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NULL CONTROLLABILITY OF THE STABILIZED KURAMOTO–SIVASHINSKY SYSTEM WITH ONE DISTRIBUTED CONTROL*

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Abstract. This paper presents a control problem for a one-dimensional nonlinear parabolic system, which consists of a Kuramoto–Sivashinsky–Korteweg de Vries equation coupled to a heat equation. We address the problem of controllability by means of a control supported in an interior open subset of the domain and acting on one equation only. The local null-controllability of the system is proved. The proof is based on a Carleman estimate for the linearized system around the origin. A local inversion theorem is applied to get the result for the nonlinear system.

Key words. parabolic system, internal control, null-controllability, Carleman estimates

AMS subject classifications. 93B05, 93C20, 35K55

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1. Introduction.

Over the last years, a lot of attention has been paid to the controllability of coupled systems of PDE's, which is in general harder to obtain than the controllability of single equations. When dealing with systems, unexpected phenomena may occur. For instance, some linear parabolic systems are controllable only if the control time is large enough; see [6]. This condition never appears for parabolic linear single equations.

As stated in the survey [4], the study of null controllability for systems of parabolic equations is rather recent. In the case of internal control of coupled reaction-diffusion equations, the articles [2, 3, 20, 26] deal with zeroth order couplings by using a Carleman estimates approach. In [14], a system of two heat equations coupled through a cubic nonlinear term is studied. They use the return method and a Carleman estimate to prove null controllability.

Concerning boundary control, there are few results for this kind of system. Some coupled one-dimensional heat equations have been considered in [18, 5, 27, 7] by applying the moment method and proving the existence of an appropriate biorthogonal family of L^2-functions. See also [10], where the approximate controllability of a one-dimensional system is studied.

The efforts of the control community have been oriented to prove control properties when there are fewer controls than equations. In this paper, we are interested in a nonlinear parabolic system formed by two PDEs and where we directly control only one of them. This parabolic system models front propagation in reaction-diffusion phenomena and combines dissipative with dispersive features and simultaneously supports stable solitary-pulse. The system consists of a one-dimensional Kuramoto–Sivashinsky–Korteweg de Vries (KS–KdV) equation, linearly coupled to

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an extra dissipative equation, and was proposed in [23] under the name of the stabilized Kuramoto–Sivashinsky system. (See also [9] for the single KS–KdV equation.) The model has the form

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} + u u_x = v_x, \\ v_t - \Gamma v_{xx} + c v_x = u_x, \end{cases} \tag{1.1}$$

where γ, a are coefficients accounting for the long-wave instability and the short-wave dissipation, respectively, $\Gamma > 0$ is the dissipative parameter, and c is the group-velocity mismatch between wave modes. Notice that the coupling is through first order terms, which is harder to deal with than zeroth order couplings. The null controllability of this system has been studied in [13] in the case where both equations are controlled from the boundary.

In this work, we are interested in the null controllability property of system (1.1) posed on the bounded interval $[0,1]$ with homogeneous boundary conditions, initial data u_0, v_0, and a control h acting in an open subset $\omega \subset (0,1)$. Thus, denoting the characteristic function on ω by $\mathbf{1}_\omega$, we can write the considered system as

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} + u u_x = v_x + h\mathbf{1}_\omega, & (x,t) \in (0,1) \times (0,T), \\ v_t - \Gamma v_{xx} + c v_x = u_x, & (x,t) \in (0,1) \times (0,T), \\ u(0,t) = 0, \quad u(1,t) = 0, & t \in (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, & t \in (0,T), \\ v(0,t) = 0, \quad v(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,1). \end{cases} \tag{1.2}$$

Here, we assume that

$$a, \gamma, \text{ and } \Gamma \text{ are positive constants while } c \text{ may have any sign.} \tag{1.3}$$

We address the problem of steering the solutions of system (1.2) to the rest. More precisely, given $T > 0$ and an appropriate space X, we say that system (1.2) is *null controllable* if for any initial condition $(u_0, v_0) \in X$, there exists an internal control h such that the solution of (1.2) satisfies $u(T,\cdot) = v(T,\cdot) = 0$. We say that the *local null controllability* holds if we can find a control as above whenever $\|(u_0,v_0)\|_X$ is small enough. In this paper, we will prove this last property for system (1.2).

Let us take a look at each equation separately. The boundary null controllability for the one-dimensional heat equation was proved in [15] by Fattorini and Russell, using the moment method. The null-controllability of the heat equation in higher dimensions and with distributed controls is due to Lebeau and Robbiano [22] and Fursikov and Imanuvilov [19]. Concerning the fourth order equation, recently the null controllability has been proved in [12] (see also [11]) with boundary controls and in [28] with a distributed control.

In this paper, we first obtain an observability inequality for the linearized system using global Carleman estimates. Then we apply an inverse function theorem for the nonlinear system in order to get the following theorem.

THEOREM 1.1. *Let $T > 0$ and ω any nonempty open subset of $(0,1)$. There exists $\delta > 0$ such that for any $(u_0, v_0) \in H^{-2}(0,1) \times H^{-1}(0,1)$ with*

$$\|u_0\|_{H^{-2}(0,1)} + \|v_0\|_{H^{-1}(0,1)} < \delta,$$

we can find a control $h \in L^2(0,T;L^2(\omega))$ such that the corresponding solution

$$(u,v) \in C([0,T], H^{-2}(0,1) \times H^{-1}(0,1)) \cap L^2(0,T; L^2(0,1) \times L^2(0,1))$$

of (1.2) *satisfies*

$$u(\cdot, T) = v(\cdot, T) = 0.$$

Let us notice that in system (1.2), the coupling occurs by means of first order terms as in [8], where a parabolic system was studied. In that work, the authors asked the control region ω to touch the boundary of the domain. Here in our paper, in order to consider a control region that is far from the boundary of the domain, we follow [21], where the author studies a parabolic system with a second order coupling.

An outline of the paper follows. In section 2, well-posedness results are stated for the considered systems. The proof of Theorem 1.1 is given in section 3. The linearized system is studied in section 3.1 by using a Carleman estimates approach. The final result for the nonlinear system is obtained by means of a local inversion theorem in section 3.2.

Remark 1.2. The control of some generalizations of the stabilized Kuramoto–Sivashinsky system to the two-dimensional case may be considered in the future. Instead of considering the KdV equation with the extra bi-Laplacian term in one dimension, we can replace the KdV equation by a Zakharov–Kuznetsov equation as in [24] or a Kadomtsev–Petviashvili equation as in [25].

2. Well-posedness. This section is devoted to the proof of well-posedness results for the equations that we are concerned with in this paper. We state results for both linear and nonlinear systems. Let us introduce, for $m, s \in \mathbb{Z}$, the notation

$$X^m(Y^s) := X^m(0, T; Y^s(0, 1)), \quad C(Y^s) := C([0, T], Y^s(0, 1)),$$

where X^m, Y^s typically stands for some Sobolev spaces.

We consider the linear system given by

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} = v_x + f_1, & (x,t) \in (0,1) \times (0,T), \\ v_t - \Gamma v_{xx} + c v_x = u_x + f_2, & (x,t) \in (0,1) \times (0,T), \\ u(0,t) = 0, \quad u(1,t) = 0, & t \in (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, & t \in (0,T), \\ v(0,t) = 0, \quad v(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,1), \end{cases} \tag{2.1}$$

and its adjoint system, which reads as

$$\begin{cases} -\varphi_t + \gamma \varphi_{xxxx} + a \varphi_{xx} - \varphi_{xxx} = -\psi_x + g_1, & (x,t) \in (0,1) \times (0,T), \\ -\psi_t - \Gamma \psi_{xx} - c \psi_x = -\varphi_x + g_2, & (x,t) \in (0,1) \times (0,T), \\ \varphi(0,t) = 0, \quad \varphi(1,t) = 0, & t \in (0,T), \\ \varphi_x(0,t) = 0, \quad \varphi_x(1,t) = 0, & t \in (0,T), \\ \psi(0,t) = 0, \quad \psi(1,t) = 0, & t \in (0,T), \\ \varphi(x,T) = \varphi_T, \quad \psi(x,T) = \psi_T, & x \in (0,1). \end{cases} \tag{2.2}$$

In this section, we prove the well-posedness of these two systems in their respective spaces.

2.1. Adjoint linear system.

PROPOSITION 2.1. *Let G denote either $L^2(0,T; L^2(0,1))^2$ or $L^1(0,T; H_0^2(0,1) \times H_0^1(0,1))$. For each $(\varphi_T, \psi_T) \in H_0^2(0,1) \times H_0^1(0,1)$ and $(g_1, g_2) \in G$, system* (2.2) *has* *a unique solution* $(\varphi, \psi) \in C(H_0^2 \times H_0^1) \cap L^2(H^4 \times H^2)$. *Moreover, there exists* $C > 0$ *such that*

$$\|(\varphi, \psi)\|_{C(H_0^2 \times H_0^1) \cap L^2(H^4 \times H^2)} \leq C \Big\{ \|(\varphi_T, \psi_T)\|_{H_0^2 \times H_0^1} + \|(g_1, g_2)\|_G \Big\}. \tag{2.3}$$

Proof. Let us perform the change of variable $t \mapsto T - t$ in (2.2) in order to have a time-forward system. Considering regular enough data, we multiply the equation on φ by φ_{xxxx} and integrate on $(0, 1)$. We obtain, for any $\varepsilon > 0$, that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_0^1 |\varphi_{xx}|^2 \, dx + \gamma \int_0^1 |\varphi_{xxxx}|^2 \, dx \\ &\quad \leq \varepsilon \int_0^1 |\varphi_{xxxx}|^2 dx + C \int_0^1 \left(|\varphi_{xx}|^2 + |\varphi_{xxx}|^2 + |\psi_x|^2 \right) dx + \int_0^1 g_1 \varphi_{xxxx} dx. \end{aligned} \tag{2.4}$$

From Ehrling's lemma, we get that

$$\int_0^1 |\varphi_{xxx}|^2 \, dx \leq \varepsilon \int_0^1 |\varphi_{xxxx}|^2 \, dx + C \int_0^1 |\varphi|^2 \, dx. \tag{2.5}$$

From (2.4) and (2.5), we have

$$\frac{d}{dt} \int_0^1 |\varphi_{xx}|^2 \, dx + \int_0^1 |\varphi_{xxxx}|^2 \, dx \leq C \int_0^1 \left(|\varphi_{xx}|^2 + |\psi_x|^2 \right) dx + \int_0^1 g_1 \varphi_{xxxx} dx. \tag{2.6}$$

Let us multiply the equation on ψ by ψ_{xx} and integrate on $(0, 1)$. We obtain, for any $\varepsilon > 0$, that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 |\psi_x|^2 \, dx + \Gamma \int_0^1 |\psi_{xx}|^2 \, dx &\leq \varepsilon \int_0^1 |\psi_{xx}|^2 dx \\ &\quad + C \int_0^1 \left(|\varphi_x|^2 + |\psi_x|^2 \right) + \int_0^1 g_2 \psi_{xx} dx \end{aligned}$$

and then

$$\frac{d}{dt} \int_0^1 |\psi_x|^2 \, dx + \int_0^1 |\psi_{xx}|^2 \, dx \leq C \int_0^1 \left(|\psi_x|^2 + |\varphi_x|^2 \right) + \int_0^1 g_2 \psi_{xx} dx. \tag{2.7}$$

Denoting $E(t) = \int_0^1 (|\varphi_{xx}|^2 + |\psi_x|^2)$, from (2.6) and (2.7) we get

$$\frac{d}{dt} E(t) + \int_0^1 \left(|\varphi_{xxxx}|^2 + |\psi_{xx}|^2 \right) \, dx \leq C E(t) + \int_0^1 g_1 \varphi_{xxxx} dx + \int_0^1 g_2 \psi_{xx} dx. \tag{2.8}$$

Let us first obtain the estimate corresponding to $(g_1, g_2) \in L^1(0, T; H_0^2(0, 1) \times H_0^1(0, 1))$. From (2.8), we have

$$\frac{d}{dt} E(t) + \int_0^1 \left(|\varphi_{xxxx}|^2 + |\psi_{xx}|^2 \right) \, dx \leq C E(t) + \int_0^1 (g_1)_{xx} \varphi_{xx} dx + \int_0^1 (g_2)_x \psi_x dx, \tag{2.9}$$

and then Gronwall's lemma implies that

$$\sup_{t \in [0, T]} E(t) \leq C \left(\int_0^1 (g_1)_{xx} \varphi_{xx} dx + \int_0^1 (g_2)_x \psi_x dx \right),$$

and therefore

$$\|(\varphi,\psi)\|_{L^\infty(0,T;H_0^2\times H_0^1(0,1))} \le C\|(g_1,g_2)\|_{L^1(0,T;H_0^2(0,1)\times H_0^1(0,1))}. \tag{2.10}$$

Integrating (2.9) in $[0,T]$ and taking into account estimate (2.10), we get that

$$\|(\varphi,\psi)\|_{L^2(0,T;H^4\times H^2(0,1))} \le C\|(g_1,g_2)\|_{L^1(0,T;H_0^2(0,1)\times H_0^1(0,1))}. \tag{2.11}$$

In order to consider the case $(g_1,g_2)\in L^2(0,T;L^2(0,1))^2$, we use the Cauchy–Schwarz estimate in the last two integrals in (2.8), and we obtain

$$\frac{d}{dt}E(t)+\frac{1}{2}\int_0^1\left(|\varphi_{xxxx}|^2+|\psi_{xx}|^2\right)dx \le CE(t)+\frac{1}{2}\int_0^1\left(|g_1|^2+|g_2|^2\right)dx. \tag{2.12}$$

By using again Gronwall inequality and proceeding as before, we get

$$\begin{aligned}\|(\varphi,\psi)\|&_{L^2(0,T;H^4\times H^2(0,1))\cap L^\infty(0,T;H_0^2\times H_0^1(0,1))}\\ &\le C\|(g_1,g_2)\|_{L^2(0,T;L^2(0,1)\times L^2(0,1))}.\end{aligned} \tag{2.13}$$

Once we get the regularity $\varphi\in L^2(0,T;H^4(0,1))$ and $\psi\in L^2(0,T;H^2(0,1))$, we can use the equations to obtain $\varphi_t\in L^2(0,T;L^2(0,1))$ and $\psi_t\in L^2(0,T;L^2(0,1))$. Thus, by the classical properties of these spaces, we conclude $(\varphi,\psi)\in C([0,T],H_0^2(0,1)\times H_0^1(0,1))$.

To recover the same spaces when the data is less regular, we use density arguments and estimate (2.13). □

2.2. Direct linear system. Using the well-posedness result for the adjoint system, we will study solutions of the direct linear system in the sense of transposition.

DEFINITION 2.2. *Let $u_0\in H^{-2}(0,1)$, $v_0\in H^{-1}(0,1)$, $f_1\in L^1(W^{-1,1})$, and $f_2\in L^2(H^{-1})$. A solution of the system* (2.1) *is a couple $(u,v)\in L^2(L^2)^2$ such that for any $g_1,g_2\in L^2(L^2)$,*

$$\begin{aligned}\int_0^T\int_0^1 u(x,t)g_1(x,t)&+\int_0^T\int_0^1 v(x,t)g_2(x,t)\\ &=\langle u_0,\varphi(\cdot,0)\rangle_{H^{-2},H_0^2}+\langle v_0,\psi(\cdot,0)\rangle_{H^{-1},H_0^1}\\ &\quad+\langle f_1,\varphi\rangle_{L^1(W^{-1,1}),L^\infty(W^{1,\infty})}+\langle f_2,\psi\rangle_{L^2(H^{-1}),L^2(H_0^1)},\end{aligned} \tag{2.14}$$

where (φ,ψ) is the solution of

$$\begin{cases}-\varphi_t+\gamma\varphi_{xxxx}+a\varphi_{xx}-\varphi_{xxx}=-\psi_x+g_1, & (x,t)\in(0,1)\times(0,T),\\ -\psi_t-\Gamma\psi_{xx}-c\psi_x=-\varphi_x+g_2, & (x,t)\in(0,1)\times(0,T),\\ \varphi(0,t)=0,\quad \varphi(1,t)=0, & t\in(0,T),\\ \varphi_x(0,t)=0,\quad \varphi_x(1,t)=0, & t\in(0,T),\\ \psi(0,t)=0,\quad \psi(1,t)=0, & t\in(0,T),\\ \varphi(x,T)=0,\quad \psi(x,T)=0, & x\in(0,1).\end{cases} \tag{2.15}$$

Remark 2.3. As usual, $\langle\cdot,\cdot\rangle_{X,Y}$ stands for the duality product between two spaces X and Y.

The next theorem establishes the existence and uniqueness of solutions for system (2.1).

THEOREM 2.4. *Let $u_0\in H^{-2}(0,1)$, $v_0\in H^{-1}(0,1)$, $f_1\in L^1(W^{-1,1})$, and $f_2\in L^2(H^{-1})$. There exists a unique solution $(u,v)\in C(H^{-2}\times H^{-1})\cap L^2(L^2)^2$ of system*

(2.1). *Moreover, there exists $C > 0$ such that*

$$\|(u,v)\|_{C(H^{-2}\times H^{-1})\cap L^2(L^2)^2} \leq C\Big\{\|f_1\|_{L^1(W^{-1,1})} + \|f_2\|_{L^2(H^{-1})} + \|(u_0,v_0)\|_{H^{-2}\times H^{-1}}\Big\}. \tag{2.16}$$

Proof. For each $(g_1,g_2) \in L^2(0,T;L^2(0,1))$, we define

$$L(g_1,g_2) = \langle u_0, \varphi(\cdot,0)\rangle_{H^{-2},H_0^2} + \langle v_0, \psi(\cdot,0)\rangle_{H^{-1},H_0^1} + \langle f_1, \varphi\rangle_{L^1(W^{-1,1}),L^\infty(W^{1,\infty})} + \langle f_2, \psi\rangle_{L^2(H^{-1}),L^2(H_0^1)}, \tag{2.17}$$

where (φ,ψ) is the solution of (2.15). From Proposition 2.1, L defines a continuous linear functional from $L^2(0,T;L^2(0,1))^2$ to $\mathbb{R}$. Then from the Riesz representation theorem, we obtain the existence and uniqueness of $(u,v) \in L^2(0,T;L^2(0,1))^2$ satisfying (2.14). Moreover, still from Riesz and from Proposition 2.1, we get

$$\begin{aligned}\|(u,v)\|_{L^2(0,T;L^2(0,1))^2} &= \|L\|_{\mathcal{L}(L^2(0,T;L^2(0,1))^2;\mathbb{R})} \\ &\leq C\Big\{\|f_1\|_{L^1(W^{-1,1})} + \|f_2\|_{L^2(H^{-1})} + \|(u_0,v_0)\|_{H^{-2}\times H^{-1}}\Big\}.\end{aligned} \tag{2.18}$$

From Proposition 2.1 we also have that $(g_1,g_2) \in L^1(0,T;H_0^2(0,1)\times H_0^1(0,1)) \mapsto L(g_1,g_2)$ given by (2.17) defines a continuous linear functional from $L^1(0,T;H_0^2(0,1)\times H_0^1(0,1))$ to $\mathbb{R}$. Hence, we get that $(u,v) \in L^\infty(0,T;H^{-2}(0,1)\times H^{-1}(0,1))$ and

$$\begin{aligned}\|(u,v)\|_{L^\infty(0,T;H^{-2}(0,1)\times H^{-1}(0,1))} &= \|L\|_{\mathcal{L}(L^\infty(0,T;H^{-2}(0,1)\times H^{-1}(0,1));\mathbb{R})} \\ &\leq C\Big\{\|f_1\|_{L^1(W^{-1,1})} + \|f_2\|_{L^2(H^{-1})} + \|(u_0,v_0)\|_{H^{-2}\times H^{-1}}\Big\}.\end{aligned} \tag{2.19}$$

Standard density arguments, starting with more regular data, allow us to get the regularity $C([0,T],H^{-2}(0,1)\times H^{-1}(0,1))$ for the solution (u,v). The key step is the following. With more regular data, we can use the equation to imply that $u_t \in L^2(0,T;H^{-4}(0,1))$ and $v_t \in L^2(0,T;H^{-2}(0,1))$. Using that $u \in L^2(0,T;L^2(0,1))$ and $v \in L^2(0,T;L^2(0,1))$, we conclude, by classical properties of theses spaces, that $(u,v) \in C([0,T],H^{-2}(0,1)\times H^{-1}(0,1))$. To recover the same spaces when the data is less regular, we use density arguments and (2.19).

From (2.18) and (2.19), we obtain (2.16), which ends the proof of Theorem 2.4. ∎

2.3. Nonlinear system.

THEOREM 2.5. *There exists a positive real number r such that for any $u_0 \in H^{-2}(0,1)$, $v_0 \in H^{-1}(0,1)$, and $h \in L^2(0,T;L^2(\omega))$ satisfying*

$$\max\{\|u_0\|_{H^{-2}}, \|v_0\|_{H^{-1}}, \|h\|_{L^2(L^2)}\} \leq r, \tag{2.20}$$

the nonlinear equation (1.2) *has a unique solution* $(u,v) \in L^2(0,T;L^2(0,1))^2$. *Moreover,* $(u,v) \in C([0,T],H^{-2}(0,1) \times H^{-1}(0,1))$.

Proof. Let us consider u_0, v_0 and $r>0$ to be chosen later. We define the map

$$\Pi : \ell \in L^2(0,T;L^2(0,1)) \longmapsto u \in L^2(0,T;L^2(0,1)), \tag{2.21}$$

where (u,v) is the solution of

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} = v_x - \ell\ell_x + h, & (x,t) \in (0,1)\times(0,T), \\ v_t - \Gamma v_{xx} + c v_x = u_x, & (x,t) \in (0,1)\times(0,T), \\ u(0,t) = 0, \quad u(1,t) = 0, & t \in (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, & t \in (0,T), \\ v(0,t) = 0, \quad v(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,1). \end{cases} \tag{2.22}$$

Let us notice that $\tilde{u}$ is a fixed point of the map Π if and only if the corresponding $(\tilde{u},\tilde{v})$ is a solution of the nonlinear control system (1.2).

From Theorem 2.4 and the fact that

$$\|\ell\ell_x\|_{L^1(W^{-1,1})} = \frac{1}{2}\left\|(\ell^2)_x\right\|_{L^1(W^{-1,1})} \leq \frac{1}{2}\|\ell\|^2_{L^2(L^2)},$$

we get

$$\|\Pi(\ell)\|_{L^2(L^2)} \leq C\Big(\|u_0\|_{H^{-2}(0,1)} + \|v_0\|_{H^{-1}(0,1)} + \|h\|_{L^2(L^2)} + \|\ell\|^2_{L^2(L^2)}\Big).$$

For each $R>0$, let us denote by $B(0,R)$ the closed ball in $L^2(0,T;L^2(0,1))$ of radius R and that is centered at the origin. We see that if $r>0$ and $R>0$ are chosen such that $C(r+R^2) \leq R$, we obtain that $\Pi|_{B(0,R)} \subset B(0,R)$. Let us verify that we can choose R such that Π is a contraction. Let ℓ and $\tilde{\ell}$ be two elements in $L^2(0,T;L^2(0,1))^2$, and we denote by (u,v) and $(\tilde{u},\tilde{v})$ the corresponding solutions of system (2.22).

The couple $(\hat{u},\hat{v})$ given by $\hat{u} = \tilde{u} - u$ and $\hat{v} = \tilde{v} - v$ is the solution of

$$\begin{cases} \hat{u}_t + \gamma \hat{u}_{xxxx} + \hat{u}_{xxx} + a \hat{u}_{xx} = \hat{v}_x + \ell\ell_x - \tilde{\ell}\tilde{\ell}_x, & (x,t) \in (0,1)\times(0,T), \\ \hat{v}_t - \Gamma \hat{v}_{xx} + c \hat{v}_x = \hat{u}_x, & (x,t) \in (0,1)\times(0,T), \\ \hat{u}(0,t) = 0, \quad \hat{u}(1,t) = 0, & t \in (0,T), \\ \hat{u}_x(0,t) = 0, \quad \hat{u}_x(1,t) = 0, & t \in (0,T), \\ \hat{v}(0,t) = 0, \quad \hat{v}(1,t) = 0, & t \in (0,T), \\ \hat{u}(x,0) = 0, \quad \hat{v}(x,0) = 0, & x \in (0,1). \end{cases}$$

From Theorem 2.4, we get

$$\|\Pi(\tilde{\ell}) - \Pi(\ell)\|_{L^2(L^2)} = \|\hat{u}\|_{L^2(L^2)} \leq C\|\tilde{\ell}\,\tilde{\ell}_x - \ell\ell_x\|_{L^1(W^{-1,1})}.$$

By using that

$$\|\tilde{\ell}\,\tilde{\ell}_x - \ell\ell_x\|_{L^1(W^{-1,1})} = \frac{1}{2}\|\tilde{\ell}^2 - \ell^2\|_{L^1(L^1)} \leq \frac{1}{2}\|\tilde{\ell} + \ell\|_{L^2(L^2)}\|\tilde{\ell} - \ell\|_{L^2(L^2)},$$

we obtain

$$\|\Pi(\tilde{\ell}) - \Pi(\ell)\|_{L^2(L^2)} \leq CR\|\tilde{\ell} - \ell\|_{L^2(L^2)},$$

and therefore the map Π is a contraction if $CR < 1$. By applying the Banach fixed point theorem, we conclude that Π has a unique fixed point u in $L^2(0,T;L^2(0,1))$. Now, once we have (u,v) as the solution of (1.2), we see that (u,v) satisfies (2.1) with source terms $f_1 = -uu_x \in L^1(0,T;W^{-1,1}(0,1))$ and $f_2 = 0$. We apply Theorem 2.4, and we get the extra regularity $(u,v) \in C([0,T],H^{-2}(0,1) \times H^{-1}(0,1))$. ∎

3. Null controllability.

3.1. Linear control system. In this section, we study the null controllability of the linear system

$$\text{(3.1)} \qquad \begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} = v_x + f_1 + h\mathbf{1}_\omega, & (x,t) \in (0,1) \times (0,T), \\ v_t - \Gamma v_{xx} + c v_x = u_x + f_2, & (x,t) \in (0,1) \times (0,T), \\ u(0,t) = 0, \quad u(1,t) = 0, & t \in (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, & t \in (0,T), \\ v(0,t) = 0, \quad v(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,1). \end{cases}$$

Let us take a well-posedness framework (U, X_1, X_2, Y, Z) for this system. By this, we mean that given $h \in U$, $(f_1, f_2) \in Y = Y_1 \times Y_2$, $u_0 \in X_1$ and $v_0 \in X_2$, there exists a unique $(u,v) \in Z = Z_1 \times Z_2$ solution of (3.1).

This system is said to be null controllable if for any state $u_0 \in X_1$, $v_0 \in X_2$ and for any $(f_1, f_2) \in Y_1 \times Y_2$, one can find a control $h \in U$ such that the solution (u,v) of (3.1) satisfies $u(T) = v(T) = 0$. It is a well-known fact that by duality, this null-controllability property is equivalent to the existence of a constant $C > 0$ such that

$$\text{(3.2)} \qquad \|(\varphi,\psi)\|_{Y^*} + \|\varphi(0,x)\|_{X_1^*} + \|\psi(0,x)\|_{X_2^*} \leq C\big(\|(g_1,g_2)\|_{Z^*} + \|\varphi\|_{U^*}\big)$$

for every $\varphi_T \in X_1^*$, $\psi_T \in X_2^*$, and $g \in Z^*$, where * stands for dual space and (φ,ψ) is the solution of the adjoint linear system (2.2). Inequality (3.2) is called an *observability inequality* for system (2.2).

Carleman estimates for the heat equation are well known (see, [19] for example). In this section, we prove a modified Carleman estimate, following [21], for system (2.2). This estimate will be very useful to deal with the first order coupling terms in our system. Then, we use it in order to prove the observability inequality (3.2) within an appropriate well-posedness framework. Thus, we get the null-controllability of system (3.1).

We shall use an abbreviated notation for the derivatives and integrals. We write, for k integer, w_{kx} instead of $\frac{\partial^k w}{\partial x^k}$ and $\iint$ instead of $\int_0^T \int_0^1$, avoiding the symbols $dxdt$ in the last case.

We take a function $\beta \in C^3([0,1])$ satisfying

$$\beta(x) > 0 \ \forall x \in (0,1), \quad \beta(0) = \beta(1) = 0 \tag{3.3}$$

and

$$|\beta'(x)| \geq \delta > 0 \ \forall \, x \in [0,1] \backslash \omega \text{ for some } \delta > 0. \tag{3.4}$$

Let us recall that (3.3) and (3.4) imply that

$$\beta'(0) > 0 \text{ and } \beta'(1) < 0. \tag{3.5}$$

Following [21], for some positive constants k, λ and $m \in \mathbb{N}$, we define

$$\alpha_m(x,t) = \frac{e^{k\frac{m+1}{m}\lambda||\beta||_\infty} - e^{\lambda(k||\beta||_\infty + \beta(x))}}{t^m(T-t)^m} \text{ and } \xi_m(x,t) = \frac{e^{\lambda(k||\beta||_\infty + \beta(x))}}{t^m(T-t)^m}. \tag{3.6}$$

In what follows, we use the estimates

$$\left|\left(e^{-2s\alpha_m}\xi_m^q\right)_x\right| \leq Cs\lambda\xi_m\left(e^{-2s\alpha_m}\xi_m^q\right), \quad \left|\left(e^{-2s\alpha_m}\xi_m^q\right)_t\right| \leq Cs\xi_m^{(1+\frac{1}{m})}\left(e^{-2s\alpha_m}\xi_m^q\right),$$

which can be easily verified for any integer $q > 0$, and where C denotes a positive constant.

THEOREM 3.1. *Let α_m and ξ_m be defined in (3.6) with $m > 3$, $k > m$, and β satisfying hypothesis (3.3) and (3.4).*

Then, there exist $\lambda_0, s_0 > 0$ such that

$$\begin{aligned} &s\lambda^2 \iint e^{-2s\alpha_m}\xi_m|\psi_{2x}|^2 + s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 \\ &\quad \leq Cs\lambda^2 \iint e^{-2s\alpha_m}\xi_m^2|L\psi|^2 + Cs\lambda^2 \iint e^{-2s\alpha_m}\xi_m^2|L\psi_x|^2 \\ &\qquad + s^3\lambda^4 \iint_\omega e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 \end{aligned} \tag{3.7}$$

for every $s \geq s_0$, $\lambda \geq \lambda_0$, and $\psi \in L^2(0,T;H^3(0,1) \cap H_0^1(0,1)) \cap H^1(0,T;H^1(0,1))$, where $L = -\partial_t - \Gamma\partial_x^2 - c\partial_x$.

We follow ideas introduced in [21, Lemmas 6 and 7]. (See also Theorem 1 in [17].) Let us define

$$\alpha_m^*(t) = \max_{x\in[0,1]} \alpha_m(x,t) \quad \text{and} \quad \xi_m^*(t) = \min_{x\in[0,1]} \xi_m(x,t). \tag{3.8}$$

We will use the following result proved in [21].

LEMMA 3.2 (Lemma 6 in [21]). *Let* $u_0 \in L^2(0,1)$ *and* $f \in L^2(0,T;L^2(0,1))$. *Then there exists a constant* $C = C(\omega) > 0$ *such that each solution* $u \in L^2(H^1) \cap L^\infty(L^2)$ *of*

$$\begin{cases} -u_t - \Delta u = f & \text{in } (0,1)\times(0,T), \\ u|_{t=T} = u_0 & \text{in } (0,1) \end{cases}$$

satisfies

$$s\lambda^2 \iint e^{-2s\alpha_m}\xi_m |\nabla u|^2 dxdt + s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3 |u|^2 dxdt$$
$$\leq C\left(s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha_m}\xi_m^3|u|^2 dxdt + \iint e^{-2s\alpha_m}|f|^2 dxdt \right.$$
$$\left. + s\lambda\int_0^T e^{-2s\alpha_m^*(t)}\xi_m^*(t)\left|\frac{\partial u}{\partial n}(1,t)\right|^2 dt - s\lambda\int_0^T e^{-2s\alpha_m^*(t)}\xi_m^*(t)\left|\frac{\partial u}{\partial n}(0,t)\right|^2 dt \right)$$

for any $\lambda \geq C$ *and* $s \geq C(T^{2m} + T^{2m-1})$.

Proof of Theorem 3.1. We apply Lemma 3.2 to function ψ_x. We obtain

$$\begin{aligned} &s\lambda^2 \iint e^{-2s\alpha_m}\xi_m|\psi_{2x}|^2 + s^3\lambda^4\iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 \\ &\quad\leq C\iint e^{-2s\alpha_m}|L\psi_x|^2 + s^3\lambda^4\iint_\omega e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + s\lambda\iint_\Sigma e^{-2s\alpha_m^*}\xi_m^*|\psi_{2x}|^2, \end{aligned} \tag{3.9}$$

where we use the notation

$$\iint_\Sigma f(x,t) = \int_0^T f(1,t)dt - \int_0^T f(0,t)dt.$$

We will find an estimate for the boundary term on the right-hand side of (3.9). For $m > 3$, we consider $r \in (\frac{5}{2}, 3)$ satisfying

$$m > \frac{r-1}{3-r} \tag{3.10}$$

and take $\theta \in (0, \frac{1}{4})$ such that $r = \theta + 3(1-\theta)$. Using trace and interpolation theorems for Sobolev spaces, we have

$$\begin{aligned} &s\lambda\iint_\Sigma e^{-2s\alpha_m^*}\xi_m^*|\psi_{2x}|^2 \\ &\leq Cs\lambda\int_0^T e^{-2s\alpha_m^*}\xi_m^*\|\psi\|^2_{H^r(0,1)} \\ &\leq Cs\lambda\int_0^T e^{-2s\alpha_m^*}\xi_m^*(\|\psi\|^\theta_{H^1(0,1)}\|\psi\|^{1-\theta}_{H^3(0,1)})^2 \\ &= C\lambda\int_0^T e^{-2s\alpha_m^*}(s\xi_m^*)^{3\theta}\|\psi\|^{2\theta}_{H^1(0,1)}(s\xi_m^*)^{1-3\theta}\|\psi\|^{2(1-\theta)}_{H^3(0,1)} \\ &\leq \frac{1}{4}s^3\lambda\int_0^T e^{-2s\alpha_m^*}(\xi_m^*)^3\|\psi\|^2_{H^1(0,1)} + Cs^p\lambda\int_0^T e^{-2s\alpha_m^*}(\xi_m^*)^p\|\psi\|^2_{H^3(0,1)} \\ &\leq \frac{1}{4}s^3\lambda^4\iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + Cs^p\lambda\int_0^T e^{-2s\alpha_m^*}(\xi_m^*)^p\|\psi\|^2_{H^3(0,1)}, \end{aligned} \tag{3.11}$$

where $p = \frac{1-3\theta}{1-\theta} = \frac{3r-7}{r-1}$.

Let us define $g(t) = s^{\frac{1}{2}-\frac{1}{m}}\lambda e^{-s\alpha_m^*}(\xi_m^*)^{\frac{1}{2}-\frac{1}{m}}$ and $\psi^* = g\psi$. Then $L\psi^* = g'\psi + gL\psi$, and from (3.10) we directly obtain that

$$s^p\lambda\int_0^T e^{-2s\alpha_m^*}(\xi_m^*)^p\|\psi\|^2_{H^3(0,1)} \leq C\|\psi^*\|^2_{L^2(H^3)}. \tag{3.12}$$

By parabolic regularity for the differential operator L and Poincaré inequality for ψ, we get

$$\begin{aligned}\|\psi^*\|^2_{L^2(H^3)} &\leq C\|L\psi^*\|^2_{L^2(H^1)}\\ &\leq C\|g'\psi_x\|^2_{L^2(L^2)} + C\|gL\psi\|^2_{L^2(H^1)}\\ &\leq \frac{1}{4}s^3\lambda^4\iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + Cs\lambda^2\iint e^{-2s\alpha_m}\xi_m^2|L\psi|^2\\ &\quad + Cs\lambda^2\iint e^{-2s\alpha_m}\xi_m^2|L\psi_x|^2.\end{aligned} \tag{3.13}$$

From (3.9), (3.11), (3.12), and (3.13), we get (3.7). Theorem 3.1 is proved. ☐

We also need the following Carleman estimate for the fourth order operator.

THEOREM 3.3. *For α_m and ξ_m defined in* (3.6), *and β satisfying hypothesis* (3.3) *and* (3.4), *there exist $\lambda_1, s_1 > 0$ such that*

$$\begin{aligned}&s^{-1}\iint \xi_m^{-1}e^{-2s\alpha_m}|\varphi_{4x}|^2 + s\lambda^2\iint \xi_m e^{-2s\alpha_m}|\varphi_{3x}|^2 + s^3\lambda^4\iint \xi_m^3 e^{-2s\alpha_m}|\varphi_{2x}|^2\\ &\quad + s^5\lambda^6\iint \xi_m^5 e^{-2s\alpha_m}|\varphi_x|^2 + s^7\lambda^8\iint \xi_m^7 e^{-2s\alpha_m}|\varphi|^2\\ &\leq C\Big(\iint e^{-2s\alpha_m}|P\varphi|^2 + s^7\lambda^8\iint_\omega \xi_m^7 e^{-2s\alpha_m}|\varphi|^2\Big)\end{aligned} \tag{3.14}$$

for every $s \geq s_1$, $\lambda \geq \lambda_1$, and $\varphi \in L^2(0,T;H^4(0,1)\cap H_0^2(0,1))\cap H^1(0,T;L^2(0,1))$, where $P = -\partial_t + \gamma\partial_x^4 - \partial_x^3 + a\partial_x^2 - \partial_x$.

Proof. A similar estimate was proved in [28]. For the sake of completeness, we give the proof in the appendix. ☐

Let us give a brief idea of the proof of our Carleman estimate. Recall that if (φ, ψ) is a solution, then $P\varphi = -\psi_x + g_1$ and $L\psi = -\varphi_x + g_2$. We apply the Carleman estimates (3.7) and (3.14) (with the same weight functions) to the corresponding equations in the system, and we add both estimates. Taking s and λ large enough, the integrals involving the coupling terms ψ_x and φ_x are absorbed by the left-hand side of the inequality. Moreover, we prove that we can eliminate one of the two observations, getting the following result.

THEOREM 3.4. *Let α_m and ξ_m be defined in* (3.6). *There exist $\lambda_2, s_2 \geq 0$ such that*

$$\begin{aligned}&s\lambda^2\iint e^{-2s\alpha_m}\xi_m|\psi_{2x}|^2 + s^3\lambda^4\iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + s^{-1}\iint \xi_m^{-1}e^{-2s\alpha_m}|\varphi_{4x}|^2\\ &\quad + s\lambda^2\iint \xi_m e^{-2s\alpha_m}|\varphi_{3x}|^2 + s^3\lambda^4\iint \xi_m^3 e^{-2s\alpha_m}|\varphi_{2x}|^2\\ &\quad + s^5\lambda^6\iint \xi_m^5 e^{-2s\alpha_m}|\varphi_x|^2 + s^7\lambda^8\iint \xi_m^7 e^{-2s\alpha_m}|\varphi|^2\\ &\leq Cs^{39}\lambda^{40}\iint_\omega \xi_m^{39}e^{-2s\alpha_m}|\varphi|^2\\ &\quad + C\iint e^{-2s\alpha_m}(s^3\lambda^4\xi_m^3|g_1|^2 + s^3\lambda^4\xi_m^3|g_2|^2 + s\lambda^2|(g_2)_x|^2)\end{aligned} \tag{3.15}$$

for every $s \geq s_2$, $\lambda \geq \lambda_2$, $(\varphi_T, \psi_T) \in H_0^2 \times H_0^1$, and $g_1 \in L^2(0,T;L^2(0,1)), g_2 \in L^2(0,T;H^2(0,1))$, where (φ, ψ) is the solution of (2.2).

Proof. For the sake of clarity, we split the proof in different steps.

Step 1 *(adding up Carleman estimates)*: Let us take ω_0 an open set such that $\overline{\omega_0} \subset \omega$. From Theorems 3.1 and 3.3 applied in ω_0 for solutions of system (2.2), we have that

$$
\begin{aligned}
(3.16)\quad & s\lambda^2 \iint e^{-2s\alpha_m}\xi_m|\psi_{2x}|^2 + s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 \\
&\quad + s^{-1}\iint \xi_m^{-1}e^{-2s\alpha_m}|\varphi_{4x}|^2 + s\lambda^2 \iint \xi_m e^{-2s\alpha_m}|\varphi_{3x}|^2 \\
&\quad + s^3\lambda^4 \iint \xi_m^3 e^{-2s\alpha_m}|\varphi_{2x}|^2 + s^5\lambda^6 \iint \xi_m^5 e^{-2s\alpha_m}|\varphi_x|^2 \\
&\quad + s^7\lambda^8 \iint \xi_m^7 e^{-2s\alpha_m}|\varphi|^2 \\
&\leq C\biggl(s\lambda^2 \iint e^{-2s\alpha_m}\xi_m^2|-\varphi_x + g_2|^2 + s\lambda^2 \iint e^{-2s\alpha_m}\xi_m^2|-\varphi_{2x} + (g_2)_x|^2 \\
&\qquad + s^3\lambda^4 \iint_{\omega_0} e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + \iint e^{-2s\alpha_m}|-\psi_x + g_1|^2 \\
&\qquad + s^7\lambda^8 \iint_{\omega_0} \xi_m^7 e^{-2s\alpha_m}|\varphi|^2\biggr).
\end{aligned}
$$

Step 2 *(absorbing coupling terms)*: It is easy to see that for any $\varepsilon > 0$, we have

$$
\begin{aligned}
s\lambda^2 \iint e^{-2s\alpha_m}\xi_m^2|\varphi_x|^2 &\leq \varepsilon s^5\lambda^6 \iint e^{-2s\alpha_m}\xi_m^5|\varphi_x|^2, \\
s\lambda^2 \iint e^{-2s\alpha_m}\xi_m^2|\varphi_{2x}|^2 &\leq \varepsilon s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3|\varphi_{2x}|^2
\end{aligned}
$$

and

$$
\iint e^{-2s\alpha_m}|\psi_x|^2 \leq \varepsilon s^3\lambda^4 \iint e^{-2s\alpha_m}|\psi_x|^2
$$

for s and λ large enough (depending on ε). Then from (3.16), we get

$$
\begin{aligned}
(3.17)\quad & s\lambda^2 \iint e^{-2s\alpha_m}\xi_m|\psi_{2x}|^2 + s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + s^{-1}\iint \xi_m^{-1}e^{-2s\alpha_m}|\varphi_{4x}|^2 \\
&\quad + s\lambda^2 \iint \xi_m e^{-2s\alpha_m}|\varphi_{3x}|^2 + s^3\lambda^4 \iint \xi_m^3 e^{-2s\alpha_m}|\varphi_{2x}|^2 \\
&\quad + s^5\lambda^6 \iint \xi_m^5 e^{-2s\alpha_m}|\varphi_x|^2 + s^7\lambda^8 \iint \xi_m^7 e^{-2s\alpha_m}|\varphi|^2 \\
&\leq C\biggl(s\lambda^2 \iint e^{-2s\alpha_m}\xi_m^2|g_2|^2 + s\lambda^2 \iint e^{-2s\alpha_m}\xi_m^2|(g_2)_x|^2 \\
&\qquad + s^3\lambda^4 \iint_{\omega_0} e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + \iint e^{-2s\alpha_m}|g_1|^2 \\
&\qquad + s^7\lambda^8 \iint_{\omega_0} \xi_m^7 e^{-2s\alpha_m}|\varphi|^2\biggr).
\end{aligned}
$$

Step 3 *(observations of φ only)*: We will prove now that the observation of ψ_x on the right-hand side of (3.16) can be eliminated. Let ω_1 be an open set such that $\omega_0 \subset\subset \omega_1 \subset\subset \omega$, and take a function $\eta \in C_0^\infty(\omega_1)$ such that $\eta = 1$ in ω_0. We have

$$\begin{aligned} s^3\lambda^4 \iint_{\omega_0} e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 \\ \le s^3\lambda^4 \iint_{\omega_1} \eta e^{-2s\alpha_m}\xi_m^3\psi_x(\varphi_t - \gamma\varphi_{4x} - a\varphi_{2x} + \varphi_{3x} + g_1) = I_1 + I_2, \end{aligned} \tag{3.18}$$

where

$$I_1 = s^3\lambda^4 \iint_{\omega_1} \eta e^{-2s\alpha_m}\xi_m^3\psi_x(\varphi_t - \Gamma\varphi_{2x}) \tag{3.19}$$

and

$$I_2 = s^3\lambda^4 \iint_{\omega_1} \eta e^{-2s\alpha_m}\xi_m^3\psi_x(-\gamma\varphi_{4x} + (\Gamma - a)\varphi_{2x} + \varphi_{3x} + g_1). \tag{3.20}$$

Claim 1. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} |I_1| \le \varepsilon s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + \varepsilon s\lambda^2 \iint e^{-2s\alpha_m}\xi_m|\psi_{2x}|^2 \\ + C_\varepsilon\bigg(s^7\lambda^8 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^7|\varphi|^2 + s^5\lambda^6 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi_x|^2 \\ + s^3\lambda^4 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^3|g_2|^2\bigg). \end{aligned} \tag{3.21}$$

Indeed, integrating by parts, we obtain

$$I_1 = s^3\lambda^4 \iint_{\omega_1} \eta e^{-2s\alpha_m}\xi_m^3\partial_x(-\psi_t - \Gamma\psi_{2x})\varphi + R, \tag{3.22}$$

where

$$\begin{aligned} R = -s^3\lambda^4 \iint_{\omega_1} \eta(e^{-2s\alpha_m}\xi_m^3)_t\psi_x\varphi + s^3\lambda^4 \iint_{\omega_1} \Gamma(\eta e^{-2s\alpha_m}\xi_m^3)_x\psi_x\varphi_x \\ - s^3\lambda^4 \iint_{\omega_1} \Gamma(\eta e^{-2s\alpha_m}\xi_m^3)_x\psi_{2x}\varphi. \end{aligned} \tag{3.23}$$

Integrating by parts in (3.22) and using system (2.2), we get that

$$\begin{aligned} I_1 = -s^3\lambda^4 \iint_{\omega_1} (\eta e^{-2s\alpha_m}\xi_m^3)_x(c\psi_x - \varphi_x + g_2)\varphi \\ - s^3\lambda^4 \iint_{\omega_1} \eta e^{-2s\alpha_m}\xi_m^3(c\psi_x - \varphi_x + g_2)\varphi_x + R. \end{aligned} \tag{3.24}$$

From the Cauchy–Schwarz inequality, we can deduce that

$$
\begin{aligned}
(3.25)\qquad |I_1| \le{}& \varepsilon s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + C_\varepsilon s^5\lambda^6 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi|^2 \\
&+ C_\varepsilon s^3\lambda^4 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^3|\varphi_x|^2 + Cs^3\lambda^4 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^3|g_2|^2 + |R|.
\end{aligned}
$$

Again from the Cauchy–Schwarz inequality in the definition of R, and using that $m > 3$, we get

$$
\begin{aligned}
(3.26)\qquad |R| \le{}& \varepsilon s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + \varepsilon s\lambda^2 \iint e^{-2s\alpha_m}\xi_m|\psi_{2x}|^2 \\
&+ C_\varepsilon s^7\lambda^8 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^7|\varphi|^2 + C_\varepsilon s^5\lambda^6 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi_x|^2.
\end{aligned}
$$

From (3.25) and (3.26), we get (3.21), i.e., Claim 1 holds.

Claim 2. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
\begin{aligned}
(3.27)\qquad |I_2| \le{}& \varepsilon s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + \varepsilon s\lambda^2 \iint e^{-2s\alpha_m}\xi_m|\psi_{2x}|^2 \\
&+ C_\varepsilon\Bigg(s^3\lambda^4 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^3|\varphi_{2x}|^2 + s^5\lambda^6 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi_{3x}|^2 \\
&\qquad + s^3\lambda^4 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^3|g_1|^2\Bigg).
\end{aligned}
$$

Integrating by parts in space the fourth order term in (3.20) and using the Cauchy–Schwarz inequality, we get (3.27), i.e., Claim 2 holds.

From (3.18), (3.21), and (3.27), we get

$$
\begin{aligned}
(3.28)\qquad & s^3\lambda^4 \iint_{\omega_0} e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 \\
&\le \varepsilon s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + \varepsilon s\lambda^2 \iint e^{-2s\alpha_m}\xi_m|\psi_{2x}|^2 \\
&\quad + Cs^7\lambda^8 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^7|\varphi|^2 + Cs^5\lambda^6 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi_x|^2 \\
&\quad + Cs^3\lambda^4 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^3|\varphi_{2x}|^2 + Cs^5\lambda^6 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi_{3x}|^2 \\
&\quad + Cs^3\lambda^4 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^3|g_1|^2 + Cs^3\lambda^4 \iint_{\omega_1} e^{-2s\alpha_m}\xi_m^3|g_2|^2.
\end{aligned}
$$

From (3.17) and (3.28), we obtain

$$
\begin{aligned}
(3.29)\qquad & s\lambda^2 \iint e^{-2s\alpha_m}\xi_m|\psi_{2x}|^2 + s^3\lambda^4 \iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 \\
&+s^{-1}\iint \xi_m^{-1}e^{-2s\alpha_m}|\varphi_{4x}|^2 + s\lambda^2\iint \xi_m e^{-2s\alpha_m}|\varphi_{3x}|^2 \\
&+s^3\lambda^4\iint \xi_m^3 e^{-2s\alpha_m}|\varphi_{2x}|^2 + s^5\lambda^6\iint \xi_m^5 e^{-2s\alpha_m}|\varphi_x|^2 \\
&+s^7\lambda^8\iint \xi_m^7 e^{-2s\alpha_m}|\varphi|^2 \\
&\le Cs^7\lambda^8\iint_{\omega_1}\xi_m^7 e^{-2s\alpha_m}|\varphi|^2 + Cs^5\lambda^6\iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi_x|^2 \\
&+Cs^3\lambda^4\iint_{\omega_1} e^{-2s\alpha_m}\xi_m^3|\varphi_{2x}|^2 + Cs^5\lambda^6\iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi_{3x}|^2 \\
&+Cs^3\lambda^4\iint e^{-2s\alpha_m}\xi_m^3|g_1|^2 + Cs^3\lambda^4\iint e^{-2s\alpha_m}\xi_m^3|g_2|^2 \\
&+Cs\lambda^2\iint e^{-2s\alpha_m}|(g_2)_x|^2.
\end{aligned}
$$

Notice that, compared to (3.16), we do not have anymore the observation of ψ_x on ω_0. This is a key point to get a control result where we only act directly on one equation. A negative point at this stage is that we have now additional observations of φ_x and φ_{3x} on the subset ω_1.

Step 4 *(only zero order observation)*: We prove that we can eliminate the higher order observation of φ in (3.29).

Claim 3. If $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
\begin{aligned}
(3.30)\qquad & s^5\lambda^6\iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi_x|^2 + s^3\lambda^4\iint_{\omega_1} e^{-2s\alpha_m}\xi_m^3|\varphi_{2x}|^2 \\
&+s^5\lambda^6\iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi_{3x}|^2 \\
&\le \varepsilon\iint e^{-2s\alpha_m}\left((s\xi_m)^{-1}|\varphi_{xxxx}|^2 + s^3\lambda^4\xi_m^3|\varphi_{2x}|^2 + s\lambda^2\xi_m|\varphi_{3x}|^2\right) \\
&+C_\varepsilon s^{39}\lambda^{40}\iint_{\omega} e^{-2s\alpha_m}\xi_m^{39}|\varphi|^2.
\end{aligned}
$$

Indeed, if ω_2 is an open set such that $\omega_1 \subset\subset \omega_2 \subset\subset \omega$ and $\eta \in C_0^\infty(\omega_2)$ is such that $\eta = 1$ in ω_1, we get

$$
\begin{aligned}
(3.31)\quad & s^5\lambda^6\iint_{\omega_1} e^{-2s\alpha_m}\xi_m^5|\varphi_{3x}|^2 \\
&\le s^5\lambda^6\iint_{\omega_2}\eta e^{-2s\alpha_m}\xi_m^5|\varphi_{3x}|^2 \\
&= -s^5\lambda^6\iint_{\omega_2}\eta e^{-2s\alpha_m}\xi_m^5\varphi_{xxxx}\varphi_{2x} + \frac{s^5\lambda^6}{2}\iint_{\omega_2}(\eta e^{-2s\alpha_m}\xi_m^5)_{2x}|\varphi_{2x}|^2 \\
&\le \varepsilon\iint e^{-2s\alpha_m}(s\xi_m)^{-1}|\varphi_{xxxx}|^2 + C_\varepsilon s^{11}\lambda^{12}\iint_{\omega_2} e^{-2s\alpha_m}\xi_m^{11}|\varphi_{2x}|^2.
\end{aligned}
$$

In the same way, if ω_3 is an open set such that $\omega_2 \subset\subset \omega_3 \subset\subset \omega$, we get that

$$
\begin{aligned}
(3.32)\quad & s^{11}\lambda^{12} \iint_{\omega_2} e^{-2s\alpha_m} \xi_m^{11} |\varphi_{2x}|^2 \\
& \leq \varepsilon s \lambda^2 \iint e^{-2s\alpha_m} \xi_m |\varphi_{3x}|^2 + C_\varepsilon s^{21}\lambda^{22} \iint_{\omega_3} e^{-2s\alpha_m} \xi_m^{21} |\varphi_x|^2,
\end{aligned}
$$

and finally

$$
\begin{aligned}
(3.33)\quad & s^{21}\lambda^{22} \iint_{\omega_3} e^{-2s\alpha_m} \xi_m^{21} |\varphi_x|^2 \\
& \leq \varepsilon s^3 \lambda^4 \iint e^{-2s\alpha_m} \xi_m^3 |\varphi_{2x}|^2 + C_\varepsilon s^{39}\lambda^{40} \iint_{\omega} e^{-2s\alpha_m} \xi_m^{39} |\varphi|^2.
\end{aligned}
$$

Inequality (3.30) is deduced from (3.31), (3.32), and (3.33).

Step 5 *(Conclusion)*: From (3.29) and (3.30), we get the Carleman estimate (3.15), which concludes the proof of Theorem 3.4. □

In order to prove an observability inequality for our system from the previous Carleman estimate, we use the following two lemmas.

LEMMA 3.5. *If $g_1, g_2 \in L^2(0,T;L^2(0,1))$, $(\varphi_T, \psi_T) \in H_0^2 \times H_0^1$, and φ, ψ are the solutions of system* (2.2)*, then*

$$
\begin{aligned}
(3.34)\quad & -\frac{d}{dt}\int_0^1 (\varphi(x,t)^2 + \psi(x,t)^2)dx \\
& \leq C\int_0^1 (\varphi(x,t)^2 + \psi(x,t)^2)dx + \int_0^1 (g_1(x,t)^2 + g_2(x,t)^2)dx
\end{aligned}
$$

for every $t \in [0,T]$.

Proof. Multiplying the first and the second equations of system (2.2) by φ and ψ, respectively, and integrating in $(0,1)$, we obtain

$$
\begin{aligned}
(3.35)\quad & -\frac{1}{2}\frac{d}{dt}\int_0^1 |\varphi(x,t)|^2 dx + \gamma \int_0^1 |\varphi_{2x}(x,t)|^2 dx + a\int_0^1 \varphi_{2x}(x,t)\varphi(x,t)dx \\
& = -\int_0^1 \psi_x(x,t)\varphi(x,t)dx + \int_0^1 \varphi(x,t)g_1(x,t)dx
\end{aligned}
$$

and

$$
\begin{aligned}
(3.36)\quad & -\frac{1}{2}\frac{d}{dt}\int_0^1 |\psi(x,t)|^2 dx + \Gamma \int_0^1 |\psi_x(x,t)|^2 dx \\
& = -\int_0^1 \varphi_x(x,t)\psi(x,t)dx + \int_0^1 \psi(x,t)g_2(x,t)dx
\end{aligned}
$$

for each $t \in [0,T]$. Adding up (3.35) and (3.36) and using that

$$
-a\int \varphi_{2x}\varphi\, dx \leq \gamma \int |\varphi_{2x}|^2\, dx + \frac{a^2}{\gamma}\int |\varphi|^2\, dx,
$$

we get (3.34). □

LEMMA 3.6. *For $g_1, g_2 \in L^2(0,T;L^2(0,1))$ and $(\varphi_T, \psi_T) \in H_0^2 \times H_0^1$, let (φ, ψ) be the solution of system (2.2). If $\Phi(t) = \int_0^1 (\varphi(t)^2 + \psi(t)^2)dx$, then*

$$\|\Phi\|_{L^\infty(0,\frac{T}{2})} \leq C \left(\|\Phi\|^2_{L^2(\frac{T}{2},\frac{3T}{4})} + \int_0^{\frac{3T}{4}} \int_0^1 (g_1^2 + g_2^2) dx dt \right). \tag{3.37}$$

Proof. Let $z \in C^\infty(0,T)$ be such that $z(t) = 1$ for all $t \in [0, T/2]$ and $z(t) = 0$ for all $t \in [3T/4, T]$. Multiplying (3.34) by z, we obtain

$$-\frac{d}{dt}(z\Phi) \leq Cz\Phi - z_t\Phi + z\int_0^1 (g_1^2 + g_2^2)dx. \tag{3.38}$$

For each $t \in [0,T]$, we apply the Gronwall inequality in $[t,T]$. Since $z(T) = 0$, we obtain

$$z(t)\Phi(t) \leq C\int_t^T h(s)ds, \tag{3.39}$$

where

$$h(t) = z(t)\int_0^1 (g_1^2 + g_2^2)dx - z_t(t)\int_0^1 (g_1^2 + g_2^2)dx.$$

From (3.39), we deduce (3.37). $\square$

In the desired observability inequality, we have to recover the solutions at $t = 0$, which is impossible with the weight function that we are working with. For this reason, we use a new weight function that is not singular at $t = 0$. We set ϕ_m defined by

$$\phi_m(t) = \begin{cases} \dfrac{4^m}{T^{2m}} & \text{if} \quad 0 \leq t < T/2, \\ \dfrac{1}{t^m(T-t)^m} & \text{if} \quad T/2 \leq t \leq T. \end{cases}$$

We denote by M_1 and M_2 the minimum and maximum, respectively, of the spatial part of α_m:

$$M_1 = \left(e^{k\frac{m+1}{m}\lambda||\beta||_\infty} - e^{\lambda(k||\beta||_\infty + \|\beta\|_\infty)} \right) \text{ and} \tag{3.40}$$

$$M_2 = \left(e^{k\frac{m+1}{m}\lambda||\beta||_\infty} - e^{\lambda(k||\beta||_\infty)} \right). \tag{3.41}$$

If k and m are chosen such that $k > m$, then we have

$$\begin{aligned} 2M_1 - M_2 &= e^{k\frac{(m+1)}{m}\lambda||\beta||_\infty} - 2e^{\lambda(k||\beta||_\infty + \|\beta\|_\infty)} + e^{\lambda(k||\beta||_\infty)} \\ &= e^{\lambda(k||\beta||_\infty + \|\beta\|_\infty)} \left(e^{\lambda(\frac{k}{m}-1)||\beta||_\infty} - 2 + e^{-\lambda(||\beta||_\infty)} \right) \\ &> 0 \end{aligned}$$

for λ large enough. Thus, we can take some $r \in \mathbb{R}$ such that

$$M_2 < r < 2M_1. \tag{3.42}$$

Proposition 3.7. *There exist $\lambda, s, C > 0$ such that the solution (φ, ψ) of (2.2) satisfies*

$$\tag{3.43}\begin{aligned}&\iint |\psi|^2 e^{-2sM_2\phi_m}\phi_m^3 + \int_0^1 |\psi(x,0)|^2 dx + \iint |\varphi|^2 e^{-2sM_2\phi_m}\phi_m^7 + \int_0^1 |\varphi(x,0)|^2 dx\\ &\quad\le C\bigg(\iint e^{-2sM_1\phi_m}(\phi_m^3|g_1|^2 + \phi_m^3|g_2|^2 + |(g_2)_x|^2) + \iint_\omega \phi_m^{39} e^{-2sM_1\phi_m}|\varphi|^2\bigg)\end{aligned}$$

for every $g_1, g_2 \in L^2(L^2)$ such that $\iint e^{-2sM_1\phi_m}(\phi_m^3|g_1|^2 + \phi_m^3|g_2|^2 + |(g_2)_x|^2) < \infty$, and every $(\varphi_T, \psi_T) \in H_0^2 \times H_0^1$.

Proof. From Proposition 2.1, Lemma 3.6, and the Poincaré inequality, we obtain

$$\tag{3.44}\begin{aligned}&\int_0^1 (|\varphi(x,0)|^2 + |\psi(x,0)|^2)dx + \int_0^{T/2}\int_0^1 |\varphi|^2 e^{-2sM_2\phi_m}\phi_m^7 dxdt\\ &\quad + \int_0^{T/2}\int_0^1 |\psi|^2 e^{-2sM_2\phi_m}\phi_m^3 dxdt\\ &\le \|(\varphi,\psi)\|^2_{L^\infty(0,T/2;L^2(0,1))^2}\\ &\le C\|(\varphi,\psi)\|^2_{L^\infty(T/2,3T/4;L^2(0,1))^2} + C\|(g_1,g_2)\|^2_{L^\infty(0,3T/4;L^2(0,1))^2}\\ &\le C\int_{T/2}^{\frac{3T}{4}}\int_0^1 (\varphi(x,t)^2 + \psi(x,t)^2)dxdt\\ &\quad + C\int_0^{\frac{3T}{4}}\int_0^1 e^{-2sM_2\phi_m}(g_1(x,t)^2 + g_2(x,t)^2)dxdt\\ &\le C\int_{T/2}^{\frac{3T}{4}}\int_0^1 (\varphi(x,t)^2 + \psi_x(x,t)^2)dxdt\\ &\quad + C\int_0^{\frac{3T}{4}}\int_0^1 e^{-2sM_2\phi_m}(g_1(x,t)^2 + g_2(x,t)^2)dxdt.\end{aligned}$$

Using again the Poincaré inequality, and Carleman estimate (3.15), we deduce that

$$\tag{3.45}\begin{aligned}&\int_{T/2}^T\int_0^1 e^{-2sM_2\phi_m}\phi_m^3|\psi|^2 + \int_{T/2}^T\int_0^1 e^{-2sM_2\phi_m}\phi_m^7|\varphi|^2\\ &\le C\iint e^{-2s\alpha_m}\xi_m^3|\psi_x|^2 + \iint \xi_m^7 e^{-2s\alpha_m}|\varphi|^2\\ &\le Cs^{39}\lambda^{40}\iint_\omega \xi_m^{39}e^{-2s\alpha_m}|\varphi|^2 + C\iint e^{-2s\alpha_m}(s^3\lambda^4\xi_m^3|g_1|^2\\ &\quad + s^3\lambda^4\xi_m^3|g_2|^2 + s\lambda^2|(g_2)_x|^2)\\ &\le Cs^{39}\lambda^{40}\iint_\omega \phi_m^{39}e^{-2sM_1\phi_m}|\varphi|^2\\ &\quad + C\iint e^{-2sM_1\phi_m}(s^3\lambda^4\phi_m^3|g_1|^2 + s^3\lambda^4\phi_m^3|g_2|^2 + s\lambda^2|(g_2)_x|^2).\end{aligned}$$

Inequality (3.43) is obtained from (3.44) and (3.45). ∎

Through the rest of the paper, we denote by ρ the function defined by

$$\rho(t) = e^{-\frac{sr}{(T-t)^m}}$$

for $t \in (0,T)$, where r satisfies (3.42).

PROPOSITION 3.8. *There exists $C > 0$ such that the solution (φ, ψ) of (2.2) satisfies*

$$\begin{aligned}(3.46)\qquad &\|(\rho\varphi, \rho\psi)\|^2_{L^\infty(H_0^2)\times L^\infty(H_0^1)} \\ &\quad\le C \iint e^{-\frac{2sM_1}{(T-t)^m}} \Big((T-t)^{-3m}|g_1|^2 + (T-t)^{-3m}|g_2|^2 + |(g_2)_x|^2\Big) \\ &\qquad + C \iint_\omega e^{-\frac{2sM_1}{(T-t)^m}} (T-t)^{-39m}|\varphi|^2\end{aligned}$$

for every g_1, g_2 such that $\iint e^{-\frac{2sM_1}{(T-t)^m}}((T-t)^{-3m}|g_1|^2 + (T-t)^{-3m}|g_2|^2 + |(g_2)_x|^2)$ is finite, and every $(\varphi_T, \psi_T) \in H_0^2 \times H_0^1$.

Proof. Let us define $\tilde\varphi = \rho\varphi$ and $\tilde\psi = \rho\psi$. Notice that $\tilde\varphi$, $\tilde\psi$ satisfy system (2.2) with $\varphi_T = \psi_T = 0$ and with the right-hand side equal to $(\rho g_1 - \rho_t\varphi)$ and $(\rho g_2 - \rho_t\psi)$, instead of g_1 and g_2, respectively. Thanks to Proposition 2.1, we have

$$\|(\rho\varphi, \rho\psi)\|_{L^\infty(H_0^2)\times L^\infty(H_0^1)} \le C\Big(\|(\rho g_1, \rho g_2)\|_{L^2(L^2)} + \|(\rho_t\varphi, \rho_t\psi)\|_{L^2(L^2)}\Big).$$

We can easily check the existence of some positive constant C such that

$$\begin{aligned}&\iint |\rho g_j|^2 \le C \iint |g_j|^2 e^{-\frac{2sr}{(T-t)^m}} \text{ for } j = 1, 2, \\ &\iint |\rho_t\varphi|^2 \le C \iint |\varphi|^2 e^{-\frac{2sr}{(T-t)^m}} (T-t)^{-2m-2} \le C \iint |\varphi|^2 e^{-\frac{2sM_2}{(T-t)^m}},\end{aligned}$$

and in the same manner

$$\iint |\rho_t\psi|^2 \le C \iint |\psi|^2 e^{-\frac{2sM_2}{(T-t)^m}}.$$

Therefore, by using (3.43), we get (3.46). ∎

Inequality (3.46) directly implies an observability inequality like (3.2) in some weighted spaces. In order to make that precise, we introduce the following notation.

DEFINITION 3.9. *Given $T > 0$, a normed vector space X, and a function $\eta : (0, T) \longrightarrow \mathbb{R}^+$, we denote*

$$L^2(\eta; X) := \left\{ f\,;\, \int_0^T \|f(t)\|_X^2 \eta(t)\, dt < \infty \right\}$$

endowed with their natural norm.

Taking into account the continuous embeddings $H_0^2(0,1) \hookrightarrow W^{1,\infty}(0,1)$ and $L^\infty(0,T) \hookrightarrow L^2(0,T)$, inequality (3.46) gives us the observability (3.2) in the spaces

$$\begin{aligned}&U = L^2\left(e^{\frac{2sM_1}{(T-t)^m}}(T-t)^{39m}; L^2(\omega)\right), \\ &X_1 = H^{-2}(0,1), \quad X_2 = H^{-1}(0,1), \\ &Z = \{(u,v)\,;\, e^{\frac{sM_1}{(T-t)^m}}(T-t)^{\frac{3m}{2}}(u,v) \in L^2(0,T; L^2(0,1) \times L^2(0,T; H^{-1}(0,1))\}, \\ &Y_1 = \{y; \rho^{-1}y \in L^1(0,T; W^{-1,1}(0,1))\}, \quad \text{and} \\ (3.47)\ &Y_2 = \{y; \rho^{-1}y \in L^2(0,T; H^{-1}(0,1))\}.\end{aligned}$$

Thus, by duality, we obtain the null controllability result in this functional framework. The next proposition makes this fact precise and gives extra information on the decay of the controlled trajectories. This decay will be important later when dealing with the nonlinear system.

PROPOSITION 3.10. *For each $f_1 \in Y_1$, $f_2 \in Y_2$, $u_0 \in H^{-2}(0,1)$, and $v_0 \in H^{-1}(0,1)$, there exists a control $h \in U$ such that the solution (u,v) of system* (3.1) *belongs to Z and satisfies*

$$(T-t)^{20m} e^{\frac{sM_1}{(T-t)^m}} (u,v) \in C([0,T]; H^{-2}(0,1) \times H^{-1}(0,1)).$$

In particular, $u(T) = v(T) = 0$.

Proof. We follow [19] (see also [16]) to get a controllability result from inequality (3.46). We denote

$$\eta_1(t) = e^{-\frac{2sM_1}{(T-t)^m}} (T-t)^{-3m}$$

and

$$\eta_2(t) = e^{-\frac{2sM_1}{(T-t)^m}} (T-t)^{-39m}.$$

Let us define, for (φ,ψ) and $(\widehat{\varphi},\widehat{\psi})$ in $C^\infty([0,1]\times[0,T])^2$ satisfying the boundary conditions of (2.2), the bilinear form given by

$$a\big((\varphi,\psi);(\widehat{\varphi},\widehat{\psi})\big) = \iint_\omega \eta_2 \varphi\widehat{\varphi} + \iint \eta_1 \Big(P^*(\varphi,\psi) P^*(\widehat{\varphi},\widehat{\psi}) \\ + L^*(\varphi,\psi) L^*(\widehat{\varphi},\widehat{\psi}) + \partial_x L^*(\varphi,\psi) \partial_x L^*(\widehat{\varphi},\widehat{\psi}) \Big),$$

where

$$P^*(\varphi,\psi) = -\varphi_t + \gamma\varphi_{xxxx} + a\varphi_{xx} - \varphi_{xxx} + \psi_x$$

and

$$L^*(\varphi,\psi) = -\psi_t - \Gamma\psi_{xx} - c\psi_x + \varphi_x.$$

By inequality (3.46), we have that $a\big((\varphi,\psi);(\varphi,\psi)\big)^{1/2}$ is a norm. Let us denote by X the completion of the pairs $(\varphi,\psi) \in C^\infty([0,1]\times[0,T])$ satisfying the boundary conditions of (2.2) with this norm.

We define from X to $\mathbb{R}$ the scalar function $\jmath$ given by

$$\jmath(\varphi,\psi) = \langle f_1, \varphi\rangle_{Y_1,Y_1^*} + \langle f_2, \psi\rangle_{Y_2,Y_2^*} + \langle u_0, \varphi(0)\rangle_{H^{-2},H_0^2} + \langle v_0, \psi(0)\rangle_{H^{-1},H_0^1},$$

which is continuous and linear. Therefore, from the Lax–Milgram theorem, we get that there exists $(\widetilde{\varphi},\widetilde{\psi}) \in X$ such that

$$a\big((\widetilde{\varphi},\widetilde{\psi});(\varphi,\psi)\big) = \jmath(\varphi,\psi) \tag{3.48}$$

for all $(\varphi,\psi) \in X$.

We set $h = -\eta_2\widetilde{\varphi}$, $u = \eta_1 P^*(\widetilde{\varphi},\widetilde{\psi})$, and finally,

$$v = \eta_1\big(L^*(\widetilde{\varphi},\widetilde{\psi}) - (L^*(\widetilde{\varphi},\widetilde{\psi}))_{xx}\big).$$

By definition of the space X, we get that $(u,v) \in Z$ and $h \in U$. From construction, we have that U and Y_1 are subspaces of $L^1(0,T;W^{-1,1}(0,1))$ and Y_2 is a subspace of

$L^2(0,T;H^{-1}(0,1))$. Hence, by Theorem 2.4, we get that system (3.1) is well posed for data in the stated spaces and for any $h \in U$. The solution satisfies $(u,v) \in L^2(0,T;L^2(0,1))^2 \cap C([0,T],H^{-2}(0,1)\times H^{-1}(0,1))$.

It is easy to see, from identity (3.48), that we have found a control h steering the solution (u,v) of the system to the null state $(0,0)$.

On the other hand, let us define $g(t) = (T-t)^{30m}e^{\frac{sM_1}{(T-t)^m}}$, $\tilde{u} = gu$, $\tilde{v} = gv$, $\tilde{h} = gh$, and $\tilde{f}_j = gf_j$ for $j = 1,2$. Then $(\tilde{u},\tilde{v})$ satisfies

$$\left\{\begin{aligned} \tilde{u}_t + \gamma\tilde{u}_{xxxx} + a\tilde{u}_{2x} + \tilde{u}_{3x} &= \tilde{v}_x + \tilde{f}_1 - g_t u + \tilde{h}1_\omega,\\ \tilde{v}_t - \Gamma\tilde{v}_{2x} + c\tilde{v}_x &= \tilde{u}_x + \tilde{f}_2 - g_t v.\end{aligned}\right. \tag{3.49}$$

It directly follows that $\tilde{h} \in L^2(0,T;L^2(0,1))$. As $f_1 \in Y_1$ and $M_1 < r$, we deduce that $\tilde{f}_1 \in L^1(0,T;W^{-1,1}(0,1))$. From the fact that $u \in Z_1$ we get $g_t u \in L^2(0,T;L^2(0,1))$. We conclude that $(\tilde{f}_1 - g_t u + \tilde{h}) \in L^1(0,T;W^{-1,1}(0,1))$.

In the same way, we prove that $\tilde{f}_2 - g_t v \in L^2(0,T;H^{-1}(0,1))$. Hence, by applying Theorem 2.4, we obtain the desired result. □

3.2. Nonlinear control system. Let us prove the local null controllability of the nonlinear system. We deduce this result from the null controllability of the linear equation by using a local inversion theorem.

In order to obtain Theorem 1.1, we use the following result.

THEOREM 3.11 (see [1]). *Let E and G be two Banach spaces and let $\Lambda : E \to G$ satisfy $\Lambda \in C^1(E;G)$. Assume that $\hat{e} \in E$, $\Lambda(\hat{e}) = \hat{g}$, and $\Lambda'(\hat{e}) : E \to G$ is surjective. Then, there exists $\delta > 0$ such that, for every $g \in G$ satisfying $\|g-\hat{g}\|_G < \delta$, there exists some $e \in E$ solution of the equation $\Lambda(e) = g$.*

Let us define some appropriate spaces E, G and a map Λ whose surjectivity is equivalent to the null controllability for our nonlinear parabolic system. We denote

$$\begin{aligned} L_1(u,v) &= u_t + \gamma u_{xxxx} + u_{3x} + au_{2x} - v_x,\\ L_2(u,v) &= v_t - \Gamma v_{2x} + cv_x - u_x.\end{aligned}$$

Keeping in mind (3.47), we define the spaces

$$\begin{aligned} E := \Big\{ &(u,v,h) \in Z \times U\ : L_1(u,v) + h\mathbf{1}_\omega \in Y_1, L_2(u,v) \in Y_2,\\ &\qquad\text{and } (T-t)^{20m}e^{\frac{sM_1}{(T-t)^m}}(u,v) \in C(H^{-2}\times H^{-1})\Big\}\end{aligned}$$

and

$$G := H^{-2}(0,1) \times Y_1 \times H^{-1}(0,1) \times Y_2.$$

The map Λ is given by

$$\begin{aligned} \Lambda : E &\longrightarrow \quad G,\\ (u,v,h) &\longmapsto \quad (u(0,\cdot), L_1(u,v) + h\mathbf{1}_\omega + uu_x, v(0,\cdot), L_2(u,v)).\end{aligned}$$

In order to prove that Λ is well-defined, we have to verify that $uu_x \in Y_1$ for each $(u,v) \in Z$. We have

$$\begin{aligned} uu_x \in Y_1 \quad &\Longleftrightarrow \quad e^{\frac{r}{(T-t)}}uu_x \in L^1(0,T;W^{-1,1}(0,1))\\ &\Longleftrightarrow \quad e^{\frac{sr}{(T-t)^m}}|u|^2 \in L^1(0,T;L^1(0,1))\\ &\Longleftrightarrow \quad \int_0^T\int_0^1 |u|^2 e^{\frac{sr}{(T-t)^m}}\,dxdt < \infty.\end{aligned}$$

Since $r < 2M_1$, we have $e^{\frac{sr}{(T-t)^m}} < e^{\frac{2sM_1}{(T-t)^m}}(T-t)^{3m}$. Hence, $(u,v) \in Z$ implies $uu_x \in Y_1$.

Notice that the map $(u_1, v), (u_2, v) \in Z \mapsto \frac{1}{2}(u_1 u_2)_x \in Y_1$ is a bilinear continuous map, and consequently Λ is a C^1 map.

Since $(u, v, h) \in E$ satisfy $u(T) = v(T) = 0$, the local surjectivity of Λ around the origin is equivalent to the local null controllability of system (1.2). Thus, by Theorem 3.11, the proof of Theorem 1.1 will be ended if we prove that the map $\Lambda'(0)$ is surjective.

PROPOSITION 3.12. *The map $\Lambda'(0) : E \to G$ is surjective.*

Proof. It is easy to see that this map is given by

$$\begin{aligned} \Lambda'(0) : E &\longrightarrow \quad G, \\ (u, v, h) &\longmapsto \quad (u(0,\cdot), L_1(u,v) + h\mathbf{1}_\omega, v(0,\cdot), L_2(u,v)) \,. \end{aligned}$$

Therefore, the surjectivity of $\Lambda'(0)$ is equivalent to the null controllability of the linearized equation with source terms lying in $Y_1 \times Y_2$ and initial data in $H^{-2}(0,1) \times H^{-1}(0,1)$ with a control $h \in U$. This control property was proved in Proposition 3.10. ∎

Appendix. Proof of Theorem 3.3. Recall the hypothesis on β and the definitions of α_m and ξ_m stated from (3.3) to (3.6). In this proof, we will denote α_m and ξ_m by α and ξ, respectively. Without loss of generality, we take the coefficient $\gamma = 1$.

We write $P = P_m + P_r$, where $P_m = -\partial_t + \partial_x^4$ and $P_r = -\partial_x^3 + a\partial_x^2 - \partial_x$. Let us define $\tilde{P}v = e^{-s\alpha}P(e^{s\alpha}v)$, the conjugate operator of P. Hence, $\tilde{P} = \tilde{P}_m + \tilde{P}_r$. From straightforward computations, we have

$$\tilde{P}_m v = P_1 v + P_2 v + Rv, \tag{A.1}$$

where

$$\begin{aligned} &P_1 v = 6\lambda^2 s^2 (\beta')^2 \xi^2 v_{2x} + \lambda^4 s^4 (\beta')^4 \xi^4 v + v_{4x} + 6\lambda^2 s^2 ((\beta')^2 \xi^2)_x v_x, \\ &P_2 v = -v_t - 4\lambda^3 s^3 (\beta')^3 \xi^3 v_x - 4\lambda s \beta' \xi v_{3x}, \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} Rv = &-s\alpha_t v - 6\lambda^3 s^3 (\beta')^2 \beta'' \xi^3 v - 6\lambda^4 s^3 (\beta')^4 \xi^3 v + 3\lambda^2 s^2 (\beta'')^2 \xi^2 v + 4\lambda^2 s^2 \beta' \beta''' \xi^2 v \\ &+ 18\lambda^3 s^2 (\beta')^2 \beta'' \xi^2 v + 7\lambda^4 s^2 (\beta')^4 \xi^2 v - \lambda s \beta_{4x} \xi v - 4\lambda s \beta''' \xi v_x - 12\lambda^2 s \beta' \beta'' \xi v_x \\ &- 6\lambda s \beta'' \xi v_{2x} - 3\lambda^2 s (\beta'')^2 \xi v - 4\lambda^2 s \beta' \beta''' \xi v - 4\lambda^3 s (\beta')^3 \xi v_x \\ &- 6\lambda^2 s (\beta')^2 \xi v_{2x} - \lambda^4 s (\beta')^4 \xi v - 6\lambda^3 s (\beta')^2 \beta'' \xi v. \end{aligned}$$

We define, for a subset $U \subset \Omega$, the following integral terms:

$$I_U(v, s, \lambda) = s^7 \lambda^8 \iint_U \xi^7 |v|^2 + s^5 \lambda^6 \iint_U \xi^5 |v_x|^2 + s^3 \lambda^4 \iint_U \xi^3 |v_{2x}|^2 + s\lambda^2 \iint_U \xi |v_{3x}|^2.$$

Then, it directly follows that

$$\|\tilde{P}_r v\|^2_{L^2(\Omega)} \leq \frac{1}{s} I_\Omega(v, s, \lambda) \tag{A.3}$$

and

$$\|Rv\|^2_{L^2(\Omega)} \leq \frac{1}{s} I_\Omega(v, s, \lambda). \tag{A.4}$$

The main part of the proof consists in estimating from below $\|P_1 v + P_2 v\|^2_{L^2(\Omega\setminus\omega)}$ by $CI_{\Omega\setminus\omega}(v,s,\lambda)$ for some positive constant C. Indeed, we have

$$\langle P_1 v, P_2 v\rangle_{L^2(\Omega)} = \sum_{i=1,j=1}^{i=4,j=3} I_{i,j},$$

where $I_{i,j}$ denotes the L^2-product between the ith term in the expression of $P_1 v$ and the jth term in the expression of $P_2 v$ (see (A.2)).

After some integrations by parts in x and t, we get:

$$\begin{aligned}
I_{11} &= -6s^2\lambda^2 \iint (\beta')^2(\xi^2)_t |v_x|^2 + 6s^2\lambda^2 \iint ((\beta')^2\xi^2)_x v_x v_t,\\
I_{12} &= 60s^5\lambda^6 \iint (\beta')^6\xi^5|v_x|^2 + 60s^5\lambda^5 \iint (\beta')^4\beta_{2x}\xi^5|v_x|^2,\\
I_{13} &= 36s^3\lambda^4 \iint (\beta')^4\xi^3|v_{2x}|^2 + 36s^3\lambda^3 \iint (\beta')^2\beta_{2x}\xi^3|v_{2x}|^2\\
&\quad - 12s^3\lambda^3 \iint_\Sigma (\beta')^3\xi^3|v_{2x}|^2,\\
I_{21} &= 2s^4\lambda^4 \iint (\beta')^4\xi^3\xi_t|v|^2,\\
I_{22} &= 14s^7\lambda^8 \iint (\beta')^8\xi^7|v|^2 + 14s^2\lambda^7 \iint (\beta')^6\beta_{2x}|v|^2,\\
I_{23} &= -28s^5\lambda^6 \iint (\beta')^6\xi^5|v_x|^2 - 30s^5\lambda^5 \iint (\beta')^4\beta_{2x}\xi^5|v_x|^2\\
&\quad + 2s^5\lambda^5 \iint ((\beta')^5\xi^5)_{3x}|v|^2,\\
I_{31} &= -\frac{1}{2}\iint (|v_{2x}|^2)_t = 0,\\
I_{32} &= -6s^3\lambda^3 \iint (\beta')^2\beta_{2x}\xi^3|v_{2x}|^2 - 6s^3\lambda^4 \iint (\beta')^4\xi^3|v_{2x}|^2\\
&\quad + 2s^3\lambda^3 \iint_\Sigma (\beta')^3\xi^3|v_{2x}|^2,\\
I_{33} &= 2s\lambda^2 \iint (\beta')^2\xi|v_{3x}|^2 + 2s\lambda \iint \beta_{2x}\xi^3|v_{3x}|^2 - 2s\lambda \iint_\Sigma \beta'\xi|v_{3x}|^2,\\
I_{41} &= -6s^2\lambda^2 \iint ((\beta')^2\xi^2)_x v_x v_t = -I_{11}^2,\\
I_{42} &= -48s^5\lambda^5 \iint (\beta')^6\xi^5|v_x|^2 - 48s^5\lambda^5 \iint (\beta')^4\beta_{2x}\xi^5|v_x|^2,\\
I_{43} &= 48s^3\lambda^4 \iint (\beta')^4\xi^3|v_{2x}|^2 + 48s^3\lambda^3 \iint (\beta')^2\xi^3\beta_{2x}|v_{2x}|^2\\
&\quad - 12s^3\lambda^3 \iint ((\beta')^2\xi(\beta_x^2\xi^2)_x)_{2x}|v_x|^2.
\end{aligned}$$

Taking into account hypothesis (3.5), we get that the sum of the boundary terms appearing above is nonnegative, i.e.,

$$-10s^3\lambda^3 \iint_\Sigma (\beta')^3\xi^3|v_{2x}|^2 - 2s\lambda \iint_\Sigma \beta'\xi|v_{3x}|^2 \geq 0. \tag{A.5}$$

Let ω_0 be an open set such that $\{\beta' = 0\} \subset \omega_0 \subset\subset \omega$. Every term in each $I_{i,j}$ is bounded, in absolute value, by a constant multiplied by $I_\Omega(v, \lambda, s)$ for s and λ large enough. (Use the corresponding term in $I_\Omega(v, \lambda, s)$ with the same derivative of v.) By using (3.4), we get that $I_{\Omega\setminus\omega_0}(v, \lambda, s)$ is positive.

Then we have, for $\lambda \geq \lambda_1$, $s \geq s_1$ and for some positive constant C, that

$$\sum_{i=1,j=1}^{i=4,j=3} I_{i,j} \geq C\, I_{\Omega\setminus\omega}(v, s, \lambda) = C\, I_\Omega(v, s, \lambda) - C\, I_{\omega_0}(v, s, \lambda). \tag{A.6}$$

In order to add a fourth order term to the inequality, let us note that from (A.2) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |v_{4x}|^2 &= \left|P_1 v - 6\lambda^2 s^2 (\beta')^2 \xi^2 v_{2x} - \lambda^4 s^4 (\beta')^4 \xi^4 v - 6\lambda^2 s^2 ((\beta')^2 \xi^2)_x v_x\right|^2 \\ &\leq C(|P_1 v|^2 + \lambda^4 s^4 \xi^4 |v_{2x}|^2 + \lambda^8 s^8 \xi^8 |v|^2 + \lambda^6 s^4 \xi^4 |v_x|^2) \end{aligned}$$

and then

$$\frac{1}{s} \iint \frac{1}{\xi} |v_{4x}|^2 \leq C \iint |P_1 v|^2 + C\, I_\Omega(v, s, \lambda). \tag{A.7}$$

From (A.1), (A.3), (A.4), (A.6), and (A.7), we get

$$\begin{aligned} \iint |P_1 v|^2 + \iint |P_2 v|^2 + \frac{1}{s} \iint \frac{1}{\xi} |v_{4x}|^2 + I_\Omega(v, s, \lambda) \\ \leq \iint |\tilde{P} v|^2 + C\, I_{\omega_0}(v, s, \lambda). \end{aligned} \tag{A.8}$$

In order to get an inequality with only an L^2 observation, let ω_1 be an open set such that $\omega_0 \subset\subset \omega_1 \subset\subset \omega$, and a cutoff function $\rho \in C_0^\infty(\omega_1)$ such that $\rho = 1$ in ω_0. Then

$$\begin{aligned} s\lambda^2 \iint_{\omega_0} \xi |v_{3x}|^2 &\leq s\lambda^2 \iint_{\omega_1} \rho \xi |v_{3x}|^2 \\ &= s\lambda^2 \left(- \iint_{\omega_0} \rho \xi v_{4x} v_2 + \frac{1}{2} \iint_{\omega_0} (\rho\xi)_{2x} |v_{2x}|^2 \right) \\ &\leq \varepsilon s^{-1} \iint_\Omega \xi^{-1} |v_{4x}|^2 + C_\varepsilon\, s^3 \lambda^4 \iint_{\omega_1} \xi^3 |v_{2x}|^2. \end{aligned}$$

In the same way, if ω_2 is an open set such that $\omega_1 \subset\subset \omega_2 \subset\subset \omega$, then

$$s^3 \lambda^4 \iint_{\omega_1} \xi |v_{2x}|^2 \leq \varepsilon s \lambda^2 \iint_\Omega \xi |v_{3x}|^2 + C_\varepsilon\, s^5 \lambda^6 \iint_{\omega_2} \xi^5 |v_x|^2$$

and

$$s^5 \lambda^6 \iint_{\omega_2} \xi^5 |v_x|^2 \leq \varepsilon s^3 \lambda^4 \iint_\Omega \xi^3 |v_{2x}|^2 + C_\varepsilon\, s^7 \lambda^8 \iint_\omega \xi^7 |v|^2. \tag{A.9}$$

From (A.8) to (A.9), we get

$$\begin{aligned} \iint \big(|P_1 v|^2 + |P_2 v|^2 + s^7 \lambda^8 \xi^7 |v|^2 + s^5 \lambda^6 \xi^5 |v_x|^2 + s^3 \lambda^4 \xi^3 |v_{2x}|^2 \\ + s\lambda^2 \xi |v_{3x}|^2 + s^{-1} \xi^{-1} |v_{4x}|^2 \big) \leq C \iint |\tilde{P} v|^2 + s^7 \lambda^8 C \iint_\omega \xi^7 |v|^2. \end{aligned} \tag{A.10}$$

Let us get the Carleman inequality for the function $\varphi = e^{s\alpha}v$. From the definitions of α and ξ, we have that

$$\begin{aligned} &e^{-2s\alpha}\big(s^7\lambda^8\xi^7|\varphi|^2 + s^5\lambda^6\xi^5|\varphi_x|^2 + s^3\lambda^4\xi^3|\varphi_{2x}|^2 + s\lambda^2\xi|\varphi_{3x}|^2 + s^{-1}\xi^{-1}|\varphi_{4x}|^2\big) \\ &\qquad \le s^7\lambda^8\xi^7|v|^2 + s^5\lambda^6\xi^5|v_x|^2 + s^3\lambda^4\xi^3|v_{2x}|^2 + s\lambda^2\xi|v_{3x}|^2 + s^{-1}\xi^{-1}|v_{4x}|^2. \end{aligned} \tag{A.11}$$

From (A.11), (A.10) and taking into account that $\tilde{P}v = e^{-s\alpha}Pw$, we get inequality (3.14), which concludes the proof of Theorem 3.3. $\square$

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