2(0, L)$, both $u = W_0(t)w_0$ and $z = \int_0^t W_0(t-\tau)f(\tau)d\tau$ belong to the space X_T and, in addition, there exists a constant $C > 0$ such that

$$\|u\|_{X_T} + \|z\|_{X_T} \le C \left(\|w_0\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))} \right).$$

For $v(t) = W_{bdr}(t)\vec{k}$, following [2], we first look for an explicit representation formula. Applying the Laplace transform with respect to t in both sides of the equation in (6.3) (i.e., $\hat{v}(s,x) = \int_0^\infty e^{-st}v(t)dt$), we obtain

$$\begin{cases} s\hat{v} + \hat{v}_{xxx} = 0, \quad x \in (0,L), \\ \hat{v}_{xx}(0,s) = \hat{k}_1(s), \quad \hat{v}(L,s) = \hat{k}_2(s), \quad \hat{v}_x(L,s) = \hat{k}_3(s). \end{cases} \tag{6.5}$$

Its solution $\hat{v}(x,s)$ can be written as $\hat{v}(x,s) = \sum_{j=1}^{3} c_j(s)e^{\lambda_j(s)x}$, where λ_j solves characteristic equation $s + \lambda^3 = 0$, i.e.,

$$\lambda_1 = i\rho, \quad \lambda_2 = -i\rho\left(\frac{1+i\sqrt{3}}{2}\right), \quad \lambda_3 = -i\rho\left(\frac{1-i\sqrt{3}}{2}\right)$$

with $s = \rho^3$. Imposition of the boundary conditions of (6.5) yields that $c_j = c_j(s)$ for $j = 1,2,3$ solves the system

$$\begin{pmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \hat{k}_1 \\ \hat{k}_2 \\ \hat{k}_3 \end{pmatrix}.$$

By Cramer's rule,

$$c_j = \frac{\Delta_j}{\Delta} \quad \text{for} \quad j = 1,2,3,$$

where

$$\Delta = \Delta(s) = \begin{vmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \end{vmatrix}$$

and $\Delta_j(s)$ is the determinant of the matrices obtained by changing the jth column of Δ by the vector $(\hat{k}_1, \hat{k}_2, \hat{k}_3)^T$ for $j = 1,2,3$. Taking the inverse Laplace transform of $\widehat{v}$ and following the same arguments as those in [1] lead us to the following representation of the solution v of the system (6.3):

$$v(x,t) = \sum_{m=1}^{3} v_m(x,t)$$

with

$$v_m(x,t) = \sum_{j=1}^{3} v_{j,m}(x,t) \quad \text{and} \quad v_{j,m}(x,t) = v_{j,m}^+(x,t) + v_{j,m}^-(x,t),$$

where for $m,\ j = 1,2,3$,

$$\begin{aligned} v_{j,m}^+(x,t) &= \frac{1}{2\pi}\int_0^\infty e^{i\rho^3 t + \lambda_j^+(\rho)x}\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}\hat{k}_m^+(\rho)3\rho^2 d\rho, \\ v_{j,m}^-(x,t) &= \overline{v_{j,m}^+(x,t)}, \end{aligned}$$

and

$$\hat{k}_m^+(\rho) = \hat{k}_m(i\rho^3),\ \Delta^+(\rho) = \Delta(i\rho^3),\ \Delta_{j,m}^+(\rho) = \Delta_{j,m}(i\rho^3),\ \lambda_j^+(\rho) = \lambda_j(i\rho^3).$$

LEMMA 6.1. *Let $T > 0$ be given. There exists a constant $C > 0$ such that for any $\vec{k} \in \mathcal{K}_T$, the system (6.3) admits a unique solution $v \in X_T$. Moreover, there exists a constant $C > 0$ such that*

$$\|v\|_{X_T} + \sum_{j=0}^{2} \|\partial_x^j v\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,T))} \leq C\|\vec{k}\|_{\mathcal{K}_T}.$$

Proof. Note that as stated above, the solution v can be written as

$$v(x,t) = v_1(x,t) + v_2(x,t) + v_3(x,t).$$

Let us prove Lemma 6.1 for v_1. First, by straightforward computation, we can list the asymptotic behavior of the ratios $\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}$ for $\rho \to +\infty$ as below:

$\frac{\Delta_{1,1}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}e^{-\frac{\sqrt{3}}{2}\rho L}$	$\frac{\Delta_{2,1}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}e^{-\sqrt{3}\rho L}$	$\frac{\Delta_{3,1}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}e^{-\sqrt{3}\rho L}$
$\frac{\Delta_{1,2}^+(\rho)}{\Delta^+(\rho)} \sim 1$	$\frac{\Delta_{2,2}^+(\rho)}{\Delta^+(\rho)} \sim e^{-\sqrt{3}\rho L}$	$\frac{\Delta_{3,2}^+(\rho)}{\Delta^+(\rho)} \sim 1$
$\frac{\Delta_{1,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}$	$\frac{\Delta_{2,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}e^{-\frac{\sqrt{3}}{2}\rho L}$	$\frac{\Delta_{3,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}$

As

$$v_1(x,t) = \frac{3}{\pi}\sum_{j=1}^{3} \mathcal{R}e \int_0^\infty e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} \hat{k}_1^+(\rho)\rho^2 d\rho,$$

we have

$$\begin{aligned}
\sup_{0<t<T} \|v_1(\cdot,t)\|_{L^2(0,L)}^2 &\leq C\int_0^\infty \rho^{-2}|\hat{k_1}^+(\rho)|^2\rho^2 d\rho \\
&\leq C\int_0^\infty \mu^{-2/3}|\hat{k_1}(i\mu)|^2 d\mu \\
&\leq C\|k_1\|_{H^{-\frac{1}{3}}(\mathbb{R}^+)}^2 \\
&\leq C\|\vec{k}\|_{\mathcal{K}_T}.
\end{aligned}$$

Furthermore, for $\ell = -1, 0, 1$, set $\mu = \rho^3,\ \theta(\mu) = \mu^{\frac{1}{3}}$,

$$\begin{aligned}
\partial_x^{\ell+1} v_1(x,t) &= \frac{3}{\pi}\sum_{j=1}^{3} \mathcal{R}e \int_0^\infty (\lambda_j^+(\rho))^{\ell+1} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} \hat{k}_1^+(\rho)\rho^2 d\rho \\
&= \frac{1}{\pi}\sum_{j=1}^{3} \mathcal{R}e \int_0^\infty (\lambda_j^+(\theta(\mu)))^{\ell+1} e^{i\mu t} e^{\lambda_j^+(\theta(\mu))x} \frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))} \hat{k}_1(i\mu) d\mu.
\end{aligned}$$

Applying the Plancherel theorem in time t yields that for any $x \in (0,L)$,

$$\begin{aligned}\|\partial_x^{\ell+1} v_1(x,\cdot)\|^2_{H^{-\frac{\ell}{3}}(0,T)} &\le C\sum_{j=1}^3 \int_0^\infty \mu^{-\frac{2\ell}{3}} \left| (\lambda_j^+(\theta(\mu)))^{\ell+1} e^{\lambda_j^+(\theta(\mu))x} \frac{\Delta^+_{j,1}(\theta(\mu))}{\Delta^+(\theta(\mu))} \hat{k}_1(i\mu)\right|^2 d\mu \\ &\le C\int_0^\infty \mu^{-\frac{2\ell}{3}} |\hat{k}_1(i\mu)|^2 d\mu \\ &\le C\|k_1\|^2_{H^{-\frac{\ell}{3}}(0,T)} \\ &\le C\|\vec{k}\|^2_{\mathcal{K}_T}\end{aligned}$$

for $\ell = -1, 0, 1$. Consequently

$$\sup_{0<x<L} \|\partial_x^{\ell+1} v_1(x,\cdot)\|_{H^{-\frac{\ell}{3}}(0,T)} \le C\|\vec{k}\|_{\mathcal{K}_T}$$

for $\ell = -1, 0, 1$. In particular,

$$\|v_1\|_{L^2(0,T;H^1(0,L))} \le C\|\vec{k}\|_{\mathcal{K}_T},$$

which ends the proof of Lemma 6.1 for v_1. The proofs for v_2 and v_3 are similar. $\square$

Now we complete the proof of Proposition 2.8. It remains to prove that

(6.6)
$$\|\partial_x^j u\|_{L_x^\infty(0,L;H^{\frac{1-j}{3}}(0,T))} + \|\partial_x^j z\|_{L_x^\infty(0,L;H^{\frac{1-j}{3}}(0,T))} \le C(\|w_0\|_{L^2(0,T)} + \|f\|_{L^1(0,T;L^2(0,L))})$$

for $j = 0, 1, 2$.

To this end, note that u and z can be written as

$$u(t) = W_R(t)\tilde{w}_0 - W_{bdr}(t)\vec{p}, \qquad z(t) = \int_0^t W_R(t-\tau)\tilde{f}(\tau)d\tau - W_{bdr}(t)\vec{q},$$

respectively. Here

(i) $\tilde{w}_0$ and $\tilde{f}$ are zero extensions of w_0 and f from $(0,L)$ to $\mathbb{R}$,

$$\tilde{w}_0(x) = \begin{cases} w_0(x), & x \in (0,L), \\ 0, & x \notin (0,L), \end{cases} \qquad \tilde{f}_0(x,t) = \begin{cases} f(x,t), & (x,t) \in (0,L)\times(0,T), \\ 0, & x \notin (0,L); \end{cases}$$

(ii) $W_R(t)$ is the C_0 semigroup associated to the initial value problem

$$\mu_t + \mu_{xxx} = 0, \quad \mu(x,0) = \tilde{w}_0(x), \qquad x \in R,\ t \in (0,T);$$

(iii) $\vec{p} = (p_1, p_2, p_3)$ with

$$p_1(t) = \mu_{xx}(0,t), \quad p_2(t) = \mu(L,t), \quad p_3(t) = \mu_x(L,t),$$

where $\mu(t) = W_R(t)\tilde{w}_0$;

(iv) $\vec{q} = (q_1, q_2, q_3)$ with

$$q_1(t) = \tilde{z}_{xx}(0,t), \quad q_2(t) = \tilde{z}(L,t), \quad q_3(t) = \tilde{z}_x(L,t),$$

where

$$\tilde{z} = \int_0^t W_R(t-\tau)\tilde{f}(\tau)d\tau.$$

According to [14], for $j = 0, 1, 2$,

$$\|\partial_x^j \mu\|_{L_x^\infty(\mathbb{R}; H^{\frac{1-j}{3}}(0,T))} \leq C\|\tilde{w}_0\|_{L^2(\mathbb{R})} \leq C\|w_0\|_{L^2(0,L)}$$

and

$$\|\partial_x^j \tilde{z}\|_{L_x^\infty(\mathbb{R}; H^{\frac{1-j}{3}}(0,T))} \leq C\|\tilde{f}_0\|_{L^1(0,T;L^2(\mathbb{R}))} \leq C\|f_0\|_{L^1(0,T;L^2(0,L))}.$$

Furthermore, by Lemma 6.1,

$$\|\partial_x^j W_{bdr}(t)\vec{p}\|_{L_x^\infty(0,L; H^{\frac{1-j}{3}}(0,T))} \leq C\|\vec{p}\|_{\mathcal{K}_T} \leq C\|w_0\|_{L^2(0,L)}$$

and

$$\|\partial_x^j W_{bdr}(t)\vec{q}\|_{L_x^\infty(0,L; H^{\frac{1-j}{3}}(0,T))} \leq C\|\vec{q}\|_{\mathcal{K}_T} \leq C\|f\|_{L^1(0,T;L^2(0,L))}.$$

The proof of Proposition 2.8 is thus complete. ☐

Proof of Lemma 4.1. As in the above proof (see also [1]), the solution μ of

$$\begin{cases} \mu_t + \mu_{xxx} = 0, \quad \mu(x,0) = 0, \ (x,t) \in (0,L) \times (0,T), \\ \mu(0,t) = 0, \quad \mu(L,t) = g_2(t), \quad \mu_x(L,t) = 0, \quad t \in (0,T), \end{cases}$$

can be written as

$$\mu(x,t) = \mu_1(x,t) + \mu_2(x,t) + \mu_3(x,t)$$

with

$$\mu_j(x,t) = \frac{3}{\pi}\mathcal{R}e\int_0^\infty e^{i\rho^3 t} e^{\lambda_j(\rho)x} S_j(\rho)\rho^2 \hat{g}_2(i\rho^3)d\rho$$

for $j = 1, 2, 3$, where

$$\lambda_1(\rho) = i\rho, \quad \lambda_2(\rho) = \frac{\sqrt{3}}{2}\rho - \frac{1}{2}i\rho, \quad \lambda_3(\rho) = -\frac{\sqrt{3}}{2}\rho - \frac{1}{2}i\rho,$$
$$S_1(\rho) \sim 1, \quad S_2(\rho) \sim e^{-\frac{\sqrt{3}}{2}L\rho}, \quad S_3(\rho) \sim 1, \quad \textit{as } \rho \to +\infty.$$

Arguing as before, $g_2 \in H^{\frac{1}{3}}(0,T)$ implies that $\mu \in X_T$. On the other hand, if $\mu \in X_T$, we show that we must have $g_2 \in H^{\frac{1}{3}}(0,T)$. First note that as

$$\begin{aligned} \mu_1(x,t) &= \frac{3}{\pi}\mathcal{R}e\int_0^\infty e^{i\rho^3 t} e^{i\rho x} S_1(\rho)\rho^2 \hat{g}_2(i\rho^3)d\rho \\ &= \frac{1}{\pi}\mathcal{R}e\int_0^\infty e^{i\nu t} e^{i\nu^{\frac{1}{3}}x} S_1(\nu^{\frac{1}{3}})\hat{g}_2(i\nu)d\nu \end{aligned}$$

and

$$\partial_x\mu_1(x,t) = \frac{1}{\pi}\mathcal{R}e\int_0^\infty e^{i\nu t} e^{i\nu^{\frac{1}{3}}x} \nu^{\frac{1}{3}} S_1(\nu^{\frac{1}{3}})\hat{g}_2(i\nu)d\nu,$$

it follows from the Plancherel theorem that for a constant $c > 0$

$$\|\partial_x\mu_1\|^2_{L_x^2(0,L;L_t^2(R))} = c\|g_2\|^2_{H^{\frac{1}{3}}(0,T)}.$$

Therefore, $\mu_1 \in L^2(0,T;H^1(0,L))$ if and only if $g_2 \in H^{\frac{1}{3}}(0,T)$. Regarding μ_2, as

$$\partial_x \mu_2(x,t) = \frac{1}{\pi}\mathcal{R}e \int_0^\infty e^{i\nu t} \exp\left(-\frac{\sqrt{3}}{2}\nu^{\frac{1}{3}}(L-x) - \frac{1}{2}ix\nu^{\frac{1}{3}}\right)\nu^{\frac{1}{3}} S_2(\nu^{\frac{1}{3}}) e^{\frac{\sqrt{3}}{2}\nu^{\frac{1}{3}}L}\hat{g}_2(i\nu)d\nu,$$

we obtain

$$\|\partial_x \mu_2(x,\cdot)\|^2_{L^2_t(\mathbb{R})} = c\int_0^\infty e^{-\sqrt{3}\nu^{\frac{1}{3}}(L-x)}\nu^{\frac{2}{3}}|\hat{g}_2(i\nu)|^2 d\nu$$

and

$$\int_0^L \|\partial_x \mu_2(x,\cdot)\|^2_{L^2_t(\mathbb{R})}dx = c\int_0^\infty \nu^{\frac{1}{3}}|\hat{g}_2(i\nu)|^2 d\nu = c\|g_2\|_{H^{\frac{1}{6}}(0,T)}.$$

Similarly, we also have

$$\int_0^L \|\partial_x \mu_3(x,\cdot)\|^2_{L^2_t(\mathbb{R})}dx = c\int_0^\infty \nu^{\frac{1}{3}}|\hat{g}_2(i\nu)|^2 d\nu = c\|g_2\|_{H^{\frac{1}{6}}(0,T)}.$$

Hence, $\mu_2 + \mu_3 \in X_T$ if and only if $g_2 \in H^{\frac{1}{6}}(0,T)$. Consequently, $\mu \in X_T$ if and only if $g_2 \in H^{\frac{1}{3}}(0,T)$. The proof of Lemma 4.1 is complete. $\square$

6.2. Proof of Lemma 3.3. Using the same arguments as that in the proof of Proposition 3.3 in [20] one can show that $N_T \neq \{0\}$ if and only if there exists $\lambda \in \mathbb{C}$ and $\varphi_0 \in H^3(0,L)\setminus\{0\}$ such that

$$\begin{cases} \lambda\varphi_0 = -\varphi_0' - \varphi_0''', \\ \varphi_0(0) = 0,\ \varphi_0'(0) = 0,\ \varphi_0(L) + \varphi_0''(L) = 0,\ \varphi_0'(L) = 0. \end{cases} \tag{6.7}$$

Thus, it is sufficient to show that $L \in \mathbb{F}$ if and only if (6.7) has a nontrivial solution for some $\lambda \in \mathbb{C}$.

Consider the characteristic equation associated to (6.7),

$$\mu + \mu^3 + \lambda = 0,$$

and let $\mu_1,\ \mu_2$, and μ_3 be its tree roots. Then

$$\begin{cases} \mu_1 + \mu_2 + \mu_3 = 0, \\ \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 = 1, \\ \mu_1\mu_2\mu_3 = -\lambda. \end{cases} \tag{6.8}$$

If there were double roots, then we must have (up to reindexing) that

$$\mu_1 = \mu_2 = \pm\sigma, \quad \mu_3 = -2\mu_1$$

with $\sigma = i\sqrt{3}/3$. In this case, a solution of (6.7) can be written as

$$\varphi_0(x) = (c_1 + c_2 x)e^{\sigma x} + c_3 e^{-2\sigma x}.$$

It follows from $\varphi_0(0) = \varphi_0'(0) = 0$ that

$$c_3 = -c_1, \quad c_3 = -3c_1\sigma,$$

and henceforth

$$\varphi_0(x) = c_1 e^{\sigma x} \left\{(1 - 3\sigma x) - e^{-3\sigma x}\right\}.$$

As $\varphi_0'(L) = 0$, one arrives at

$$2e^{-3\sigma L} = 2 + 3\sigma L,$$

which implies that

$$1 + \frac{3}{4}L^2 = 1.$$

Consequently, one must have $L = 0$.

If all roots are simple, then a solution of (6.7) can be written as

$$\varphi_0(x) = C_1 e^{\mu_1 x} + C_2 e^{\mu_2 x} + C_3 e^{\mu_3 x},$$

where the constants C_j for $j = 1, 2, 3$ are the solutions of the system

$$\begin{pmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ (1+\mu_1^2)e^{\mu_1 L} & (1+\mu_2^2)e^{\mu_2 L} & (1+\mu_3^2)e^{\mu_3 L} \\ \mu_1 e^{\mu_1 L} & \mu_2 e^{\mu_2 L} & \mu_3 e^{\mu_3 L} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let us denote, $a = L\mu_1$, $b = L\mu_2$, and $c = L\mu_3$. Then by (6.8),

$$c = -(a + b), \qquad L^2 = -(a^2 + ab + b^2).$$

Reducing the rows of the matrix, one obtains the new one

$$M := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{c-a}{b-a} \\ 0 & 0 & A \\ 0 & 0 & B \end{pmatrix}$$

with

$$A = abe^c - bce^a - \frac{c-a}{b-a}\left(ace^b - bce^a\right), \quad B = ce^c - ae^a - \frac{c-a}{b-a}\left(be^b - ae^a\right).$$

The system has nonzero solutions if and only if

$$A = 0, \qquad B = 0,$$

or equivalently

$$\text{(6.9)} \qquad \begin{cases} (b-c)bce^a + (c-a)ace^b = (b-a)abe^c, \\ (b-c)ae^a + (c-a)be^b = (b-a)ce^c, \end{cases}$$

from which one can arrives at $c \neq 0$ and

$$\text{(6.10)} \qquad e^a = \frac{a^2}{c^2}e^c, \qquad e^b = \frac{b^2}{c^2}e^c.$$

Indeed, as

$$det \begin{pmatrix} (b-c)bc & (c-a)ac \\ (b-c)a & (c-a)b \end{pmatrix} = (c-a)(b-c)(b-a)(b+a)c \neq 0,$$

it is straightforward to verify that the system

$$\begin{cases} (b-c)bcX + (c-a)acY = (b-a)abe^c, \\ (b-c)aX + (c-a)bY = (b-a)ce^c \end{cases} \tag{6.11}$$

possess a unique solution

$$X = \frac{a^2}{c^2}e^c, \qquad Y = \frac{b^2}{c^2}e^c.$$

Therefore, the set of nonzero solutions is empty if and only if L does not belong to

$$\mathbb{F} = \left\{ L \in \mathbb{R}^+ : L^2 = -(a^2+ab+b^2) \text{ with } a, b \in \mathbb{C}^2 \text{ satisfying } \quad \frac{e^a}{a^2} = \frac{e^b}{b^2} = \frac{e^{-(a+b)}}{(a+b)^2} \right\},$$

which concludes the proof of Lemma 3.3.

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