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A NOTE ON THE PAPER "ON THE CONTROLLABILITY OF A COUPLED SYSTEM OF TWO KORTEWEG–DE VRIES EQUATIONS"*

EDUARDO CERPA

Departamento de Matemática
Universidad Técnica Federico Santa María
Avda. España 1680, Valparaíso, Chile
eduardo.cerpa@usm.cl

ADEMIR F. PAZOTO

Instituto de Matemática
Universidade Federal do Rio de Janeiro
P. O. Box 68530, CEP 21945-970
Rio de Janeiro, RJ, Brazil
ademir@im.ufrj.br

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This note concerns the paper "On the controllability of a coupled system of two Korteweg–de Vries equations" by Micu *et al.* [2]. They study a nonlinear coupled system of two Korteweg–de Vries equations and prove that the system is controllable by using four boundary controls. Here, we prove that in some cases it is possible to get the controllablity of the system by using only two controls. This can be done depending on both the spatial domain and the control time.

Keywords: Korteweg–de Vries equation; coupled system; boundary control; controllability.

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In [2], the authors consider a nonlinear control system coupling two Korteweg–de Vries (KdV) equations on a spatial interval $(0, L)$ and in a control time $T > 0$. It has the form

$$\begin{cases} u_t + uu_x + u_{xxx} + a_1 vv_x + a_2(uv)_x + a_3 v_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ cv_t + rv_x + vv_x + ba_3 u_{xxx} + v_{xxx} + ba_1(uv)_x = 0, & \text{in } (0,T)\times(0,L), \\ u(t,0) = 0, \quad u(t,L) = h_1(t), \quad u_x(t,L) = h_2(t), & \text{in } (0,T), \\ v(t,0) = 0, \quad v(t,L) = g_1(t), \quad v_x(t,L) = g_2(t), & \text{in } (0,T), \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & \text{in } (0,L), \end{cases} \tag{1}$$

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where a_1, a_2, a_3, c, b, r are real constants satisfying $c > 0, b > 0$ and $a_3^2 b < 1$. The functions h_1, h_2, g_1, g_2 are the control inputs and u_0, v_0 the initial data. This kind of system has been introduced in [1] to model the interactions of weakly nonlinear gravity waves.

The main purpose in [2] is to address the exact controllability problem for system (1). More precisely, their work is devoted to prove the following result.

Theorem 1 (See [2]). *Let $L > 0$ and $T > 0$. Then, there exists a constant $\delta > 0$ such that for any initial and final data $u_0, v_0, u_1, v_1 \in L^2(0, L)$ verifying*

$$\|(u_0, v_0)\|_{(L^2(0,L))^2} \le \delta \quad \text{and} \quad \|(u_1, v_1)\|_{(L^2(0,L))^2} \le \delta,$$

there exist four control functions $h_1, g_1 \in H_0^1(0, T)$ and $h_2, g_2 \in L^2(0, T)$, such that the solution

$$(u, v) \in C([0, T]; (L^2(0, L))^2) \cap L^2(0, T; (H^1(0, L))^2)$$

of (1) satisfies

$$u(T, \cdot) = u_1(\cdot), \quad v(T, \cdot) = v_1(\cdot). \tag{2}$$

In order to prove Theorem 1, the main step is to get the controllability of the linearized control system around the origin, that corresponds to take $a_1 = a_2 = 0$ in (1). By putting a instead of a_3, we can write the corresponding linear system as

$$\begin{cases} u_t + u_{xxx} + a v_{xxx} = 0, & \text{in } (0, T) \times (0, L), \\ c v_t + r v_x + b a u_{xxx} + v_{xxx} = 0, & \text{in } (0, T) \times (0, L), \\ u(t, 0) = 0, \quad u(t, L) = h_1(t), \quad u_x(t, L) = h_2(t), & \text{in } (0, T), \\ v(t, 0) = 0, \quad v(t, L) = g_1(t), \quad v_x(t, L) = g_2(t), & \text{in } (0, T), \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & \text{in } (0, L). \end{cases} \tag{3}$$

By the duality controllability-observability, the exact controllability of system (3) is equivalent to an observability property for the same linear system with homogeneous boundary conditions. More precisely, the controllability of (3) in $L^2(0, L)$ with controls $h_1, g_1 \in H_0^1(0, T)$ and $h_2, g_2 \in L^2(0, T)$ is equivalent to prove the existence of a constant $K > 0$, such that for any $u_0, v_0 \in L^2(0, L)$,

$$\begin{aligned} &\|(u_0, v_0)\|_{L^2(0,L)^2} \\ &\quad \le K \|(u_x(\cdot, 0), v_x(\cdot, 0))\|_{L^2(0,T)^2} + \|(u_{xx}(\cdot, 0), v_{xx}(\cdot, 0))\|_{H^{-1}(0,T)^2} \end{aligned} \tag{4}$$

where (u, v) is the solution of the homogeneous system

$$\begin{cases} u_t + u_{xxx} + a v_{xxx} = 0, & \text{in } (0, T) \times (0, L), \\ c v_t + r v_x + b a u_{xxx} + v_{xxx} = 0, & \text{in } (0, T) \times (0, L), \\ u(t, 0) = 0, \quad u(t, L) = 0, \quad u_x(t, L) = 0, & \text{in } (0, T), \\ v(t, 0) = 0, \quad v(t, L) = 0, \quad v_x(t, L) = 0, & \text{in } (0, T), \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & \text{in } (0, L). \end{cases} \tag{5}$$

By following the work of Rosier [3], the authors of [2] use a compactness-uniqueness argument to reduce the proof of the observability inequality (4) to the following lemma.

Lemma 1 (See [2]). *Let $\lambda \in \mathbb{C}$. If (w, z) is a solution of*

$$\begin{cases} \lambda w + w''' + az''' = 0, & \text{in } (0, L), \\ c\lambda z + rz' + baw''' + z''' = 0, & \text{in } (0, L), \end{cases} \tag{6}$$

which satisfies the boundary conditions

$$\begin{cases} w(0) = w(L) = w'(0) = w'(L) = w''(0) = 0, \\ z(0) = z(L) = z'(0) = z'(L) = z''(0) = 0. \end{cases} \tag{7}$$

Then $w = z = 0$.

This lemma holds because (w, z) is the solution of a linear third-order ordinary differential system (6) with three null conditions at the same point $x = 0$, see (7).

In this note, we intend to prove Theorem 1 by using only two controls h_2, g_2 and fixing $h_1 = g_1 = 0$ in (1). Thus, the new control system considers homogeneous Dirichlet boundary conditions and is controlled by means of the Neumann boundary conditions at the right end-point of the interval. In this case, the observability inequality to be proved is the following one

$$\begin{gathered} \exists\, K > 0, \quad \forall\, u_0, v_0 \in L^2(0, L), \\ \|(u_0, v_0)\|_{L^2(0,L)^2} \leq K \|(u_x(\cdot, 0), v_x(\cdot, 0))\|_{L^2(0,T)^2}, \end{gathered} \tag{8}$$

where (u, v) is the solution of (5).

The same approach as in [2, 3] leads to prove Lemma 1 with the boundary conditions

$$\begin{cases} w(0) = w(L) = w'(0) = w'(L) = 0, \\ z(0) = z(L) = z'(0) = z'(L) = 0, \end{cases} \tag{9}$$

instead of (7). Let us note that the previous argument proving Lemma 1 fails because we do not have anymore three null initial conditions for the system satisfied by (w, z). In fact, this lemma is no longer true with the boundary conditions (9). A very simple counterexample is given by $w = -a(1 - \cos(x))$ and $z = (1 - \cos(x))$ in the case $L = 2\pi$, $\lambda = 0$ and $r = 1 - ba^2$.

Rosier in [3] studied the controllability of a single KdV equation with one control on the Neumann data at the right-end point. In order to prove the analogous of Lemma 1 with boundary conditions (9), he used the Fourier transform and complex analysis to obtain a characterization of all the lengths L for which this lemma holds. In the case of a system of KdV equations, this analysis is much more complicated and it is not clear at all that such a characterization is possible. For that reason, we try a direct approach based on the multiplier technique that gives us the observability inequality (8) for small values of the length L and large time of control T. We focus on the controllability of the linear system and prove the observability inequality (8). The fixed point argument, as well as the existence and regularity results needed in order to consider the nonlinear system, run exactly in the same way as in the paper [2].

Let (u, v) be the solution of the adjoint system (5). If we multiply the first equation by bxu, the second one by xv and integrate in $(0,T)\times(0,L)$ we deduce that

$$\begin{aligned}\int_0^T\int_0^L (bu_x^2 + 2abu_xv_x + v_x^2) &= \frac{r}{3}\int_0^T\int_0^L v^2 - \frac{1}{3}\int_0^L x(bu^2(t,x) + cv^2(t,x))|_0^T \\ &\le \frac{r}{3}\int_0^T\int_0^L v^2 + \frac{1}{3}\int_0^L x(bu_0^2(x) + cv_0^2(x)) \\ &\le \frac{r}{3}\int_0^T\int_0^L v^2 + \frac{L}{3}\int_0^L (bu_0^2(x) + cv_0^2(x)). \qquad (10)\end{aligned}$$

Under the condition $a^2b < 1$, we can chose $\varepsilon > 0$ such that $\sqrt{a^2b} < \varepsilon < 1$. We can write

$$bu_x^2 + 2abu_xv_x + v_x^2 \ge b(1-\varepsilon^2)u_x^2 + v_x^2\left(1 - \frac{a^2b}{\varepsilon^2}\right) \qquad (11)$$

and therefore we get

$$\begin{aligned}\int_0^T\int_0^L (u_x^2 + v_x^2) \le &\frac{1}{\min\left\{b(1-\varepsilon^2), \left(1 - \frac{a^2b}{\varepsilon^2}\right)\right\}} \\ &\times\left[\frac{r}{3}\int_0^T\int_0^L v^2 + \frac{L}{3}\int_0^L (bu_0^2 + cv_0^2)\right]. \qquad (12)\end{aligned}$$

Let us add the first equation in (5) multiplied by bu, and the second equation multiplied by v. We integrate in $(0,T)\times(0,L)$ and deduce that

$$\begin{aligned}0 &= \frac{b}{2}\int_0^L u^2(T,x) - \frac{b}{2}\int_0^L u^2(0,x) - \int_0^T\int_0^L (bu_{xx} + abv_{xx})u_x \\ &\quad + \frac{1}{2}\int_0^L cv^2(T,x) - \frac{1}{2}\int_0^L cv^2(0,x) - \int_0^T\int_0^L (v_{xx} + abu_{xx})v_x \\ &= \frac{b}{2}\int_0^L u^2(T,x) - \frac{b}{2}\int_0^L u^2(0,x) + \frac{1}{2}\int_0^L cv^2(T,x) - \frac{1}{2}\int_0^L cv^2(0,x) \\ &\quad + \int_0^T \left[\frac{b}{2}u_x^2(t,0) + abv_x(t,0)u_x(t,0) + \frac{1}{2}v_x^2(t,0)\right], \qquad (13)\end{aligned}$$

that is,

$$\begin{aligned}b\int_0^L u^2(T,x) + c\int_0^L v^2(T,x) - b\int_0^L u^2(0,x) - c\int_0^L v^2(0,x) \\ = -\int_0^T [bu_x^2(t,0) + 2abv_x(t,0)u_x(t,0) + v_x^2(t,0)]. \qquad (14)\end{aligned}$$

By using the same ε such that $1 > \varepsilon > \sqrt{a^2 b}$, we obtain

$$\begin{aligned} 2abv_x(t,0)u_x(t,0) &= 2\left(\frac{1}{\varepsilon}\sqrt{a^2 b}\, v_x(t,0)\right)(\varepsilon\sqrt{b}\, u_x(t,0)) \\ &\geq -\varepsilon^2 b u_x^2(t,0) - \frac{a^2 b}{\varepsilon^2} v_x^2(t,0), \end{aligned} \tag{15}$$

and therefore, the right-hand side in (14) is non-positive. Then,

$$b\int_0^L u^2(T,x) + c\int_0^L v^2(T,x) \leq b\int_0^L u^2(0,x) + c\int_0^L v^2(0,x). \tag{16}$$

By combining (12) and (16), we obtain

$$\int_0^T\int_0^L (u_x^2 + v_x^2) \leq C_0\left[b\int_0^L u^2(0,x) + c\int_0^L v^2(0,x)\right], \tag{17}$$

with

$$C_0 := \frac{1}{\min\left\{b(1-\varepsilon^2), \left(1 - \frac{a^2 b}{\varepsilon^2}\right)\right\}}\left(\frac{rT}{3c} + \frac{L}{3}\right).$$

On the other hand, a similar argument with the multipliers $(T-t)u$ and $(T-t)v$ leads to the identity

$$\begin{aligned} \int_0^L (bu_0^2 + cv_0^2) &= \frac{1}{T}\int_0^T\int_0^L (bu^2 + cv^2) + \frac{1}{T}\int_0^T (T-t)(bu_x^2(t,0) \\ &\quad + v_x^2(t,0) + 2abv_x(t,0)u_x(t,0)), \end{aligned} \tag{18}$$

which implies that

$$\int_0^L (bu_0^2 + cv_0^2) \leq \frac{1}{T}\int_0^T\int_0^L (bu^2 + cv^2) + C_1\left[\int_0^T |u_x(t,0)|^2 + |v_x(t,0)|^2\right], \tag{19}$$

where

$$C_1 = \max\left\{b(1+\varepsilon^2), \left(1 + \frac{a^2 b}{\varepsilon^2}\right)\right\}.$$

From (19), Poincare's inequality and (17), we can write

$$\begin{aligned} \int_0^L (bu_0^2 + cv_0^2) &\leq \frac{L^2}{T\pi^2} C_0 \max\{b,c\}\int_0^L (bu_0^2 + cv_0^2) \\ &\quad + C_1\left[\int_0^T |u_x(t,0)|^2 + |v_x(t,0)|^2\right]. \end{aligned} \tag{20}$$

Thus, we finally get the observability inequality

$$\int_0^L (u_0^2+v_0^2) \leq \frac{C_1}{\min\{b,c\}}\left(1-\frac{L^2C_0\max\{b,c\}}{T\pi^2}\right)^{-1}\left[\int_0^T |u_x(t,0)|^2+|v_x(t,0)|^2\right]$$

under the condition

$$1 > \frac{\max\{b,c\}}{\min\left\{b(1-\varepsilon^2),\left(1-\dfrac{a^2b}{\varepsilon^2}\right)\right\}}\left\{\frac{rL^2}{3c\pi^2}+\frac{L^3}{3T\pi^2}\right\}, \tag{21}$$

which makes positive the observability constant K in (8). In order to write a condition without ε, we minimize with respect to ε the right-hand side of (21) over $(\sqrt{a^2b},1)$. Thus we get the condition (21) with ε replaced by $\hat{\varepsilon}=\sqrt{\frac{-(1-b)+\sqrt{(1-b)^2+4a^2b^2}}{2b}}$.

Depending on the constants a,b,c, this condition can be satisfied for small values of L and large enough control time T.

In this way, we are able to prove the controllability result with only two controls for the original nonlinear control system. Hence, the main result of this note can be stated as follows:

Theorem 2. *Let a_1,a_2,a_3,c,b,r be real constants satisfying $c>0, b>0$ and $a_3^2b<1$. Let us suppose that $T,L>0$ satisfy*

$$1 > \frac{\max\{b,c\}}{\min\left\{b(1-\hat{\varepsilon}^2),\left(1-\dfrac{a_3^2b}{\hat{\varepsilon}^2}\right)\right\}}\left\{\frac{rL^2}{3c\pi^2}+\frac{L^3}{3T\pi^2}\right\} \tag{22}$$

where

$$\hat{\varepsilon}=\sqrt{\frac{-(1-b)+\sqrt{(1-b)^2+4a_3^2b^2}}{2b}}.$$

Then, there exists a constant $\delta>0$ such that for any initial and final data $u_0,v_0,u_1,v_1\in L^2(0,L)$ verifying

$$\|(u_0,v_0)\|_{(L^2(0,L))^2}\leq\delta, \quad \textit{and} \quad \|(u_1,v_1)\|_{(L^2(0,L))^2}\leq\delta,$$

there exist two control functions $h,g\in L^2(0,T)$ such that the solution

$$(u,v)\in C([0,T];(L^2(0,L))^2)\cap L^2(0,T;(H^1(0,L))^2)$$

of

$$\begin{cases} u_t+uu_x+u_{xxx}+a_3v_{xxx}+a_1vv_x+a_2(uv)_x=0, & \text{in } (0,T)\times(0,L),\\ cv_t+rv_x+vv_x+ba_3u_{xxx}+v_{xxx}+ba_1(uv)_x=0, & \text{in } (0,T)\times(0,L),\\ u(t,0)=0, \quad u(t,L)=0, \quad u_x(t,L)=h(t), & \text{in } (0,T),\\ v(t,0)=0, \quad v(t,L)=0, \quad v_x(t,L)=g(t), & \text{in } (0,T),\\ u(0,x)=u^0(x), \quad v(0,x)=v^0(x), & \text{in } (0,L), \end{cases} \tag{23}$$

satisfies

$$u(T, \cdot) = u^1(\cdot), \quad v(T, \cdot) = v^1(\cdot). \tag{24}$$

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References

[1] J. A. Gear and R. Grimshaw, Weak and strong interactions between internal solitary waves, *Stud. Appl. Math.* **70** (1984) 235–258.
[2] S. Micu, A. F. Pazoto and J. H. Ortega, On the controllability of a coupled system of two Korteweg–de Vries equations, *Commun. Contemp. Math.* **11**(5) (2009) 799–827.
[3] L. Rosier, Exact boundary controllability for the Korteweg–de Vries equation on a bounded domain, *ESAIM Control Optim. Cal. Var.* **2** (1997) 33–55.