Singular perturbation analysis of a coupled system involving the wave equation

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Abstract

This article considers a system coupling an ordinary differential equation with a wave equation through its boundary data. The existence of a small parameter in the wave equation suggests the idea of applying a singular perturbation method to get the stability of the full system by analyzing the stability of some appropriate subsystems given by the method. However, for infinite-dimensional systems it is known that in some cases this method does not work. Indeed, you can not be sure of the stability of the full system even if the given subsystems are stable. In this paper we prove that the singular perturbation method works for the system under study. Using this strategy we get the stability of the system and a Tikhonov theorem, which is the first of this kind for systems involving the wave equation. Simulations are performed to show the applicability of our results.

I. Introduction

The singular perturbation method (SPM) is a classical tool to study stability properties of coupled systems where there appear some small parameters. Roughly speaking, the idea is to deduce the stability of the original system by using the behavior of the system when those parameters are chosen to be zero. Depending on the applications and the particular equations, the parameters can play the role of different time scales allowing the modeling of different physical situations. As an example we can mention the Saint-Venant–Exner equations described in [12] and in [2, Section 1.5]. This hyperbolic system is used to study the dynamics of the flow in a reach, coupled with the sediment dynamics. The sediment dynamics has, by nature, a very slow dynamic with respect to the velocity flow in the fluid. Thus this model is a singularly perturbed hyperbolic system, as studied in [15] (see also [8] for control results on this system). Other examples of systems with different time scales appear when considering infinite-dimensional control systems with dynamics at the boundaries, as introduced for instance in [2, Section 3.4]. One naturally obtains partial differential equations (PDE) coupled to ordinary differential equations (ODE) at different time scales. In [21, Chapter 2] a slow ODE coupled with a fast PDE appears, and in [18] a fast ODE coupled with a slow PDE is studied.

As usual, the literature on singularly perturbed systems has first grown up for finite-dimensional systems (see in particular the seminal works [11], [13]). For infinite-dimensional systems we find [9], [10] where delay systems are studied. Closer to the present contribution, let us cite [3] where a parabolic singularly perturbed PDE is considered. Regarding coupled hyperbolic PDEs, we mention [19] and [20] dealing with conservation laws and balance laws, respectively. In both papers, Lyapunov function approaches are useful to analyze stability properties.

However, the validity of the SPM in an infinite-dimensional framework depends on the system. Even linear systems can show unexpected behavior. This was shown in [18] for a first order hyperbolic system with different time scales and in [4] for a second order hyperbolic system coupled to a ODE. In these papers there are examples of unstable systems for which the SPM does not work. More precisely, the SPM says that we can deduce the stability of the full system (for small parameters) when some particular subsystems (reduced and boundary layer systems) are stable. In [18], [4] we find exemples of unstable full system giving stable subsystems. Thus, we can not deduce the stability of the full system by applying a SPM.

The main goal of this paper is to establish stability and Tikhonov results for an infinite-dimensional system by applying the SPM.

We consider as a model the wave equation coupled to an ordinary differential equation through boundary data (see [22], [7], [1] for similar couplings). More precisely, our system is given by

$$\begin{cases} \varepsilon^2 w_{tt}(t,x) - w_{xx}(t,x) = 0, & t \geq 0,\, 0 < x < 1, \\ w(t,0) = cz(t), & t \geq 0, \\ w_x(t,1) = -\varepsilon d w_t(t,1), & t \geq 0, \\ \dot{z}(t) = az(t) + bw(t,1), & t \geq 0, \end{cases} \tag{1}$$

with a, b, c, d constant values and a positive value $\varepsilon > 0$. When $\varepsilon > 0$ is small then the dynamics (1) have two different time scales and couplings. We consider usual initial condition for (1) given by w^0 in $H^1(0,1)$, w^1 in $L^2(0,1)$ and z^0 in $\mathbb{R}$,

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that is

$$\begin{cases} w(0,x) = w^0(x), & 0 < x < 1, \\ w_t(0,x) = w^1(x), & 0 < x < 1, \\ z(0) = z^0. \end{cases} \tag{2}$$

It is important to note here that the stability of this system can not be deduced from the results in [18]. Even if one-dimensional wave equations can be written in terms of first-order hyperbolic equations, system (1) presents troubles with the boundary conditions. By rewriting system (1) in Riemann coordinates, we get a system of conservation laws coupled with an ODE. However, in Riemann coordinates, the boundary conditions obtained from second and four lines in (1) are different to the ones in [18]. See Appendix A for the precise writing of system (1) in Riemann coordinates. In this way, we can see that the results in [18] do not apply for (1) and consequently our developments are a true contribution with respect to the existing literature in the topic.

When applying SPM, we have to obtain what are called the reduced and the boundary layer systems. In fact, as it will be explained in Section II below, the reduced system is

$$\tfrac{d}{dt}\bar{z} = (a+bc)\bar{z}, \quad t \geq 0, \tag{3}$$

while the boundary layer system for $\tau = t/\varepsilon$ is

$$\begin{cases} \bar{w}_{\tau\tau}(\tau,x) - \bar{w}_{xx}(\tau,x) = 0, & \tau \geq 0,\ 0 < x < 1, \\ \bar{w}(\tau,0) = 0, & \tau \geq 0, \\ \bar{w}_x(\tau,1) = -d\bar{w}_\tau(\tau,1), & \tau \geq 0. \end{cases} \tag{4}$$

We consider usual initial condition for (3) and (4) given by $\bar{w}^0$ in $H^1(0,1)$, $\bar{w}^1$ in $L^2(0,1)$ and $\bar{z}^0$ in $\mathbb{R}$, that is

$$\begin{cases} \bar{w}(0,x) = \bar{w}^0(x), & 0 < x < 1, \\ \bar{w}_t(0,x) = \bar{w}^1(x), & 0 < x < 1 \end{cases} \tag{5}$$

and

$$\bar{z}(0) = \bar{z}^0. \tag{6}$$

Let us notice that the stability of the reduced system (3) is equivalent to $(a+bc) < 0$ and the stability of the boundary layer (4) is equivalent to $d > 0$. Thus, our first result is concern with the stability of system (1) when the subsystems are stable and ε is small enough. It is worth to mention that Theorem 1 below appeared in the conference paper [4] but we give here a different proof. The used approach allows to unify the stability analysis in Theorem 1 with the Tikhonov result in Theorem 2.

Theorem 1: *Let $d > 0$ and a, b, c such that $a + bc < 0$. There exists $\varepsilon^\star > 0$ such that for any $\varepsilon \in (0, \varepsilon^\star)$ the full system (1) is exponentially stable, that is, there exists $C_0 > 0$ such that, for all (w^0, w^1, z^0) in $H^1(0,1) \times L^2(0,1) \times \mathbb{R}$ satisfying the compatibility condition $w^0(0) = cz^0$, the solution $z \in C([0,+\infty))$ and $w \in C^0([0,+\infty); H^1(0,1)) \cap C^1([0,+\infty); L^2(0,1))$ to (1)-(2) satisfies, for all $t \geq 0$,*

$$\|(w(t), w_t(t), z(t))\|_{H^1(0,1)\times L^2(0,1)\times\mathbb{R}} \leq C_0 e^{\frac{(a+bc)}{2}t} \|(w^0, w^1, z^0)\|_{H^1(0,1)\times L^2(0,1)\times\mathbb{R}}.$$

The second result of this paper states that the SPM gives us a Tikhonov approximation: the dynamics of (1) can be approximated by those of the boundary layer system (4) and of the reduced system (3).

Theorem 2: *Let $d > 0$ and a, b, c such that $\delta := a + bc + \sqrt{3}|bc| < 0$. There exists $\varepsilon^\star > 0$ such that for any $\varepsilon \in (0, \varepsilon^\star)$, w^0 in $H^1(0,1)$, w^1 in $L^2(0,1)$, z^0 in $\mathbb{R}$, $\bar{w}^0$ in $H^1(0,1)$, $\bar{w}^1$ in $L^2(0,1)$, $\bar{z}^0$ in $\mathbb{R}$ satisfying the compatibility conditions $w^0(0) = cz^0$, $\bar{w}^0(0) = 0$, and smallness conditions*

$$\|w^0 - c\bar{z}^0 - \bar{w}^0\|_{H^1(0,1)} + \|w^1 - (a+bc)c\bar{z}^0 - \bar{w}^1\|_{L^2(0,1)} + |z^0 - \bar{z}^0| = O(\varepsilon^2)$$

$$\|\bar{w}^0\|_{H^1(0,1)} + \|\bar{w}^1\|_{L^2(0,1)} = O(\varepsilon^{3/2}), \quad |\bar{z}^0| = O(\varepsilon^{3/2}),$$

the solution w in $C([0,\infty); H^1(0,1)) \cap C^1([0,\infty); L^2(0,1))$ and z in $C^1([0,\infty))$ to (1)-(2) satisfies, for all $t \geq 0$,

$$\|w(t) - c\bar{z}(t) - \bar{w}(t/\varepsilon)\|_{H^1(0,1)} + \|w_t(t) - c(a+bc)\bar{z}(t) - \bar{w}_\tau(t/\varepsilon)\|_{L^2(0,1)} = e^{\delta t} O(\varepsilon), \tag{7}$$

and

$$|z(t) - \bar{z}(t)| = e^{\delta t} O(\varepsilon^{3/2}), \tag{8}$$

where $\bar{w}$ in $C([0,\infty); H^1(0,1)) \cap C^1([0,\infty); L^2(0,1))$ is the solution to (4)-(5) and $\bar{z}$ in $C^1([0,\infty))$ is the solution to (3)-(6).

Note that the assumptions on coefficients a, b, c are more restrictive in Theorem 2 than in Theorem 1 since $\delta > a + bc$. Thus, the exponential decay rate in Theorem 2 is weaker than the one in Theorem 1. More importantly, we note that regarding the solution w there is a $O(\varepsilon^2)$ in the initial condition hypothesis and a $O(\varepsilon)$ in the conclusions of Theorem 2. This is due to the fact that our Lyapunov function depends on ε. See the proof of Theorem 2 for more details.

The remaining part of the paper is organized as follows. In Section II we prove Theorem 1. Section III is devoted to the proof of Theorem 2. Section IV contains numerical simulations illustrating the stability and the Tikhonov approximation stated in our theorems. Finally, we give in Section V some conclusions. Appendix A shows as our system is written in Riemann coordinates while Appendix B is concerned with an important technical lemma.

II. Stability analysis

The goal of this section is to prove Theorem 1. We apply the SPM, which leads us to find some appropriate subsystems called the reduced order system and the boundary layer system. Theorem 1 makes sure that the full system is stable when the previous subsystems are stable and the parameter ε is small enough.

To formally compute the reduced order system, let $\varepsilon = 0$ in (1). We get from the boundary condition at $x = 1$ that $w_x(t, 1) = 0$ which gives $w_x = 0$ when using $w_{xx} = 0$ (coming from the PDE). From the boundary condition at $x = 0$, it follows that $w(t, x) = cz(t)$ for all $t \geq 0$ and for all $x \in (0, 1)$. Thus, the reduced order system is

$$\tfrac{d}{dt}\bar{z} = (a + bc)\bar{z}, \quad t \geq 0. \tag{9}$$

Let us compute now the boundary layer system. We introduce $\tau = t/\varepsilon$ and the new variable $\bar{w}(\tau, x) = w(\tau, x) - cz(\tau)$. We compute $\frac{d}{d\tau}\bar{w} = \frac{d}{d\tau}w - c\varepsilon\frac{d}{dt}z = \frac{d}{d\tau}w$ by letting $\varepsilon = 0$ and by using the z dynamics. Moreover, $\frac{d^2}{d\tau^2}\bar{w} = \frac{d^2}{d\tau^2}w$, $\frac{d}{dx}\bar{w} = \frac{d}{dx}w$, and $\frac{d^2}{dx^2}\bar{w} = \frac{d^2}{dx^2}w$. Therefore $\bar{w}_{\tau\tau} - \bar{w}_{xx} = 0$. To compute the boundary conditions for the variable $\bar{w}$, let us note that $\bar{w}_x(\tau, 1) = w_x(\tau, 1) = -d\varepsilon w_t(\tau, 1) = -dw_\tau(\tau, 1) = -d\bar{w}_\tau(\tau, 1)$ by approximating ε by 0 in the last equation. To sum up, the boundary layer system is written as

$$\begin{cases} \bar{w}_{\tau\tau}(\tau, x) - \bar{w}_{xx}(\tau, x) = 0, & \tau \geq 0,\ 0 < x < 1, \\ \bar{w}(\tau, 0) = 0, & \tau \geq 0, \\ \bar{w}_x(\tau, 1) = -d\bar{w}_\tau(\tau, 1), & \tau \geq 0. \end{cases} \tag{10}$$

The boundary layer system is known to be exponentially stable. In fact, this system is called a passive damped wave equation. As the following computation is used a couple of times later, we explain it here. Let us consider the following Lyapunov function

$$V_1(\bar{w}) = \int_0^1 e^{\mu x}(\bar{w}_x + \bar{w}_\tau)^2 dx + \int_0^1 e^{-\mu x}(\bar{w}_x - \bar{w}_\tau)^2 dx, \tag{11}$$

with $\mu > 0$ to be fixed later. This Lyapunov function appeared for the wave equation in [17] and is related to a Lyapunov function for first-order hyperbolic equations studied in [6].

Along the solutions to (10), it holds

$$\begin{aligned} \tfrac{d}{d\tau}V_1 &= 2\int_0^1 e^{\mu x}(\bar{w}_\tau + \bar{w}_x)(\bar{w}_{\tau\tau} + \bar{w}_{x\tau})dx \\ &\quad +2\int_0^1 e^{-\mu x}(\bar{w}_\tau - \bar{w}_x)(\bar{w}_{\tau\tau} - \bar{w}_{x\tau})dx\ , \\ &= 2\int_0^1 e^{\mu x}(\bar{w}_\tau + \bar{w}_x)(\bar{w}_{xx} + \bar{w}_{x\tau})dx \\ &\quad -2\int_0^1 e^{-\mu x}(\bar{w}_\tau - \bar{w}_x)(\bar{w}_{x\tau} - \bar{w}_{xx})dx\ , \\ &= -\mu\int_0^1 e^{\mu x}(\bar{w}_\tau + \bar{w}_x)^2 dx \\ &\quad +[e^{\mu x}(\bar{w}_\tau + \bar{w}_x)^2]_{x=0}^{x=1} \\ &\quad -\mu\int_0^1 e^{-\mu x}(\bar{w}_\tau - \bar{w}_x)^2 dx \\ &\quad -[e^{-\mu x}(\bar{w}_\tau - \bar{w}_x)^2]_{x=0}^{x=1}\ . \end{aligned}$$

Now, note that the boundary condition in the second line of (10) implies that $\bar{w}_\tau(\tau, 0) = 0$ and thus, for all $\tau \geq 0$,

$$\begin{aligned} &[e^{\mu x}(\bar{w}_\tau + \bar{w}_x)^2](\tau, 0) - [e^{-\mu x}(\bar{w}_\tau - \bar{w}_x)^2](\tau, 0) \\ &\qquad = \bar{w}_x^2(\tau, 0) - \bar{w}_x^2(\tau, 0) = 0. \end{aligned}$$

Therefore, we get

$$\frac{d}{d\tau}V_1 = -\mu V_1 + e^{\mu}(\bar{w}_\tau(\tau, 1) + \bar{w}_x(\tau, 1))^2 - e^{-\mu}(\bar{w}_\tau(\tau, 1) - \bar{w}_x(\tau, 1))^2\ ,$$

and thus with the boundary condition in the last line of (10):

$$\begin{aligned} \frac{d}{d\tau}V_1 &= -\mu V_1 + e^{\mu}(\bar{w}_\tau(\tau, 1) - d\bar{w}_\tau(\tau, 1))^2 \\ &\quad -e^{-\mu}(\bar{w}_\tau(\tau, 1) + d\bar{w}_\tau(\tau, 1))^2\ , \\ &= -\mu V_1 + \Big(e^{\mu}(1 - d)^2 - e^{-\mu}(1 + d)^2\Big)\bar{w}_\tau(\tau, 1)^2\ . \end{aligned} \tag{12}$$

We obtain the exponential stability by choosing μ such that $e^{\mu}(1-d)^2 < e^{-\mu}(1+d)^2$, which is possible due to $d > 0$.

Let us now define the following variable $\tilde{w} = w - cz$. We compute successively

$$\begin{array}{rcl} w_t &=& \tilde{w}_t + (a+bc)cz + bc\tilde{w}(t,1) \ , \\ w_{tt} &=& \tilde{w}_{tt} + (abc + b^2c^2)\tilde{w}(t,1) \\ && +bc\tilde{w}_t(t,1) + (a^2c + 2abc^2 + b^2c^3)z \ , \\ \tilde{w}_x &=& w_x \ , \\ \tilde{w}_{xx} &=& w_{xx}. \end{array}$$

Therefore we get the following dynamics, equivalent to (1)

$$\begin{cases} \varepsilon^2\tilde{w}_{tt} - \tilde{w}_{xx} + \varepsilon^2(abc + b^2c^2)\tilde{w}(t,1) \\ \qquad +\varepsilon^2 bc\tilde{w}_t(t,1) + \varepsilon^2(a^2c + 2abc^2 + b^2c^3)z(t) = 0, \\ \tilde{w}(t,0) = 0, \\ \tilde{w}_x(t,1) = \varepsilon bcd\tilde{w}(t,1) - d\varepsilon\tilde{w}_t(t,1) - \varepsilon d(a+bc)cz(t), \\ \dot{z}(t) = (a+bc)z(t) + b\tilde{w}(t,1). \end{cases} \tag{13}$$

We are now in position to prove Theorem 1 by studying system (13).

Proof: In order to prove that system (13) is exponentially stable, we apply Lemma 1 in Appendix B. More precisely, we use (23) with $A = -(a^2c + 2abc^2 + b^2c^3)$, $B = -(abc + b^2c^2)$, $C = -bc$, $D = -d$, $E = -d(a+bc)c$, $F = bcd$, $G = a + bc$, $H = b$, $M = 0$, $d_1(t) = d_2(t) = d_3(t) = 0$, for all $t \geq 0$. Thus, defining $V(\tilde{w}, z) = V_1(\tilde{w}) + V_2(z)$, with

$$V_1 = \int_0^1 e^{\mu x}(\tilde{w}_x + \varepsilon\tilde{w}_t)^2 dx + \int_0^1 e^{-\mu x}(\tilde{w}_x - \varepsilon\tilde{w}_t)^2 dx$$

and $V_2 = z^2$, we get that, along the solutions to (13),

$$\begin{aligned} \frac{d}{dt}V \leq &\Big[-\frac{\mu}{\varepsilon} + e^{\mu}(\kappa_1 + \kappa_2 + \kappa_3) + \frac{2\varepsilon^2 B^2 C_1}{\kappa_2} + 3\varepsilon C_1 F^2(e^{\mu} + e^{-\mu})(1 + \frac{|D|+1}{\kappa_5}) + \frac{C_1 H^2}{\kappa_6}\Big]V_1 \\ &+ \Big[2G + \kappa_6 + \kappa_7 + \frac{2\varepsilon^2 A^2}{\kappa_1} + 3\varepsilon E^2(e^{\mu} + e^{-\mu})(1 + \frac{|D|+1}{\kappa_5})\Big]V_2 \\ &+ \Big[\varepsilon e^{\mu}(D+1)^2 - \varepsilon e^{-\mu}(D-1)^2 + 3\varepsilon\kappa_5(e^{\mu} + e^{-\mu})(|D|+1) + \frac{2\varepsilon^2 C^2}{\kappa_3}\Big]\beta_t(t,1)^2 \ . \end{aligned}$$

for all positive values κ_i, $i = 1, \ldots, 7$.

Note that, under the assumptions of Theorem 1, $D < 0$ and consequently $-(D-1)^2 + (D+1)^2 < 0$. Now letting

$$\kappa_5 = \frac{1}{12}\frac{|(D+1)^2 - (D-1)^2|}{(|D|+1)} \tag{14}$$

we get the existence of $\mu^\star$ such that, for all μ in $(0, \mu^\star]$,

$$e^{\mu}(D+1)^2 - e^{-\mu}(D-1)^2 + 3\kappa_5(e^{\mu} + e^{-\mu})(|D|+1) < 0.$$

Then for any positive κ_3 we get the existence of $\varepsilon^\star$ such that, for all ε in $(0, \varepsilon^\star)$,

$$\varepsilon e^{\mu}(D+1)^2 - \varepsilon e^{-\mu}(D-1)^2 + 3\varepsilon\kappa_5(e^{\mu} + e^{-\mu})(|D|+1) + \frac{2\varepsilon^2 C^2}{\kappa_3} < 0.$$

In this way, we obtain, along the solutions to (13),

$$\begin{aligned} \frac{d}{dt}V \leq &\Big[-\frac{\mu}{\varepsilon} + e^{\mu}(\kappa_1 + \kappa_2 + \kappa_3 + 3\kappa_4) + \frac{2\varepsilon^2 B^2 C_1}{\kappa_2} + 3\varepsilon C_1 F^2(e^{\mu} + e^{-\mu})(1 + \frac{|D|+1}{\kappa_5}) + \frac{C_1 H^2}{\kappa_6}\Big]V_1 \\ &+ \Big[2G + \kappa_6 + \kappa_7 + \frac{2\varepsilon^2 A^2}{\kappa_1} + 3\varepsilon E^2(e^{\mu} + e^{-\mu})(1 + \frac{|D|+1}{\kappa_5})\Big]V_2 \ . \end{aligned} \tag{15}$$

We make negative the factor multiplying V_2 in (15). As $G < 0$, $A = O(1)$, $D = O(1)$, $E = O(1)$, for any positive value κ_1, there exist sufficiently small values $\kappa_6 > 0$, $\kappa_7 > 0$ and $\varepsilon^\star$, such that for all ε in $(0, \varepsilon^\star)$ (up to reducing $\varepsilon^\star$) and for all μ in $(0, \mu^\star)$, it holds

$$2G + \kappa_6 + \kappa_7 + \frac{2\varepsilon^2 A^2}{\kappa_1} + 3\varepsilon E^2(e^{\mu} + e^{-\mu})(1 + \frac{|D|+1}{\kappa_5}) < \frac{3}{2}G < 0.$$

Remark 1: Of course, in previous line the factor $\frac{3}{2}$ multiplying G can be changed to any factor in the interval $(0, 2)$. We can get a factor as near to 2 as we want. $\circ$

Thanks to the term $-\mu/\varepsilon$ we can make negative the factor multiplying V_1 in (15). As $B = O(1)$, $D = O(1)$, $F = O(1)$, and $H = O(1)$, for any positive value $\kappa_2 > 0$ we have that for all ε in $(0, \varepsilon^\star)$ (up to reducing $\varepsilon^\star$) and for all μ in $(0, \mu^\star)$, it holds

$$-\frac{\mu}{\varepsilon} + e^{\mu}(\kappa_1 + \kappa_2 + \kappa_3 + 3\kappa_4) + \frac{2\varepsilon^2 B^2 C_1}{\kappa_2} + 3\varepsilon C_1 F^2 (e^{\mu} + e^{-\mu})(1 + \frac{|D| + 1}{\kappa_5}) + \frac{C_1 H^2}{\kappa_6} < -\frac{\mu}{2\varepsilon} \ . \tag{16}$$

We arrive in this way to

$$\frac{d}{dt}V(t) \leq -\frac{\mu}{2\varepsilon}V_1(t) + \frac{3}{2}GV_2(t) \ \leq -\min\left\{\frac{\mu}{2\varepsilon}, \frac{3}{2}G\right\} V(t) \ \leq -\frac{3}{2}GV(t) \ ,$$

for all $\varepsilon \in (0, \varepsilon^\star)$ with a sufficiently small positive value $\varepsilon^\star$.

Therefore, the function V decreases to zero exponentially fast, along the solutions to (13). Note that this exponential decreasing for the Lyapunov function $V(\tilde{w}, z)$ is equivalent, up to a factor ε^2, to the exponential decreasing of the usual norm in $H^1(0,1) \times \ L^2(0,1) \times \mathbb{R}$ thanks to the fact that $\tilde{w}(t, 0) = 0$.

This concludes the proof of Theorem 1. ■

III. Tikhonov theorem

Until now we have seen that the SPM gives us the reduced order system and the boundary layer, whose stability imply the stability of the full system. The goal of this section is to prove Theorem 2, that uses the previous subsystems to give a better approximation of the full system. The idea is not only saying that the system goes to zero but trying to explain how it does when the parameter ε is small enough.

Let us introduce, for all $t \geq 0$ and x in $[0, 1]$,

$$\alpha(t) = z(t) - \bar{z}(t)$$

and

$$\beta(t, x) = w(t, x) - c\bar{z}(t) - \bar{w}(\frac{t}{\varepsilon}, x).$$

From (1) and (9), we get

$$\begin{aligned} \dot{\alpha}(t) &= az(t) + bw(t, 1) - (a + bc)\bar{z}(t) \\ &= (a + bc)\alpha(t) - bcz(t) + bw(t, 1) \\ &= (a + bc)\alpha(t) + b\beta(t, 1) + b\bar{w}(\tfrac{t}{\varepsilon}, 1). \end{aligned}$$

Moreover, we compute successively

$$\beta_t(t, x) = w_t(t, x) - c(a + bc)\bar{z} - \frac{1}{\varepsilon}\bar{w}_\tau(\frac{t}{\varepsilon}, x) \ ,$$

$$\beta_{tt}(t, x) = w_{tt}(t, x) - c(a + bc)^2\bar{z} - \frac{1}{\varepsilon^2}\bar{w}_{\tau\tau}(\frac{t}{\varepsilon}, x) \ ,$$

$$\beta_x(t, x) = w_x(t, x) - \bar{w}_x(\frac{t}{\varepsilon}, x) \ ,$$

and

$$\beta_{xx}(t, x) = w_{xx}(t, x) - \bar{w}_{xx}(\frac{t}{\varepsilon}, x) \ .$$

Therefore, from (1) and (10) we get

$$\left\{ \begin{array}{l} \varepsilon^2\beta_{tt} - \beta_{xx} = -\varepsilon^2 c(a + bc)^2\bar{z}, \\ \beta(t, 0) = c(z(t) - \bar{z}(t)), \\ \beta_x(t, 1) = -d\varepsilon w_t(t, 1) + d\bar{w}_\tau(\frac{t}{\varepsilon}, 1). \end{array} \right. \tag{17}$$

The last boundary condition of (17) is

$$\beta_x(t, 1) = -d\varepsilon\beta_t(t, 1) - d\varepsilon c(a + bc)\bar{z}(t)$$

where the expression of β_t has been used. To sum up the dynamics of α and β can be rewritten as

$$\left\{ \begin{array}{rcl} \varepsilon^2\beta_{tt} - \beta_{xx} &=& -\varepsilon^2 c(a + bc)^2\bar{z}(t) \\ \beta(t, 0) &=& c\alpha(t) \\ \beta_x(t, 1) &=& -d\varepsilon\beta_t(t, 1) - d\varepsilon c(a + bc)\bar{z}(t) \\ \dot{\alpha}(t) &=& (a + bc)\alpha(t) + b\beta(t, 1) + b\bar{w}(\frac{t}{\varepsilon}, 1). \end{array} \right. \tag{18}$$

We are now in position to prove Theorem 2 by studying stability of system (18).

Proof: Let us apply Lemma 1 with $A = B = C = E = F = 0$, $D = -d$, $G = a + bc$, $H = b$, $M = c$,

$$d_1(t) = -c(a + bc)^2 \bar{z}(t),$$

$$d_2(t) = -cd(a + bc)\bar{z}(t),$$

and

$$d_3(t) = b\bar{w}(\frac{t}{\varepsilon}, 1).$$

Therefore (24) holds along the solutions to (18) where V is defined by $V(\beta, \alpha) = V_1(\beta) + V_2(\alpha)$ with

$$V_1(\beta) = \int_0^1 e^{\mu x}(\beta_x + \varepsilon\beta_t)^2 dx + \int_0^1 e^{-\mu x}(\beta_x - \varepsilon\beta_t)^2 dx$$

and $V_2(\alpha) = \alpha^2$.

We select κ_i, $i = 1, \dots, 5$ and κ_7 by adapting the proof of Theorem 1, and we get

$$\begin{aligned} \frac{d}{dt}V &\leq (2G + \kappa_6 + 3M^2\frac{H^2}{\kappa_6})V + \frac{2\varepsilon^2}{\kappa_4}d_1(t)^2 + 3\varepsilon(e^{\mu} + e^{-\mu})(1 + \frac{|D| + 1}{\kappa_5})d_2(t)^2 + \Big(6M^2\varepsilon + \frac{1}{\kappa_7}\Big)d_3(t)^2 \\ &\leq (2G + 2\sqrt{3}|MH|)V + \frac{2\varepsilon^2}{\kappa_4}d_1(t)^2 + 3\varepsilon(e^{\mu} + e^{-\mu})(1 + \frac{|D| + 1}{\kappa_5})d_2(t)^2 + \Big(6M^2\varepsilon + \frac{1}{\kappa_7}\Big)d_3(t)^2, \end{aligned} \tag{19}$$

with $\kappa_6 = \sqrt{3}|MH|$ (minimizing $\kappa_6 + 3M^2\frac{H^2}{\kappa_6}$). Moreover, due to (9), it holds $|\bar{z}(t)| \leq e^{Gt}|\bar{z}(0)|$ and we note that there exists a constant value $C_2 > 0$ such that

$$d_1(t)^2 + d_2(t)^2 \leq C_2 e^{2Gt}\bar{z}(0)^2. \tag{20}$$

Concerning $d_3(t)$, from (12) we can easily see that there exists $C_3 > 0$ such that, for all $t \geq 0$

$$d_3(t)^2 \leq C_3 e^{-\frac{\mu t}{\varepsilon}}(\|\bar{w}^0\|_{H^1(0,1)} + \|\bar{w}^1\|_{L^2(0,1)})^2. \tag{21}$$

Inspecting the choices of κ_4 and κ_5 done in (16) and (14) respectively, we have that $\kappa_4 = O(1)$, $\kappa_5 = O(1)$ and $\kappa_7 = O(1)$. Note that $2\delta = 2G + 2\sqrt{3}|MH|$ where δ in defined in the statement of Theorem 2. We first bound (19) using (20)-(21) to get

$$\frac{d}{dt}V \leq 2\delta V + O(\varepsilon^2)e^{2Gt}\bar{z}(0)^2 + O(\varepsilon)e^{2Gt}\bar{z}(0)^2 + O(1)e^{-\frac{\mu t}{\varepsilon}}(\|\bar{w}^0\|_{H^1(0,1)} + \|\bar{w}^1\|_{L^2(0,1)})^2,$$

and then we integrate between 0 and t to obtain (with $\delta > G$)

$$\begin{aligned} V &\leq e^{2\delta t}(V(0) + \bar{z}(0)^2 + (\|\bar{w}^0\|_{H^1(0,1)} + \|\bar{w}^1\|_{L^2(0,1)})^2), \\ &\leq e^{2\delta t}O(\varepsilon^3) \end{aligned}$$

where we used the hypothesis on the initial conditions. We have now to come back to the norm. To do that we use that there exists a positive constant C_4, not depending on ε, such that

$$\varepsilon^2 C_4\Big(\|f\|_{H^1(0,1)} + \|g\|_{L^2(0,1)}\Big)^2 \leq \int_0^1 e^{\mu x}(f_x + \varepsilon g)^2\, dx + \int_0^1 e^{-\mu x}(f_x - \varepsilon g)^2\, dx\ .$$

From here we deduce Theorem 2. ■

IV. Numerical simulations

In this section, we illustrate Theorems 1 and 2, by some numerical simulations. We apply a Lax-Friedrichs method [16] to get the numerical solutions. The codes are available on [5].

Concerning Theorem 1, we simulate system (1) with $a = -2$, $b = 1$, $c = -2$, and $d = 0.5$ and the initial conditions $w^0(x) = 2\pi\sin(2\pi x)$, $w^1(x) = 2$, for all x in $(0, 1)$ and $z^0 = w^0(1)/c$, so that the assumptions of Theorem 1 hold. Pick $\varepsilon = 0.1$ for the time scale. We can check on Figure 1 (Left) that the Lyapunov function V (with $\mu = 0$) in the proof of Theorem 1 decreases exponentially fast to zero. We also see that the norm of the solution goes to zero. This is consistent with the conclusions of Theorem 1 giving the exponential stability of the full coupled system.

To illustrate Theorem 2, we also compute the numerical solutions to the reduced system (3), and to the boundary layer system (4) with the initial conditions $\bar{z}^0 = z^0$, $\bar{w}^0 = w^0 - c\bar{z}^0$, and $\bar{w}^1(x) = w^1 - c(a+bc)\bar{z}^0$, for all x in $(0, 1)$. We show the norms of the Tikhonov approximations as given in (7). We compare a first approximation of w given by the reduced system $c\bar{z}(t)$ with a second approximation given by the sum of the reduced system and the boundary layer $c\bar{z}(t) + \bar{w}(t/\varepsilon, x))$. We check on Figure 1 (Right) that the second approximation is better than the first one, specially for small times, which is natural because the contribution of the boundary layer $\bar{w}(t/\varepsilon, x)$ is relevant for small times. Thus, the interest of the Tikhonov approximation is confirmed as stated in Theorem 2.

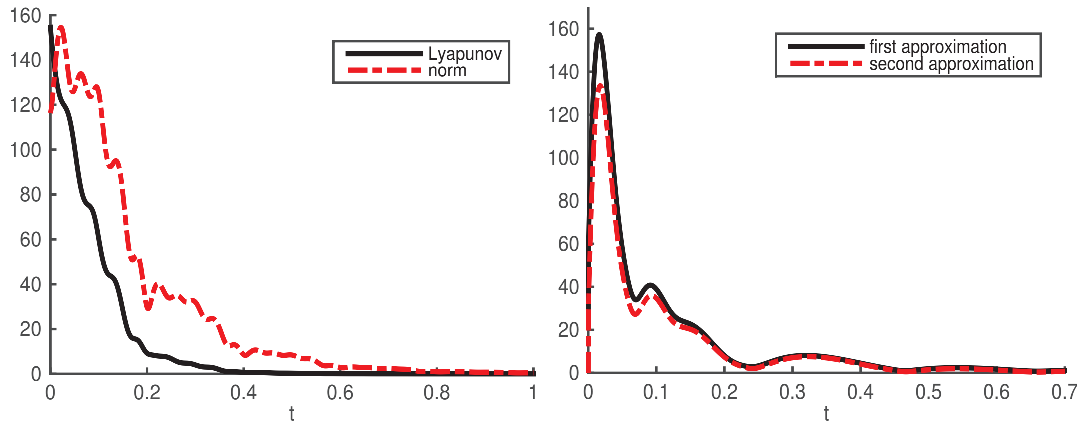

Fig. 1. (Left) Time evolution of the Lyapunov function V and the norm along the solution to (1). (Right) Time evolution of the norm of Tikhonov approximations

V. Conclusions

In this paper we have considered an infinite-dimensional system coupling a wave equation with a linear ODE. Some small parameters appear in the partial differential equation, which can be interpreted as having different time scales. Having in mind what is known for finite-dimensional systems, it is natural to apply the singular perturbation method. However, this does not always work in an infinite-dimensional framework as shown in recent literature. In this context, we prove here that this kind of analysis can be performed for the system under study and we obtain stability and Tikhonov results. Our main tool is the use of Lyapunov functions. Simulations illustrating our results are presented.

Interesting open problems arise. For instance, performing similar analysis for other infinite-dimensional systems, maybe involving nonlinearities. Other possible research line is to relax the assumptions on the initial conditions in Theorem 2 with respect to their dependence on ε. This would ask for looking for a Lyapunov function not depending on ε.

Appendix

A. Rewriting system (1) in Riemann coordinates

Denote $v_1(t,x) = \varepsilon w_t(t,x) + w_x(t,x)$, $v_2(t,x) = w_x(t,x) - \varepsilon w_t(t,x)$, and $Z(t) = \dot{z}(t)$. By differentiating the second and the fourth line of (1), we obtain from (1) the following system, for all $t \geq 0$, and for all $0 < x < 1$,

$$\left\{ \begin{array}{l} \varepsilon v_{1t}(t,x) - v_{1x}(t,x) = 0 \ , \ \varepsilon v_{2t}(t,x) + v_{2x}(t,x) = 0, \\ v_1(t,0) - v_2(t,0) = 2\varepsilon c Z(t), \\ v_1(t,1) + v_2(t,1) = -d v_1(t,1) + d v_2(t,1), \\ \dot{Z}(t) = aZ(t) + \frac{b}{2\varepsilon} v_1(t,1) - \frac{b}{2\varepsilon} v_2(t,1). \end{array} \right. \tag{22}$$

System (22) is a system of fast conservation laws coupled with an ODE. However, due to the presence of ε in the denominator of the last line of (22) and of the presence of ε in the second line, it differs from the class of coupled systems of conservation laws and ODE studied in [18]. Therefore the results in [18] do not apply to (22), as claimed in the Introduction.

B. A technical lemma and its proof

In this section, we prove a technical lemma that is instrumental for the proof of Theorems 1 and 2.

Lemma 1: *Let ε, A, B, C, D, E, F, G, H, M be constant values, and d_1, d_2, d_3 be functions in $C([0,\infty))$. Let us consider the system*

$$\left\{ \begin{array}{l} \varepsilon^2 \beta_{tt} - \beta_{xx} = \varepsilon^2 A\alpha(t) + \varepsilon^2 B\beta(t,1) + \varepsilon^2 C\beta_t(t,1) + \varepsilon^2 d_1(t), \\ \beta(t,0) = M\alpha(t), \\ \beta_x(t,1) = \varepsilon D\beta_t(t,1) + \varepsilon E\alpha(t) + \varepsilon F\beta(t,1) + \varepsilon d_2(t), \\ \dot{\alpha}(t) = G\alpha(t) + H\beta(t,1) + d_3(t), \end{array} \right. \tag{23}$$

and the Lyapunov function candidate $V(\beta,\alpha) = V_1(\beta) + V_2(\alpha)$, where $V_2(\alpha) = \alpha^2$ and

$$V_1(\beta) = \int_0^1 e^{\mu x}(\beta_x + \varepsilon\beta_t)^2 dx + \int_0^1 e^{-\mu x}(\beta_x - \varepsilon\beta_t)^2 dx.$$

Then, there exists $C_1 > 0$ such that, for any positive values κ_i, $i = 1, \dots, 7$, we have along all solutions to (23)

$$\begin{aligned}\frac{d}{dt}V(\beta,\alpha) \leq & \Big[-\frac{\mu}{\varepsilon}+e^{\mu}(\kappa_1+\kappa_2+\kappa_3+\kappa_4)+\frac{2\varepsilon^2B^2C_1}{\kappa_2}+3\varepsilon C_1F^2(e^{\mu}+e^{-\mu})(1+\frac{|D|+1}{\kappa_5})+6M^2C_1\varepsilon H^2+\frac{C_1H^2}{\kappa_6}\Big]V_1(\beta)\\ &+\Big[2G+\kappa_6+\kappa_7+3M^2\frac{H^2}{\kappa_6}+\frac{2\varepsilon^2A^2}{\kappa_1}+3\varepsilon E^2(e^{\mu}+e^{-\mu})(1+\frac{|D|+1}{\kappa_5})+6M^2\varepsilon C_1G^2\\ &+\Big(\frac{2\varepsilon^2B^2}{\kappa_2}+3\varepsilon F^2(e^{\mu}+e^{-\mu})(1+\frac{|D|+1}{\kappa_5})+6M^2\varepsilon H^2\Big)3M^2\Big]V_2(\alpha)\\ &+\frac{2\varepsilon^2}{\kappa_4}d_1(t)^2+3\varepsilon(e^{\mu}+e^{-\mu})(1+\frac{|D|+1}{\kappa_5})d_2(t)^2+(6M^2\varepsilon+\frac{1}{\kappa_7})d_3(t)^2\\ &+\Big[\varepsilon e^{\mu}(D+1)^2-\varepsilon e^{-\mu}(D-1)^2+3\varepsilon\kappa_5(e^{\mu}+e^{-\mu})(|D|+1)+\frac{2\varepsilon^2C^2}{\kappa_3}\Big]\beta_t(t,1)^2 \ . \end{aligned} \tag{24}$$

Proof: Along the solutions to (23), we compute

$$\begin{aligned}\tfrac{d}{dt}V_1(\beta) &= 2\int_0^1 e^{\mu x}(\beta_{xt}+\varepsilon\beta_{tt})(\beta_x+\varepsilon\beta_t)dx\\ &\quad +2\int_0^1 e^{-\mu x}(\beta_{xt}-\varepsilon\beta_{tt})(\beta_x-\varepsilon\beta_t)dx\\ &= \tfrac{2}{\varepsilon}\int_0^1 e^{\mu x}(\varepsilon\beta_{xt}+\beta_{xx})(\beta_x+\varepsilon\beta_t)dx\\ &\quad -\tfrac{2}{\varepsilon}\int_0^1 e^{-\mu x}(-\varepsilon\beta_{xt}+\beta_{xx})(\beta_x-\varepsilon\beta_t)dx\\ &\quad +2(\varepsilon A\alpha(t)+\varepsilon B\beta(t,1)+\varepsilon C\beta_t(t,1)+\varepsilon d_1(t))\int_0^1 e^{\mu x}(\beta_x+\varepsilon\beta_t)dx\\ &\quad -2(\varepsilon A\alpha(t)+\varepsilon B\beta(t,1)+\varepsilon C\beta_t(t,1)+\varepsilon d_1(t))\int_0^1 e^{-\mu x}(\beta_x-\varepsilon\beta_t)dx\end{aligned}$$

and thus, using integrations by parts,

$$\begin{aligned}\tfrac{d}{dt}V_1(\beta) &= -\tfrac{\mu}{\varepsilon}\int_0^1 e^{\mu x}(\beta_x+\varepsilon\beta_t)^2dx\\ &\quad -\tfrac{\mu}{\varepsilon}\int_0^1 e^{-\mu x}(\beta_x-\varepsilon\beta_t)^2dx\\ &\quad +\tfrac{1}{\varepsilon}\left[e^{\mu x}(\beta_x+\varepsilon\beta_t)^2-e^{-\mu x}(\beta_x-\varepsilon\beta_t)^2\right]_0^1\\ &\quad +2(\varepsilon A\alpha(t)+\varepsilon B\beta(t,1)+\varepsilon C\beta_t(t,1)+\varepsilon d_1(t))\int_0^1 e^{\mu x}(\beta_x+\varepsilon\beta_t)dx\\ &\quad -2(\varepsilon A\alpha(t)+\varepsilon B\beta(t,1)+\varepsilon C\beta_t(t,1)+\varepsilon d_1(t))\int_0^1 e^{-\mu x}(\beta_x-\varepsilon\beta_t)dx \ .\end{aligned}$$

Now using the inequalities $2ef \leq \frac{e^2}{k} + kf^2$, $(e+f+g)^2 \leq 3(e^2+f^2+g^2)$ (for any values e, f, and g and any positive value k), and (23), we get

$$\begin{aligned}\tfrac{d}{dt}V_1(\beta) &\leq -\tfrac{\mu}{\varepsilon}V_1(\beta)+\{[\int_0^1 e^{\mu x}(\beta_x+\varepsilon\beta_t)dx]^2+[\int_0^1 e^{-\mu x}(\beta_x-\varepsilon\beta_t)dx]^2\}(\kappa_1+\kappa_2+\kappa_3+\kappa_4)\\ &\quad +\tfrac{2\varepsilon^2A^2}{\kappa_1}\alpha(t)^2+\tfrac{2\varepsilon^2B^2}{\kappa_2}\beta(t,1)^2+\tfrac{2\varepsilon^2C^2}{\kappa_3}\beta_t(t,1)^2+\tfrac{2\varepsilon^2}{\kappa_4}d_1(t)^2\\ &\quad +\tfrac{e^{\mu}}{\varepsilon}[\varepsilon(D+1)\beta_t(t,1)+\varepsilon E\alpha(t)+\varepsilon F\beta(t,1)+\varepsilon d_2(t)]^2\\ &\quad -\tfrac{e^{-\mu}}{\varepsilon}[\varepsilon(D-1)\beta_t(t,1)+\varepsilon E\alpha(t)+\varepsilon F\beta(t,1)+\varepsilon d_2(t)]^2\\ &\quad +\tfrac{2}{\varepsilon}M^2\varepsilon^2(G\alpha(t)+H\beta(t,1)+d_3(t))^2\\ &\leq -\tfrac{\mu}{\varepsilon}V_1(\beta)+\{e^{\mu}\int_0^1 e^{\mu x}(\beta_x+\varepsilon\beta_t)^2dx+e^{-\mu}\int_0^1 e^{-\mu x}(\beta_x-\varepsilon\beta_t)^2dx\}(\kappa_1+\kappa_2+\kappa_3+\kappa_4)\\ &\quad +\Big(\tfrac{2\varepsilon^2A^2}{\kappa_1}+6M^2\varepsilon G^2\Big)\alpha(t)^2+\Big(\tfrac{2\varepsilon^2B^2}{\kappa_2}+6M^2\varepsilon H^2\Big)\beta(t,1)^2+\tfrac{2\varepsilon^2C^2}{\kappa_3}\beta_t(t,1)^2+\tfrac{2\varepsilon^2}{\kappa_4}d_1(t)^2\\ &\quad +\varepsilon e^{\mu}\{(D+1)^2\beta_t(t,1)^2+(E\alpha(t)+F\beta(t,1)+d_2(t))^2\\ &\quad +2(E\alpha(t)+F\beta(t,1)+d_2(t)(D+1)\beta_t(t,1)\}\\ &\quad -\varepsilon e^{-\mu}\{(D-1)^2\beta_t(t,1)^2+(E\alpha(t)+F\beta(t,1)+d_2(t))^2\\ &\quad +2(E\alpha(t)+F\beta(t,1)+d_2(t))(D-1)\beta_t(t,1)\}+6M^2\varepsilon d_3(t)^2\\ &\leq [-\tfrac{\mu}{\varepsilon}+e^{\mu}(\kappa_1+\kappa_2+\kappa_3+\kappa_4)]V_1(\beta)\\ &\quad +[\tfrac{2\varepsilon^2A^2}{\kappa_1}+3\varepsilon E^2(e^{\mu}+e^{-\mu})(1+\tfrac{|D|+1}{\kappa_5})+6M^2\varepsilon G^2]\alpha(t)^2\\ &\quad +[\tfrac{2\varepsilon^2B^2}{\kappa_2}+3\varepsilon F^2(e^{\mu}+e^{-\mu})(1+\tfrac{|D|+1}{\kappa_5})+6M^2\varepsilon H^2]\beta(t,1)^2\\ &\quad +\tfrac{2\varepsilon^2}{\kappa_4}d_1(t)^2+6M^2\varepsilon d_3(t)^2+3\varepsilon(e^{\mu}+e^{-\mu})(1+\tfrac{|D|+1}{\kappa_5})d_2(t)^2\\ &\quad +[\varepsilon e^{\mu}(D+1)^2-\varepsilon e^{-\mu}(D-1)^2+3\varepsilon\kappa_5(e^{\mu}+e^{-\mu})(|D|+1)+\tfrac{2\varepsilon^2C^2}{\kappa_3}]\beta_t(t,1)^2 \ ,\end{aligned}$$

for any positive values κ_i, $i = 1, \dots, 5$. Using first the Agmon inequality (see Appendix A in [14]) and one boundary condition of β in (23), it holds

$$\begin{aligned}\beta(t,1)^2 &\leq \beta(t,0)^2+2\|\beta(t)\|_{L^2(0,1)}\|\beta_x(t)\|_{L^2(0,1)} \ ,\\ &\leq M^2\alpha(t)^2+\|\beta(t)\|^2_{L^2(0,1)}+\|\beta_x(t)\|^2_{L^2(0,1)} \ .\end{aligned} \tag{25}$$

Moreover, with the Poincaré inequality (see again Appendix A in [14]) it holds, for any positive value κ in $(0,1)$,

$$\begin{array}{rcl}\|\beta(t)\|^2_{L^2(0,1)} & = & \int_0^1(\beta(t,x)-\beta(t,0))^2dx+2\beta(t,0)\int_0^1\beta(t,x)dx-\beta(t,0)^2\\ & \leq & \frac{4}{\pi^2}\|\beta_x(t)\|^2_{L^2(0,1)}+\frac{1}{\kappa}\beta(t,0)^2+\kappa\|\beta(t)\|^2_{L^2(0,1)}-\beta(t,0)^2\\ & \leq & \frac{4}{\pi^2}\|\beta_x(t)\|^2_{L^2(0,1)}+(\frac{1}{\kappa}-1)\beta(t,0)^2+\kappa\|\beta(t)\|^2_{L^2(0,1)}\end{array}$$

Thus letting $\kappa=1/2$ and one boundary condition of β in (23), it is deduced

$$\|\beta(t)\|^2_{L^2(0,1)} \quad \leq \quad \frac{8}{\pi^2}\|\beta_x(t)\|^2_{L^2(0,1)}+2M^2\alpha(t)^2$$

It follows with (25) that

$$|\beta(t,1)|^2\leq C_1V_1(\beta)+3M^2\alpha(t)^2, \tag{26}$$

with $C_1=\left(\frac{8}{\pi^2}+1\right)$. Therefore, we get

$$\begin{array}{r}\frac{d}{dt}V_1(\beta)\leq[-\frac{\mu}{\varepsilon}+e^{\mu}(\kappa_1+\kappa_2+\kappa_3+\kappa_4)+\frac{2\varepsilon^2B^2C_1}{\kappa_2}+3\varepsilon C_1F^2(e^{\mu}+e^{-\mu})(1+\frac{|D|+1}{\kappa_5})+6M^2C_1\varepsilon H^2]V_1(\beta)\\ +\Big[\frac{2\varepsilon^2A^2}{\kappa_1}+3\varepsilon E^2(e^{\mu}+e^{-\mu})(1+\frac{|D|+1}{\kappa_5})+6M^2C_1\varepsilon G^2\\ +\Big(\frac{2\varepsilon^2B^2}{\kappa_2}+3\varepsilon F^2(e^{\mu}+e^{-\mu})(1+\frac{|D|+1}{\kappa_5})+6M^2\varepsilon H^2\Big)3M^2\Big]\alpha(t)^2\\ +\frac{2\varepsilon^2}{\kappa_4}d_1(t)^2+3\varepsilon(e^{\mu}+e^{-\mu})(1+\frac{|D|+1}{\kappa_5})d_2(t)^2+6M^2\varepsilon d_3(t)^2\\ +[\varepsilon e^{\mu}(D+1)^2-\varepsilon e^{-\mu}(D-1)^2+3\varepsilon\kappa_5(e^{\mu}+e^{-\mu})(|D|+1)+\frac{2\varepsilon^2C^2}{\kappa_3}]\beta_t(t,1)^2\ .\end{array} \tag{27}$$

Moreover, using the fourth line of (23), inequality (26) and $2fg\leq\frac{f^2}{k}+kg^2$ (for any values f and g and any positive value k), we have

$$\begin{array}{rcl}\frac{d}{dt}V_2(\alpha) & = & 2\alpha(t)(G\alpha(t)+H\beta(t,1)+d_3(t))\\ & = & 2GV_2(\alpha)+2H\alpha(t)\beta(t,1)+2\alpha(t)d_3(t)\\ & \leq & 2GV_2(\alpha)+\frac{H^2}{\kappa_6}\beta(t,1)^2+\kappa_6\alpha(t)^2+\kappa_7\alpha(t)^2+\frac{1}{\kappa_7}d_3(t)^2\\ & \leq & 2GV_2(\alpha)+\frac{H^2}{\kappa_6}(C_1V_1(\beta)+3M^2\alpha(t)^2)+\kappa_6\alpha(t)^2+\kappa_7\alpha(t)^2+\frac{1}{\kappa_7}d_3(t)^2\\ & \leq & (2G+\kappa_6+\kappa_7+3M^2\frac{H^2}{\kappa_6})V_2(\alpha)+\frac{C_1H^2}{\kappa_6}V_1(\beta)+\frac{1}{\kappa_7}d_3(t)^2\ ,\end{array}$$

for any positive values κ_6 and κ_7. Combining the previous inequality with (27) we readily obtain (24). This concludes the proof of Lemma 1. ■

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