J. Differential Equations 250 (2011) 2024–2044

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Local exact controllability to the trajectories of the 1-D Kuramoto–Sivashinsky equation ☆

Eduardo Cerpa *, Alberto Mercado

Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile

ARTICLE INFO

Article history:
Received 18 November 2009
Revised 19 May 2010
Available online 30 December 2010

MSC:
93B05
35K55
93C10

Keywords:
Kuramoto–Sivashinsky equation
Null controllability
Carleman estimates

ABSTRACT

We are concerned with the boundary controllability to the trajectories of the Kuramoto–Sivashinsky equation. By using a Carleman estimate, we obtain the null controllability of the linearized equation around a given solution. From a local inversion theorem we get the local controllability to the trajectories of the nonlinear system.

1. Introduction

In this paper we consider the following Kuramoto–Sivashinsky (KS) control system

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} + y y_x = 0, & x \in (0,1),\ t > 0, \\ y(t,0) = h_1(t), \qquad y(t,1) = 0, & t > 0, \\ y_x(t,0) = h_2(t), \qquad y_x(t,1) = 0, & t > 0, \end{cases} \tag{1}$$

where the state is given by $y = y(t,x)$ and the time-dependent functions h_1, h_2 are boundary controls. This equation, where the real positive number γ is called the "anti-diffusion" parameter, was

☆ This work has been partially supported by FONDECYT 11090161 (E. Cerpa), FONDECYT 11080130 (A. Mercado) and CMM-BASAL grants.

* Corresponding author.
E-mail addresses: eduardo.cerpa@usm.cl (E. Cerpa), alberto.mercado@usm.cl (A. Mercado).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.12.015

derived independently by Kuramoto and Tsuzuki in [19,20,18] as a model for phase turbulence in reaction–diffusion systems and by Sivashinsky in [26] as a model for plane flame propagation. This nonlinear partial differential equation describes incipient instabilities in a variety of physical and chemical systems (see, for instance, [6] and [16]). From a mathematical point of view, well-posedness and dynamical properties of KS equations have attracted a lot of attention since the pioneer articles [22,23,11].

We are interested in controllability properties of system (1). For parabolic control problems, in general it is not possible to steer the system to an arbitrary prescribed state. Thus, we do not expect the exact controllability to be true for the KS control system.

In this work, we address the problem of steering the solutions of system (1) to a given trajectory of the KS equation. More precisely, given $T > 0$ and an appropriate space X, we say that system (1) is *exactly controllable to the trajectories* if for any initial condition $y_0 \in X$ and for any trajectory u satisfying

$$\begin{cases} u_t + u_{xxxx} + \gamma u_{xx} + uu_x = 0, & x \in (0,1),\ t > 0, \\ u(t,0) = 0, \quad u(t,1) = 0, & t > 0, \\ u_x(t,0) = 0, \quad u_x(t,1) = 0, & t > 0, \end{cases} \tag{2}$$

there exist boundary controls h_1, h_2 such that the solution of (1) with $y(0,\cdot) = y_0$ satisfies $y(T,\cdot) = u(T,\cdot)$. We say that the *local exact controllability to the trajectories* holds if we can find controls as above whenever $\|y_0 - u(0,\cdot)\|_X$ is small enough. In this paper we will prove this last property for system KS.

Let y, u be solutions of (1) and (2) respectively. The function $q := y - u$ satisfies

$$\begin{cases} q_t + q_{xxxx} + \gamma q_{xx} + uq_x + u_x q + q_x q = 0, \\ q(t,0) = h_1(t), \quad q(t,1) = 0, \\ q_x(t,0) = h_2(t), \quad q_x(t,1) = 0. \end{cases} \tag{3}$$

Given a state q_0, we wonder if there exist some controls h_1, h_2 such that the solution $q = q(t,x)$ of (3) with initial condition $q(0,x) = q_0(x)$ satisfies $q(T,x) = 0$. If this controls exist for any state q_0 lying in an appropriate space, we say that (3) is *null controllable* in time T. We can easily see that the controllability to the trajectories of (1) is equivalent to the null controllability of (3). Therefore from now on, we focus on the proof of the latter property.

In order to study the control system (3), we first prove that the following linear KS equation is null controllable

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} + uy_x + u_x y = 0, \\ y(t,0) = h_1(t), \quad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \quad y_x(t,1) = 0. \end{cases} \tag{4}$$

To do that, we obtain a global Carleman estimate for the adjoint system of (4). From this estimate, we deduce an observability inequality which is equivalent to the null controllability of the direct system (4). Then, we show that the local null controllability holds for the nonlinear control system (3). It will be done by using an inverse function argument.

The main result of this paper is the following.

Theorem 1.1. *Let $T > 0$. Let $u \in L^\infty(0,T;H_0^2(0,1))$ be a solution of system (2). There exists $r > 0$ such that for any $y_0 \in H^{-2}(0,1)$ with $\|y_0 - u(0,\cdot)\|_{H^{-2}(0,1)} \leqslant r$, there exist $h_1, h_2 \in L^2(0,T)$ and $y \in C([0,T], H^{-2}(0,1)) \cap L^2(0,T;L^2(0,1))$ satisfying (1), $y(0,\cdot) = y_0$ and $y(T,\cdot) = u(T,\cdot)$.*

Remark 1.2. The reference trajectory u is required to be more regular than the solution y. This is due to the fact that u will also appear into the adjoint equation of (4) which will be studied in a more regular framework.

Remark 1.3. The linear control system (4) with $u = 0$ has been studied in [4] by using a spectral approach. This system is null controllable with two controls. Moreover, if one can acts on the system with just one control input, the system is still controllable if and only if the "anti-diffusion" parameter γ does not belong to a countably infinite discrete set of critical values. If γ belongs to this set, there exists a finite-dimensional space of initial conditions that cannot be driven to zero with only one control. A similar situation was found by Rosier in [24] for a Korteweg–de Vries (KdV) equation. Later, in [8,3,5] the authors proved that the KdV equation is exactly controllable in despite of the lack of this property for the linearized system. In the KS context, an interesting open problem would be to study these critical cases. We could wonder if the null controllability holds for the nonlinear system (1) with only one control.

Remark 1.4. Other control topics for the KS equation have been studied in the literature. For instance [2,7,21] deal with the stabilization problem and [17] is concerned with the robust control problem.

Remark 1.5. The null controllability and the controllability to the trajectories for other nonlinear partial differential equations has been studied by other authors. It has been done by using either internal controls (see [9,14,15,10,13]) or boundary controls (see [12,25]).

This article is organized as follows. First, in Section 2 we establish the well-posedness framework used in this paper. Next, Section 3 is devoted to the linear control system. We prove a global Carleman estimate in Section 3.1 and we use it in Section 3.2 in order to obtain the observability inequality. Thus, we obtain the null controllability of the linearized KS equation. Finally, Section 4 is concerned with the nonlinear system. We get the local null controllability and consequently the local exact controllability to the trajectories by means of an inverse function theorem.

2. The Cauchy problem

In this section we prove the well-posedness results we need along this paper for both linear and nonlinear equations. We can restrict ourselves to the case $u = 0$. The general case can be proved by using a classical fixed point argument thanks to the regularity asked to the reference trajectory.

In order to consider boundary conditions in $L^2(0,T)$, we will define the solution of (4) by transposition. Therefore, we have to study the corresponding adjoint equation (see (5) below).

2.1. Adjoint equation

It is not difficult to see that the self-adjoint operator

$$A : w \in D(A) \subset L^2(0,1) \longmapsto -w'''' - \gamma w'' \in L^2(0,1),$$
$$D(A) := H^4(0,1) \cap H_0^2(0,1),$$

has a compact resolvent. Hence the spectrum of A is a discrete set $\sigma(A) = \{\sigma_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ consisting only of eigenvalues, which satisfy $\lim_{k\to\infty} \sigma_k = -\infty$. Furthermore, the eigenfunctions define an orthonormal basis of $L^2(0,1)$. Thanks to classical semigroup theory, we have that if $g \in L^1(0,T;L^2(0,1))$, then the solution of

$$\begin{cases} -w_t + w_{xxxx} + \gamma w_{xx} = g, \\ w(t,0) = 0, \qquad w(t,1) = 0, \\ w_x(t,0) = 0, \qquad w_x(t,1) = 0, \\ w(T,x) = 0 \end{cases} \tag{5}$$

satisfies $w \in C([0,T];L^2(0,1))$. Moreover, if $g \in C^1([0,T];L^2(0,1))$, then $w \in C([0,T];D(A)) \cap C^1([0,T];L^2(0,1))$. As we need more precise information about the regularity of the solutions, we obtain the following results, by using energy estimates as in [17].

Proposition 2.1. *Let G be either $L^2(0,T;L^2(0,1))$ or $L^1(0,T;H_0^2(0,1))$. If $g \in G$, then the solution of* (5) *satisfies $w \in L^2(0,T;H^4(0,1)) \cap C([0,T];H_0^2(0,1))$. Moreover there exists a constant $C > 0$ such that*

$$\|w\|_{L^2(0,T;H^4(0,1))\cap C([0,T];H_0^2(0,1))} \leqslant C\|g\|_G$$

for every $g \in G$.

Proof. We consider g regular enough. Let us replace t by $T-t$. System (5) becomes

$$w_t + w_{xxxx} + \gamma w_{xx} = g \tag{6}$$

with homogeneous boundary conditions and null initial data $w(0,x) = 0$. Let us multiply the equation by w_{xxxx} and integrate on $(0,1)$. We obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 |w_{xx}|^2\,dx + \int_0^1 |w_{xxxx}|^2\,dx &= -\gamma\int_0^1 w_{xx}w_{xxxx}\,dx + \int_0^1 g w_{xxxx}\,dx \\ &\leqslant \frac{1}{2}\int_0^1 |w_{xxxx}|^2\,dx + \frac{\gamma^2}{2}\int_0^1 |w_{xx}|^2 + \int_0^1 g w_{xxxx}\,dx\end{aligned}$$

and then

$$\frac{d}{dt}\int_0^1 |w_{xx}|^2\,dx + \int_0^1 |w_{xxxx}|^2\,dx \leqslant C\int_0^1 |w_{xx}|^2 + C\int_0^1 g w_{xxxx}\,dx. \tag{7}$$

Let us first obtain the norm corresponding to the space $G = L^1(0,T;H_0^2(0,1))$. From (7) we have

$$\frac{d}{dt}\int_0^1 |w_{xx}|^2\,dx \leqslant C\int_0^1 |w_{xx}|^2 + C\int_0^1 g_{xx}w_{xx}\,dx, \tag{8}$$

and Gronwall's lemma implies the existence of $C > 0$ such that

$$\|w\|^2_{L^\infty([0,T];H_0^2(0,1))} \leqslant C\int_0^T\int_0^1 |g_{xx}w_{xx}|\,dx\,dt$$

and then

$$\|w\|_{L^\infty([0,T];H_0^2(0,1))} \leqslant \|g\|_{L^1([0,T];H_0^2(0,1))}.$$

Taking into account this estimate, we integrate (7) on $(0,T)$ and we obtain $C > 0$ such that

$$\|w\|_{L^2((0,T);H^4(0,1))} \leqslant C\|g\|_{L^1((0,T);H_0^2(0,1))}.$$

Thus, by a density argument we can prove that $g \in L^1(0,T;H_0^2(0,1))$ implies that the solution w lies in $L^2(0,T;H^4(0,1)) \cap C([0,T];H_0^2(0,1))$.

In order to get the norm of the space $G = L^2(0,T;L^2(0,1))$, let us note that the second term in the right-hand side of inequality (7) can be bounded from above by

$$\frac{C}{2}\int_0^1 |g|^2\,dx + \frac{1}{2}\int_0^1 |w_{xxxx}|^2\,dx$$

and in this way the term $\frac{1}{2}\int_0^1 |w_{xxxx}|^2\,dx$ can be absorbed into the left-hand side. As before, by applying the Gronwall lemma, we obtain the existence of a constant $C > 0$ such that

$$\|w\|_{L^2(0,T;H^4(0,1))\cap C([0,T];H_0^2(0,1))} \leqslant C\|g\|_{L^2(0,T;L^2(0,1))}.$$

Using a density argument we end the proof of Proposition 2.1. □

Corollary 2.2. *If either $g \in L^2(0,T;L^2(0,1))$ or $g \in L^1(0,T;H_0^2(0,1))$, then $w \in L^\infty(0,T;W^{1,\infty}(0,1))$.*

Proof. Obvious from the fact that $H_0^2(0,1)$ embeds continuously into $W^{1,\infty}(0,1)$. □

In order to be able to define the solution of (4) with boundary conditions $h_1, h_2 \in L^2(0,T)$, we need the following result.

Corollary 2.3. *If either $G = L^2(0,T;L^2(0,1))$ or $G = L^1(0,T;H_0^2(0,1))$, then there exists $C > 0$ such that for any $g \in G$, the solution w of (5) satisfies*

$$\left\|w_{xx}(\cdot,0)\right\|_{L^2(0,T)} + \left\|w_{xxx}(\cdot,0)\right\|_{L^2(0,T)} \leqslant C\|g\|_G.$$

Proof. This inequality is a direct consequence of Proposition 2.1 and the continuous embedding of $H^4((0,1))$ into $C^3([0,1])$. □

Remark 2.4. If $g \in L^2(0,T;L^2(0,1))$, then the solution satisfies $w \in L^2(0,T;H^4(0,1))$ and therefore $w_{xxx} \in L^2(0,T;H^1(0,1))$. Moreover, by using the equation, we get $w \in H^1(0,T;L^2(0,1))$ and therefore $w_{xxx} \in H^1(0,T;H^{-3}(0,1))$. By interpolation, we obtain $w_{xxx}(t,0) \in H^{\frac{1}{8}-\epsilon}(0,T)$ for any positive number ϵ. In the same way we find $w_{xx}(t,0) \in H^{\frac{3}{8}-\epsilon}(0,T)$. This regularity would allow us to consider the direct Cauchy problem with boundary data $h_1 \in H^{-\frac{1}{8}+\epsilon}(0,T)$ and $h_2 \in H^{-\frac{3}{8}+\epsilon}(0,T)$. However, we will stay within L^2-regularity because of the control framework we will consider later.

2.2. Direct linear equation

Let us define what we mean by a solution of the linear KS equation.

Definition 2.5. Let $y_0 \in H^{-2}(0,1)$, $f \in L^1(0,T;W^{-1,1}(0,1))$ and $h_1, h_2 \in L^2(0,T)$. A solution of the equation

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = f, \\ y(t,0) = h_1(t), \qquad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \qquad y_x(t,1) = 0, \\ y(0,x) = y_0(x), \end{cases} \tag{9}$$

is a function $y \in L^2(0,T;L^2(0,1))$ such that for any $g \in L^2(0,T;L^2(0,1))$,

$$\int_0^T \int_0^1 y(t,x)g(t,x)\,dx\,dt = \langle y_0, w(0,x)\rangle_{H^{-2}(0,1),H_0^2(0,1)} - \int_0^T h_1(t)w_{xxx}(t,0)\,dt + \int_0^T h_2(t)w_{xx}(t,0)\,dt$$
$$+ \langle f, w\rangle_{L^1(0,T;W^{-1,1}(0,1)),L^\infty(0,T;W^{1,\infty}(0,1))}, \tag{10}$$

where $w = w(t,x)$ is the solution of (5).

Next theorem establishes that the solutions of (9) belong to the space

$$\mathcal{B} := C\big([0,T]; H^{-2}(0,1)\big) \cap L^2\big(0,T; L^2(0,1)\big).$$

Theorem 2.6. *Let $y_0 \in H^{-2}(0,1)$, $f \in L^1(0,T;W^{-1,1}(0,1))$, and $h_1, h_2 \in L^2(0,T)$. Then there exists a unique solution $y \in \mathcal{B}$ of Eq. (9).*

Proof. From Proposition 2.1 and Corollary 2.3, the right-hand side of (10) defines, for each $h_1, h_2 \in L^2(0,T)$, $f \in L^1(0,T;W^{-1,1}(0,1))$ and $y_0 \in H^{-2}(0,1)$, a linear bounded functional

$$L_h^0 : g \in L^2\big(0,T;L^2(0,1)\big) \longmapsto L_h^0(g) \in \mathbb{R},$$

and therefore, from the Riesz representation theorem, we obtain the existence and uniqueness of a solution $y \in L^2(0,T;L^2(0,1))$. By using the same results with $g \in L^1(0,T;H_0^2(0,1))$, we prove that the same L_h^0 defines a linear bounded functional on $L^1(0,T;H_0^2(0,1))$. Thus, we see that in fact $y \in \mathcal{B}$. □

2.3. Nonlinear equation

Theorem 2.7. *There exists a positive real number r such that for any $y_0 \in H^{-2}(0,1)$, $h_1, h_2 \in L^2(0,T)$ and $f \in L^1(0,T;W^{-1,1}(0,1))$ satisfying*

$$\|y_0\|_{H^{-2}(0,1)} + \|h_1\|_{L^2(0,T)} + \|h_2\|_{L^2(0,T)} + \|f\|_{L^1(0,T;W^{-1,1}(0,1))} \leqslant r, \tag{11}$$

the nonlinear equation

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} + yy_x = f, \\ y(t,0) = h_1(t), \qquad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \qquad y_x(t,1) = 0, \\ y(0,x) = y_0(x), \end{cases} \tag{12}$$

has a unique solution $y \in \mathcal{B}$.

Proof. Let us consider $y_0 \in H^{-2}(0,1)$, $h_1, h_2 \in L^2(0,T)$ and $f \in L^1(0,T;W^{-1,1}(0,1))$ satisfying (11) for $r > 0$ to be chosen later.

We define the following map

$$\Pi : \ell \in L^2\big(0,T;L^2(0,1)\big) \longmapsto y \in L^2\big(0,T;L^2(0,1)\big) \tag{13}$$

where y is the solution of

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = f - \ell\ell_x, \\ y(t,0) = h_1(t), \qquad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \qquad y_x(t,1) = 0, \\ y(0,x) = y_0(x). \end{cases} \tag{14}$$

Let us notice that a function y is a fixed point of this map Π if and only if y is a solution of our nonlinear KS equation (12).

From Theorem 2.6 and by using $\|\ell\ell_x\|_{L^1(0,T;W^{-1,1})} \leqslant \frac{1}{2}\|\ell\|^2_{L^2(0,T;L^2(0,1))}$, we get

$$\left\|\Pi(\ell)\right\|_{L^2(0,T;L^2(0,1))} \leqslant C\big(\|y_0\|_{H^{-2}(0,1)} + \|h_1\|_{L^2(0,T)} + \|h_2\|_{L^2(0,T)} + \|f\|_{L^1(0,T;W^{-1,1}(0,1))} + \|\ell\|^2_{L^2(0,T;L^2(0,1))}\big).$$

For each $R > 0$, let us denote the ball of radius R and centered at the origin by

$$B(0,R) := \big\{\ell \in L^2\big(0,T;L^2(0,1)\big);\ \|\ell\|_{L^2(0,T;L^2(0,1))} \leqslant R\big\}.$$

We see that if $r > 0$ and $R > 0$ are chosen such that $C(r + R^2) \leqslant R$, we obtain that $\Pi|_{B(0,R)} \subset B(0,R)$. Let us verify that we can choose R such that Π is a contraction. Let $\ell, \bar{\ell} \in L^2(0,T;L^2(0,1))$. The function $\hat{y} := (\Pi(\bar{\ell}) - \Pi(\ell))$ is the solution of

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = \bar{\ell}\bar{\ell}_x - \ell\ell_x, \\ y(t,0) = 0, \qquad y(t,1) = 0, \\ y_x(t,0) = 0, \qquad y_x(t,1) = 0, \\ y(0,x) = 0. \end{cases} \tag{15}$$

From Theorem 2.6, we get

$$\left\|\Pi(\bar{\ell}) - \Pi(\ell)\right\|_{L^2(0,T;L^2(0,1))} \leqslant C\|\bar{\ell}\bar{\ell}_x - \ell\ell_x\|_{L^1(0,T;W^{-1,1}(0,1))}$$

and using that

$$\|\bar{\ell}\bar{\ell}_x - \ell\ell_x\|_{L^1(0,T;W^{-1,1}(0,1))} = \frac{1}{2}\left\|\bar{\ell}^2 - \ell^2\right\|_{L^1(0,T;L^1(0,1))} \leqslant \frac{1}{2}\|\bar{\ell} + \ell\|_{L^2(0,T;L^2(0,1))}\|\bar{\ell} - \ell\|_{L^2(0,T;L^2(0,1))}$$

we obtain

$$\left\|\Pi(\bar{\ell}) - \Pi(\ell)\right\|_{L^2(0,T;L^2(0,1))} \leqslant CR\|\bar{\ell} - \ell\|_{L^2(0,T;L^2(0,1))}$$

and therefore the map Π is a contraction if $CR < 1$. By applying the Banach fixed point theorem, we can conclude that Π has a unique fixed point which is the solution of Eq. (12). □

3. Linear control system

In this section, we study the boundary control of the linear system

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} + uy_x + u_x y = f, \\ y(t,0) = h_1(t), \qquad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \qquad y_x(t,1) = 0, \\ y(0,x) = y_0(x), \end{cases} \tag{16}$$

where $u = u(t,x)$ is a given function.

Let us take a well-posedness framework (U_1, U_2, X, Y, Z) for this system. By this we mean that given $h_1 \in U_1$, $h_2 \in U_2$, $f \in Y$ and $y_0 \in X$, then there exists a unique $y \in Z$ solution of Eq. (16). This system is said to be null controllable if for any state $y_0 \in X$ and for any $f \in Y$, one can find controls $h_1 \in U_1$, $h_2 \in U_2$ such that the solution y of (16) satisfies $y(T) = 0$. It is a well-known fact that by duality, this null-controllability property is equivalent to the existence of a constant $C > 0$ such that

$$\|w\|_{Y^*} + \big\|w(0,x)\big\|_{X^*} \leqslant C\big(\|g\|_{Z^*} + \big\|w_{xxx}(t,0)\big\|_{U_1^*} + \big\|w_{xx}(t,0)\big\|_{U_2^*}\big) \tag{17}$$

for every $w_T \in X^*$ and $g \in Z^*$, where $*$ stands for dual space and w is the solution of the adjoint linear system given by

$$\begin{cases} -w_t + w_{xxxx} + \gamma w_{xx} - u w_x = g, \\ w(t,0) = 0, \qquad w(t,1) = 0, \\ w_x(t,0) = 0, \qquad w_x(t,1) = 0, \\ w(T,x) = w_T(x). \end{cases} \tag{18}$$

Inequality (17) is called an *observability inequality* for Eq. (18).

In this section we prove a Carleman estimate for Eq. (18). Then we use it in order to prove the observability inequality (17) within an appropriate framework (U_1, U_2, X, Y, Z). Thus, we get the null controllability of Eq. (16).

3.1. Carleman estimate

In this part of the work we shall use an abbreviated notation for the derivatives and integrals. We write, for k integer, w_{kx} instead of $\frac{\partial^k w}{\partial x^k}$ and $\iint$ instead of $\int_0^T \int_0^1$, avoiding the symbols $dx\,dt$ in the last case.

In order to deduce a Carleman estimate for the differential operator

$$Pw = -w_t + w_{4x} + \gamma w_{2x} - u w_x,$$

we take a function $\beta \in C^4([0,1])$ and define $\varphi(x,t) := \frac{\beta(x)}{t(T-t)}$, called the weight function. For each $\lambda > 0$ let us consider the space

$$W_\lambda := \big\{e^{-\lambda\varphi} w;\ w \text{ is solution of (18) with } g \in L^2\big(0,T; L^2(0,1)\big)\big\}$$

and define for each $v \in W_\lambda$,

$$P_\varphi v = e^{-\lambda\varphi} P\big(e^{\lambda\varphi} v\big). \tag{19}$$

Thus, we can write $P_\varphi v = P_1 v + P_2 v + R v$, where

$$\begin{aligned} P_1 v &= 6\lambda^2\varphi_x^2 v_{2x} + \lambda^4\varphi_x^4 v + v_{4x} + 12\lambda^2\varphi_x\varphi_{2x} v_x, \\ P_2 v &= -v_t + 4\lambda^3\varphi_x^3 v_x + 4\lambda\varphi_x v_{3x} + 5\lambda^3\varphi_x^2\varphi_{2x} v, \\ Rv &= \lambda\varphi_{4x} v + 2\lambda^2\varphi_{3x}\varphi_x v - \lambda\varphi_t v + 3\lambda^2\varphi_{2x}^2 v + 2\lambda^2\varphi_x\varphi_{3x} v + \lambda\gamma\varphi_{2x} v + \lambda^2\gamma\varphi_x^2 v \\ &\quad + 6\lambda\varphi_{2x} v_{2x} + 4\lambda\varphi_{3x} v_x - u v_x + 2\lambda\gamma\varphi_x v_x - \lambda u\varphi_x v + \gamma v_{2x}. \end{aligned} \tag{20}$$

In the following lemma, we develop the L^2-product between $P_1 v$ and $P_2 v$.

Lemma 3.1. *With the notation stated above, we have the decomposition*

$$\iint P_1 v P_2 v = I(v) + I(v_x) + I(v_{xx}) + I(v_{3x}) + I_x$$

where

$$\begin{aligned}
I(v) &= -9\lambda^7 \iint |v|^2 \varphi_x^6 \varphi_{xx} + 2\lambda^4 \iint |v|^2 \varphi_x^3 \varphi_{xt} + 5\lambda^5 \iint |v|^2 \big(\varphi_x^4 \varphi_{2x}\big)_{2x} \\
&\quad - 30\lambda^5 \iint |v|^2 \big(\varphi_x^3 \varphi_{2x}^2\big)_x + \frac{5}{2}\lambda^3 \iint |v|^2 \big(\varphi_x^2 \varphi_{2x}\big)_{4x}, \\
I(v_x) &= -12\lambda^5 \iint |v_x|^2 \varphi_x^4 \varphi_{xx} - 6\lambda^2 \iint |v_x|^2 \varphi_x \varphi_{xt} + 8\lambda^3 \iint |v_x|^2 \big(\varphi_x^2 \varphi_{2x}\big)_{2x}, \\
I(v_{2x}) &= -61\lambda^3 \iint |v_{2x}|^2 \varphi_x^2 \varphi_{xx}, \\
I(v_{3x}) &= -2\lambda \iint |v_{3x}|^2 \varphi_{xx},
\end{aligned}$$

and

$$\begin{aligned}
I_x &= 10\lambda^3 \int_0^T \big|v_{2x}(1,t)\big|^2 \varphi_x^3(1,t)\,dt - 10\lambda^3 \int_0^T \big|v_{2x}(0,t)\big|^2 \varphi_x^3(0,t)\,dt \\
&\quad + 2\lambda \int_0^T \big|v_{3x}(1,t)\big|^2 \varphi_x(1,t)\,dt - 2\lambda \int_0^T \big|v_{3x}(0,t)\big|^2 \varphi_x(0,t)\,dt.
\end{aligned}$$

Proof. See Appendix A. □

The next step is to estimate from below the L^2-norm of $P_\varphi v$. In order to do that, we ask the function β to satisfy

$$0 < \eta \leqslant \frac{d^k \beta}{dx^k}(x), \quad \forall x \in [0,1], \text{ for } k = 0, 1 \tag{21}$$

and

$$\frac{d^2 \beta}{dx^2}(x) \leqslant -\eta < 0, \quad \forall x \in [0,1], \tag{22}$$

for some positive constant η.

Under this hypothesis, it is straightforward to see that the function $\varphi(x,t) = \frac{\beta(x)}{t(T-t)}$ satisfies

$$\frac{1}{C}\big|\partial_x^k \varphi(x,t)\big| \leqslant \varphi(x,t) \leqslant C\big|\partial_x^k \varphi(x,t)\big|, \quad \forall (x,t) \in [0,1] \times (0,T), \text{ for } k = 1, 2 \tag{23}$$

and also

$$\big|\partial_x^k \partial_t \varphi(x,t)\big| \leqslant C\varphi^2(x,t), \quad \forall (x,t) \in [0,1] \times (0,T), \text{ for } k = 1, \dots, 4, \tag{24}$$

for some positive constant C.

Remark 3.2. It is not difficult to show that such a weight function β actually exists. Take for instance $\beta(x) = -(x-2)^2 + 8$.

Proposition 3.3. *Let $\beta \in C^4([0,1])$ satisfying (21) and (22). There exist $\delta > 0$ and $\lambda_0 > 0$ such that for every $\lambda \geqslant \lambda_0$ and $v \in W_\lambda$, we have*

$$\int_0^T \int_0^1 P_1 v P_2 v \, dx \, dt \geqslant \delta \|v\|_{\varphi,\lambda}^2 + I_x \tag{25}$$

where φ and I_x are defined above, and we have denoted

$$\|v\|_{\varphi,\lambda}^2 = \lambda^7 \iint |v|^2 \varphi^7 + \lambda^5 \iint |v_x|^2 \varphi^5 + \lambda^3 \iint |v_{2x}|^2 \varphi^3 + \lambda \iint |v_{3x}|^2 \varphi. \tag{26}$$

Proof. Following the notation in Lemma 3.1 we have that

$$\iint P_1 v P_2 v = \sum_{k=0}^{3} I(v_{kx}) + I_x. \tag{27}$$

We will bound from below each integral at the right-hand side of (27). Taking into account (23) and (24) it follows that there exist positive constants δ, C such that

$$\begin{aligned}
I(v) &= 9\lambda^7 \iint |v|^2 \left|\varphi_x^6 \varphi_{xx}\right| \\
&\quad + \iint |v|^2 \left(2\lambda^4 \varphi_x^3 \varphi_{xt} + 5\lambda^5 \left(\varphi_x^4 \varphi_{2x}\right)_{2x} - 30\lambda^5 \left(\varphi_x^3 \varphi_{2x}^2\right)_x + (5/2)\lambda^3 \left(\varphi_x^2 \varphi_{2x}\right)_{4x}\right) \\
&\geqslant 2\delta\lambda^7 \iint |v|^2 \varphi^7 - C \iint |v|^2 \left(\lambda^4 \varphi^5 + \lambda^5 \varphi^5 + \lambda^5 \varphi^5 + \lambda^3 \varphi^3\right) \\
&\geqslant 2\delta\lambda^7 \iint |v|^2 \varphi^7 - C\lambda^5 \iint |v|^2 \varphi^7 \\
&\geqslant \delta\lambda^7 \iint |v|^2 \varphi^7
\end{aligned} \tag{28}$$

for λ large enough.

In the same way we have

$$\begin{aligned}
I(v_x) &= -12\lambda^5 \iint |v_x|^2 \varphi_x^4 \varphi_{xx} + \iint |v_x|^2 \left(-6\lambda^2 \varphi_x \varphi_{xt} + 8\lambda^3 \left(\varphi_x^2 \varphi_{2x}\right)_{2x}\right) \\
&\geqslant 2\delta\lambda^5 \iint |v_x|^2 \varphi^5 - C \iint |v_x|^2 \varphi^3 \left(\lambda^2 + \lambda^3\right) \\
&\geqslant \delta\lambda^5 \iint |v_x|^2 \varphi^5,
\end{aligned} \tag{29}$$

$$I(v_{2x}) \geqslant \delta\lambda^3 \iint |v_{2x}|^2 \varphi^3 \tag{30}$$

and

$$I(v_{3x}) \geqslant \delta\lambda \iint |v_{3x}|^2\varphi \tag{31}$$

for λ large enough.

From inequalities (27) to (31) we get (25). □

Proposition 3.4. *Under the same hypothesis than Proposition* 3.3*, there exist $C > 0$, $\lambda_0 > 0$ such that*

$$\|P_1 v\|^2_{L^2((0,T)\times(0,1))} + \|P_2 v\|^2_{L^2((0,T)\times(0,1))} + \|v\|^2_{\lambda,\varphi} \leqslant C\|P_\varphi v\|^2_{L^2((0,T)\times(0,1))} - CI_x \tag{32}$$

for every $\lambda \geqslant \lambda_0$ and $v \in W_\lambda$.

Proof. With the notation stated in (20) and Proposition 3.3 it is not difficult to check that

$$\begin{aligned}\|Rv\|^2_{L^2((0,T)\times(0,1))} &\leqslant C\left(\lambda^4 \iint \varphi^4|v|^2 + \lambda^2 \iint \varphi^2|v_x|^2 + \lambda^2 \iint \varphi^2|v_{xx}|^2\right)\\ &\leqslant C\lambda^{-1}\|v\|^2_{\lambda,\varphi}.\end{aligned} \tag{33}$$

Then we have

$$\begin{aligned}\|P_1 v\|^2_{L^2} + 2\iint P_1 v P_2 v + \|P_2 v\|^2_{L^2} &= \|P_\varphi v - Rv\|^2_{L^2}\\ &\leqslant 2\|P_\varphi v\|^2_{L^2} + 2\|Rv\|^2_{L^2}\\ &\leqslant 2\|P_\varphi v\|^2_{L^2} + C\lambda^{-1}\|v\|^2_{\lambda,\varphi}\\ &\leqslant 2\|P_\varphi v\|^2_{L^2} + \frac{\delta}{2}\|v\|^2_{\lambda,\varphi}\end{aligned} \tag{34}$$

for λ large enough. From (34) and Proposition 3.3 we get (32). □

We are now able to get a Carleman estimate for the solutions of Eq. (18).

Theorem 3.5. *Let $u \in L^\infty(0,T; H^2_0(0,1))$ and $\varphi(t,x) = \frac{\beta(x)}{t(T-t)}$ with $\beta \in C^4([0,1])$ satisfying* (21) *and* (22)*. There exist $C > 0$ and $\lambda_0 > 0$ such that*

$$\begin{aligned}&\lambda^7 \iint |w|^2 e^{-2\lambda\varphi}\varphi^7 + \lambda^5 \iint |w_x|^2 e^{-2\lambda\varphi}\varphi^5 + \lambda^3 \iint |w_{2x}|^2 e^{-2\lambda\varphi}\varphi^3 + \lambda \iint |w_{3x}|^2 e^{-2\lambda\varphi}\varphi\\ &\quad\leqslant C\left(\iint |g|^2 e^{-2\lambda\varphi} + \lambda^3 \int_0^T \left|w_{2x}(0,t)\right|^2 e^{-2\lambda\varphi(0,t)}\varphi_x^3(0,t)\,dt\right.\\ &\qquad\left. + \lambda \int_0^T \left|w_{3x}(0,t)\right|^2 e^{-2\lambda\varphi(0,t)}\varphi_x(0,t)\,dt\right)\end{aligned} \tag{35}$$

for every $\lambda \geqslant \lambda_0$ and $g \in L^2((0,T); L^2(0,1))$, where w is the solution of Eq. (18)*.*

Proof. This is a direct consequence of Proposition 3.4, after realizing that $w = e^{\lambda\varphi} v$ with $v \in W_\lambda$. Indeed, developing the derivatives of $(e^{\lambda\varphi} v)$ and having in mind property (23) it is not difficult to prove that

$$\iint e^{-2\lambda\varphi}\left(\lambda^7\varphi^7\left|e^{\lambda\varphi}v\right|^2 + \lambda^5\varphi^5\left|\left(e^{\lambda\varphi}v\right)_x\right|^2 + \lambda^3\varphi^3\left|\left(e^{\lambda\varphi}v\right)_{2x}\right|^2 + \lambda\varphi\left|\left(e^{\lambda\varphi}v\right)_{3x}\right|^2\right) \leqslant C\|v\|^2_{\lambda,\varphi}. \tag{36}$$

On the other hand, $P_\varphi v = e^{-\lambda\varphi} P w = e^{-\lambda\varphi} g$, and then from Proposition 3.4 we get

$$\|v\|^2_{\lambda,\varphi} \leqslant \iint C|P_\varphi v|^2 - CI_x = C\iint e^{-2\lambda\varphi}|g|^2 - CI_x. \tag{37}$$

Finally, since $\frac{d\beta}{dx} \geqslant 0$, we have $\varphi_x(1,t) \geqslant 0$ for any $t \in (0,T)$ and therefore

$$-I_x \leqslant \lambda^3 \int_0^T \left|w_{2x}(0,t)\right|^2 e^{-2\lambda\varphi(0,t)}\varphi_x^3(0,t)\,dt + \lambda\int_0^T \left|w_{3x}(0,t)\right|^2 e^{-2\lambda\varphi(0,t)}\varphi_x(0,t)\,dt. \tag{38}$$

From (36), (37) and (38) we get inequality (35). □

Remark 3.6. We asked the function β to be increasing. The only place we use this hypothesis is in (38), which allows us to obtain the Carleman inequality with boundary terms at $x = 0$. With the choice of a decreasing function β, we would obtain an inequality with boundary terms at $x = 1$. As we shall see below, the boundary terms in the Carleman inequality are related with the location of the control in Eq. (16).

3.2. Null controllability

Recall that the reference trajectory u belongs to the space $L^\infty(0,T; H_0^2(0,1))$, in particular u_x lies in $L^\infty((0,T)\times(0,1))$. We prove the following energy estimate for Eq. (18).

Lemma 3.7. *If $g \in L^2((0,T); L^2(0,1))$ and w is the solution of Eq.* (18) *then*

$$-\frac{d}{dt}\int_0^1 \left|w(x,t)\right|^2 dx \leqslant \left(\left\|u_x(\cdot,t)\right\|_{L^\infty(0,1)} + \gamma^2 + 1\right)\int_0^1 \left|w(x,t)\right|^2 dt + \int_0^1 \left|g(x,t)\right|^2 dt \tag{39}$$

for every $t \in [0,T]$.

Proof. Multiplying Eq. (18) by w and integrating in $(0,1)$ we obtain

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_0^1 \left|w(x,t)\right|^2 dx + \int_0^1 \left|w_{xx}(x,t)\right|^2 dx + \gamma\int_0^1 w_{xx}(x,t)w(x,t)\,dx + \frac{1}{2}\int_0^1 u_x(x,t)\left|w(x,t)\right|^2 dx \\
&\quad = \int_0^1 w(x,t)g(x,t)\,dx
\end{aligned} \tag{40}$$

for each $t \in [0,T]$. By using $\int w_{xx} w \leqslant \frac{1}{2\gamma}\int |w_{xx}|^2 + \frac{\gamma}{2}\int |w|^2$, we get that

$$-\frac{d}{dt}\int_0^1 \left|w(x,t)\right|^2 dx + \int_0^1 |w_{xx}(x,t)|^2\,dx$$

$$\leqslant \left(\gamma^2 + \|u_x\|_{L^\infty(0,1)}\right)\int_0^1 \left|w(x,t)\right|^2 dx + 2\int_0^1 w(x,t)g(x,t)\,dx, \tag{41}$$

for every $t \in [0,T]$. From this last inequality, (39) directly follows. $\square$

In order to get a Carleman inequality with the norm of $w(0,x)$ at the left-hand side, we introduce a new weight function $\psi(x,t) = \beta(x)\psi_0(t)$ where $\beta \in C^4([0,1])$ satisfies hypothesis (21)–(22) and ψ_0 is defined by

$$\psi_0(t) = \begin{cases} \frac{4}{T^2} & \text{if } 0 \leqslant t < T/2, \\ \frac{1}{t(T-t)} & \text{if } T/2 \leqslant t \leqslant T. \end{cases}$$

Proposition 3.8. *There exist $\lambda, C > 0$ such that the solution w of* (18) *satisfies*

$$\lambda^7 \iint |w|^2 e^{-2\lambda\psi}\psi^7 + \int_0^1 \left|w(x,0)\right|^2 dx$$

$$\leqslant C\left(\iint |g|^2 e^{-2\lambda\psi} + \lambda^3\int_0^T \left|w_{2x}(0,t)\right|^2 e^{-2\lambda\psi(0,t)}\psi_x(0,t)^3\,dt\right.$$

$$\left. + \lambda\int_0^T \left|w_{3x}(0,t)\right|^2 e^{-2\lambda\psi(0,t)}\psi_x(0,t)\,dt\right) \tag{42}$$

for every g such that $\iint |g|^2 e^{-2\lambda\psi} < \infty$.

Proof. Let $\eta \in C^\infty(0,T)$ be such that $\eta(t) = 1$ for all $t \in [0,T/2]$ and $\eta(t) = 0$ for all $t \in [3T/4,T]$. Multiplying inequality (39) by η we obtain

$$-\frac{d}{dt}\int_0^1 \eta(t)\left|w(x,t)\right|^2 dx \leqslant \left(\left\|u_x(t,\cdot)\right\|_{L^\infty(0,1)} + \gamma^2 + 1\right)\eta(t)\int_0^1 \left|w(x,t)\right|^2 dx$$

$$+ \eta(t)\int_0^1 \left|g(x,t)\right|^2 dx - \eta_t(t)\int_0^1 \left|w(x,t)\right|^2 dx.$$

For each $t \in [0,T]$ we apply Gronwall inequality in $[t,T]$. Taking into account that $\eta(T) = 0$ and that $\|u_x(t,\cdot)\|_{L^\infty(0,1)} \leqslant \|u_x(t,\cdot)\|_{H^1(0,1)}$, we get

$$\int_0^1 \eta(t)\left|w(t)\right|^2 dx \leqslant e^{T(\|u_x\|_{L^\infty(0,T;H^1(0,1))}+\gamma^2+1)} \int_t^T h(s)\,ds \tag{43}$$

where $h(t) = \eta(t)\int_0^1 |g(x,t)|^2\,dx - \eta_t(t)\int_0^1 |w(x,t)|^2\,dx$.

Then we deduce that

$$\|w\|^2_{L^\infty(0,\frac{T}{2};L^2(0,1))} \leqslant e^{T(\|u_x\|_{L^\infty(0,T;H^1(0,1))}+\gamma^2+1)}\left(C\|w\|^2_{L^2(\frac{T}{2},\frac{3T}{4};L^2(0,1))} + \|g\|^2_{L^2(0,\frac{3T}{4};L^2(0,1))}\right). \tag{44}$$

Recall that $\psi(x,t) \leqslant \varphi(x,t)$ if $t \in [0,T/2]$ and $\psi(x,t) = \varphi(x,t)$ for $t \in [T/2,T]$. Also, we have $e^{-2\lambda\varphi(x,t)}\varphi_x(x,t)|_{t=0} = e^{-2\lambda\varphi(x,t)}\varphi_x^3(x,t)|_{t=0} = 0$. From all this and (35) we get that

$$\int_{T/2}^T \int_0^1 |w|^2 e^{-2\lambda\psi}\psi^7\,dx\,dt \leqslant C\iint |g|^2 e^{-2\lambda\psi}\,dx\,dt + \lambda^3\int_0^T \left|w_{2x}(0,t)\right|^2 e^{-2\lambda\psi(0,t)}\psi_x^3(0,t)\,dt$$
$$+ \lambda\int_0^T \left|w_{3x}(0,t)\right|^2 e^{-2\lambda\psi(0,t)}\psi_x(0,t)\,dt \tag{45}$$

for λ large enough.

On the other hand, using (44) and that $e^{-2\lambda\psi(x,t)}$ is strictly positive in $[0,3T/4]$ we obtain

$$\begin{aligned}\int_0^{T/2}\int_0^1 |w|^2 e^{-2\lambda\psi}\psi^7 + \int_0^1 \left|w(x,0)\right|^2 dx &\leqslant C\|w\|^2_{L^\infty(0,T/2;L^2(0,1))}\\ &\leqslant C\left(\|w\|^2_{L^2(T/2,3T/4;L^2(0,1))} + \|g\|^2_{L^2(0,3T/4;L^2(0,1))}\right)\\ &\leqslant C\left(\int_{T/2}^{3T/4}\int_0^1 |w|^2 e^{-2\lambda\psi}\psi^7 + \int_0^{3T/4}\int |g|^2 e^{-2\lambda\psi}\right).\end{aligned} \tag{46}$$

Combining (45) and (46) we deduce (42). □

We will need an additional property of the weight function. We will ask the function $\beta(x)$ to satisfy

$$\max_{x\in[0,1]}\beta(x) < 2\min_{x\in[0,1]}\beta(x), \tag{47}$$

as well as the stated conditions (21), (22). Thus, for λ given by Proposition 3.8, we define

$$k_1 := \frac{\lambda}{T}\min_{x\in[0,1]}\beta(x), \qquad k_2 := \frac{\lambda}{T}\max_{x\in[0,1]}\beta(x) \tag{48}$$

which satisfy the relationship $k_2 < 2k_1$, which will be used later in Section 4.

Remark 3.9. The function $\beta(x) = -(x-2)^2 + 8$, introduced in Remark 3.2, satisfies (47).

Proposition 3.10. *There exists $C>0$ such that the solutions w of* (18)*, satisfy*

$$\max_{t\in[0,T]}\left\|we^{-\frac{k_2}{(T-t)}}(T-t)^{-3/2}\right\|^2_{W^{1,\infty}(0,1)}+\int_0^1\left|w_{xx}(0,x)\right|^2dx$$
$$\leqslant C\left\{\iint|g|^2e^{-\frac{2k_1}{T-t}}+\int_0^T\left|w_{2x}(t,0)\right|^2e^{-\frac{2k_1}{T-t}}(T-t)^{-3}\,dt+\int_0^T\left|w_{3x}(t,0)\right|^2e^{-\frac{2k_1}{T-t}}(T-t)^{-1}\,dt\right\} \tag{49}$$

for every g such that $\iint|g|^2e^{-\frac{2k_1}{T-t}}<\infty$.

Proof. Since $H_0^2(0,1)$ embeds continuously into $W^{1,\infty}(0,1)$, we will be done if we are able to get inequality (42), with the term $\|w(t,x)e^{\frac{-k_2}{T-t}}(T-t)^{-\frac{3}{2}}\|_{L^\infty(0,T;H_0^2(0,1))}$ at the left-hand side. Let us denote $\xi(t)=e^{\frac{-k_2}{T-t}}(T-t)^{-\frac{3}{2}}$, and define $\tilde{w}(t,x)=w(t,x)\xi(t)$. Notice that $\tilde{w}$ satisfies (18) with $w_T=0$ and right-hand side equals to $(\xi g-\xi_t w)$. Thanks to Proposition 2.1 we can write

$$\|\tilde{w}\|_{L^\infty(0,T;H_0^2(0,1))}\leqslant C\left(\|\xi g\|_{L^2(0,T;L^2(0,1))}+\|\xi_t w\|_{L^2(0,T;L^2(0,1))}\right).$$

We can easily check the existence of some positive constants C_1 and C_2 such that

$$\iint|\xi g|^2\leqslant C_1\iint|g|^2e^{-\frac{2k_1}{T-t}}$$

and

$$\iint|\xi_t w|^2\leqslant C_1\iint|w|^2e^{-\frac{2k_2}{T-t}}(T-t)^{-7}.$$

Therefore, by using (42), we get (49). □

Inequality (49) directly implies an observability inequality like (17) in some weighted spaces. In order to precise that, we introduce the following notations.

Definition 3.11. Given $T>0$ and a function $\rho:(0,T)\to\mathbb{R}^+$, we denote

$$L_t^2(\rho):=\left\{f;\ \int_0^T\left|f(t)\right|^2\rho(t)\,dt<\infty\right\}$$

and

$$L_{tx}^2(\rho):=\left\{f;\ \int_0^T\int_0^1\left|f(x,t)\right|^2\rho(t)\,dx\,dt<\infty\right\}$$

endowed with their natural norms.

Inequality (49) gives us the observability (17) in the spaces

$$U_1 = L_t^2\big(e^{\frac{2k_1}{T-t}}(T-t)\big), \qquad U_2 = L_t^2\big(e^{\frac{2k_1}{T-t}}(T-t)^3\big), \qquad X = H^{-2}(0,1), \qquad Z = L_{tx}^2\big(e^{\frac{2k_1}{T-t}}\big)$$

and

$$Y = \Big\{y;\ (T-t)^{3/2}e^{\frac{k_2}{(T-t)}}y \in L^1\big(0,T;W^{-1,1}(0,1)\big)\Big\}. \tag{50}$$

As explained before, we get the null controllability for the linear system in the framework given by these spaces. Moreover, we get further decreasing properties of the controlled solution, which will be useful in dealing with the nonlinear equation. More precisely, we have the following result.

Proposition 3.12. *For each $f \in Y$ and $y_0 \in H^{-2}(0,1)$ there exist controls*

$$h_1 \in L_t^2\big(e^{\frac{2k_1}{T-t}}(T-t)\big), \qquad h_2 \in L_t^2\big(e^{\frac{2k_1}{T-t}}(T-t)^3\big) \tag{51}$$

such that the solution of system (16) *satisfies $y \in L_{tx}^2(e^{\frac{2k_1}{T-t}})$. Furthermore, this solution fulfills*

$$y \in E_1 := \Big\{y \in L_{tx}^2\big(e^{\frac{2k_1}{T-t}}\big);\ (T-t)^2 e^{\frac{k_1}{T-t}}y \in \mathcal{B}\Big\}. \tag{52}$$

In particular, $y(T) = 0$.

Proof. Given the inequality (49) and the controllability-observability duality, we get the existence of controls $h_1 \in U_1$ and $h_2 \in U_2$. Thus, the only fact we have to prove is $(T-t)^2 e^{\frac{k_1}{T-t}}y \in \mathcal{B}$. Let us define $\tilde{y} := (T-t)^2 e^{\frac{k_1}{T-t}}y$, $\tilde{h}_1 := (T-t)^2 e^{\frac{k_1}{T-t}}h_1$, $\tilde{h}_2 := (T-t)^2 e^{\frac{k_1}{T-t}}h_2$ and $\tilde{f} := (T-t)^2 e^{\frac{k_1}{T-t}}f$. We can easily check that

$$\tilde{y}_t + \tilde{y}_{xxxx} + \gamma\tilde{y}_{xx} + u\tilde{y}_x + u_x\tilde{y} = \tilde{f} - 2(T-t)e^{\frac{k_1}{T-t}}y + k_1 y e^{\frac{k_1}{T-t}}. \tag{53}$$

Furthermore, from the regularity of controls we have $\tilde{h}_1, \tilde{h}_2 \in L^2(0,T)$. By using that $y \in L_{tx}^2(e^{\frac{2k_1}{T-t}})$ we get

$$(T-t)e^{\frac{k_1}{(T-t)}}y \in L^2\big(0,T;L^2(0,1)\big),$$

and therefore the right-hand side of (53) is in $L^1(0,T;W^{-1,1}(0,1))$. From Theorem 2.6, we conclude that $\tilde{y} \in \mathcal{B}$, which ends the proof of this proposition. □

4. Nonlinear control system

In this section we prove the null controllability of the nonlinear system. As usual in this kind of problems, we use the null controllability of the linear equation and a local inversion theorem.

In order to obtain Theorem 1.1, we use the following result.

Theorem 4.1. *(See [1].) Let E and G be two Banach spaces and let $\Lambda: E \to G$ satisfy $\Lambda \in C^1(E;G)$. Assume that $\hat{e} \in E$, $\Lambda(\hat{e}) = \hat{g}$, and $\Lambda'(\hat{e}): E \to G$ is surjective. Then, there exists $r > 0$ such that, for every $g \in G$ satisfying $\|g - \hat{g}\|_G < r$, there exists some $e \in E$ solution of the equation $\Lambda(e) = g$.*

Let us define the spaces E, G and a map Λ whose surjectivity will be equivalent to the null controllability for the KS equation. We denote

$$Ly = y_t + y_{xxxx} + \gamma y_{xx} + u y_x + u_x y.$$

Keeping in mind (50) and (52) we define the spaces

$$E := \{y \in E_1:\ Ly \in Y\} \quad \text{and} \quad G := H^{-2}(0,1) \times Y.$$

The map Λ is given by

$$\begin{aligned} \Lambda : E &\longrightarrow G, \\ y &\longmapsto \big(y(0,\cdot), Ly + y y_x\big). \end{aligned}$$

To see that Λ is well defined, we have to verify that $y y_x \in Y$ for each $y \in E$. We have the following equivalences

$$\begin{aligned} y y_x \in Y \quad &\Longleftrightarrow \quad e^{\frac{k_2}{(T-t)}} (T-t)^{3/2} y y_x \in L^1\big(0,T; W^{-1,1}(0,1)\big) \\ &\Longleftrightarrow \quad e^{\frac{k_2}{(T-t)}} (T-t)^{3/2} |y|^2 \in L^1\big(0,T; L^1(0,1)\big) \\ &\Longleftrightarrow \quad \int_0^T \int_0^1 |y|^2 e^{\frac{k_2}{(T-t)}} (T-t)^{3/2}\, dx\, dt < \infty. \end{aligned}$$

Therefore, as $k_2 < 2k_1$ (see (47)–(48)) and $y \in L^2_{xt}(e^{\frac{2k_1}{(T-t)}})$, we see that $y \in E$ implies $y y_x \in Y$.

Notice that $(y,z) \in E \times E \mapsto \frac{1}{2}(yz)_x \in Y$ is a bilinear continuous map and then Λ is a C^1 map.

As the functions $y \in E$ satisfy $y(T) = 0$, the local surjectivity of Λ around the origin is equivalent to the local null controllability of the KS equation. Thus, by Theorem 4.1, the proof of Theorem 1.1 will be ended if we prove that the map $\Lambda'(0)$ is surjective.

Proposition 4.2. *The map $\Lambda'(0) : E \to G$ is surjective.*

Proof. It is easy to see that this map is given by

$$\begin{aligned} \Lambda'(0) : E &\longrightarrow G, \\ y &\longmapsto \big(y(0,\cdot), Ly\big), \end{aligned}$$

and therefore its surjectivity is equivalent to the null controllability of the linearized equation with a right-hand side lying in Y. This control property was proved in Proposition 3.12. □

Appendix A. Proof of Lemma 3.1

Let us start by writing

$$\iint P_1 v P_2 v = \sum_{i,j=1}^{4} I_{i,j},$$

where $I_{i,j}$ denotes the L^2-product between the i-th term in the expression of $P_1 v$ and the j-th term in the expression of $P_2 v$ in (20).

Integrating by parts in x we have

$$I_{1,1} = -6\lambda^2 \iint v_t \varphi_x^2 v_{xx} = 12\lambda^2 \iint v_x v_t \varphi_x \varphi_{xx} + 6\lambda^2 \iint v_{xt} \varphi_x^2 v_x$$

and integrating by parts the second term with respect to t we deduce that

$$I_{1,1} = 12\lambda^2 \iint v_x v_t \varphi_x \varphi_{xx} - 6\lambda^2 \iint |v_x|^2 \varphi_x \varphi_{xt}.$$

The first term above will be simplified with $I_{4,1}$, and the other one is part of $I(v_x)$.

Integrating by parts in x we are able to write

$$I_{1,2} = 24\lambda^5 \iint v_x v_{xx} \varphi_x^5 = -60\lambda^5 \iint |v_x|^2 \varphi_x^4 \varphi_{xx}$$

which is part of the dominant term in $I(v_x)$, that is, it has the largest power of λ in that expression.

In the same way we obtain

$$\begin{aligned} I_{1,3} = 24\lambda^3 \iint v_{2x} v_{3x} \varphi_x^3 &= -36\lambda^3 \iint |v_{2x}|^2 \varphi_x^2 \varphi_{2x} \\ &\quad + 12\lambda^3 \int_0^T \left|v_{2x}(1,t)\right|^2 \varphi_x(1,t)^3 \, dt - 12\lambda^3 \int_0^T \left|v_{2x}(0,t)\right|^2 \varphi_x(0,t)^3 \, dt. \end{aligned}$$

We will list the first term into $I(v_{2x})$ and the two trace terms into I_x.

We integrate by parts in x and get

$$I_{1,4} = 30\lambda^5 \iint v_{2x} v \varphi_x^4 \varphi_{2x} = -30\lambda^5 \iint |v_x|^2 \varphi_x^4 \varphi_{2x} + 15\lambda^5 \iint |v|^2 \left(\varphi_x^4 \varphi_{2x}\right)_{2x}.$$

The first term belongs to the dominant part in $I(v_x)$ and the second one goes to $I(v)$.

Writing $v v_t = \frac{1}{2}\partial_t |v|^2$ and integrating by parts in t we obtain

$$I_{2,1} = -\lambda^4 \iint v v_t \varphi_x^4 = 2\lambda^4 \iint |v|^2 \varphi_x^3 \varphi_{xt},$$

which is listed in $I(v)$.

By using the identity $v v_x = \frac{1}{2}\partial_x |v|^2$, we see that

$$I_{2,2} = 4\lambda^7 \iint v v_x \varphi_x^7 = -14\lambda^7 \iint |v|^2 \varphi_x^6 \varphi_{2x},$$

which is part of the dominant term in $I(v)$.

Integrating by parts with respect to x we get

$$\begin{aligned} I_{2,3} = 4\lambda^5 \iint v v_{3x} \varphi_x^5 &= -4\lambda^5 \iint v_{2x} v_x \varphi_x^5 - 20\lambda^5 \iint v v_{2x} \varphi_x^4 \varphi_{xx} \\ &= 30\lambda^5 \iint |v_x|^2 \varphi_x^4 \varphi_{xx} - 10\lambda^5 \iint |v|^2 \left(\varphi_x^4 \varphi_{2x}\right)_{2x}, \end{aligned}$$

terms which will appear in the dominant term of $I(v_x)$ and in $I(v)$.

We get directly that

$$I_{2,4} = 5\lambda^7 \iint |v|^2 \varphi_x^6 \varphi_{xx}$$

is part of the dominant term in $I(v)$. When adding $I_{2,4}$ with $I_{2,2}$, we will get the coefficient $-9\lambda^7$ appearing at $I(v)$.

We integrate by parts twice in x and once in t to get

$$I_{3,1} = -\iint v_t v_{4x} = -\iint v_{2xt} v_{2x} = -\frac{1}{2}\iint \partial_t |v_{2x}|^2 = 0.$$

From two integrations by parts in x we deduce that

$$\begin{aligned} I_{3,2} &= 4\lambda^3 \iint v_x v_{4x} \varphi_x^3 = -4\lambda^3 \iint v_{2x} v_{3x} \varphi_x^3 - 12\lambda^3 \iint v_x v_{3x} \varphi_x^2 \varphi_{2x} \\ &= 18\lambda^3 \iint |v_{2x}|^2 \varphi_x^2 \varphi_{xx} - 2\lambda^3 \int_0^T \left|v_{2x}(1,t)\right|^2 \left|\varphi_x(1,t)\right|^3 \\ &\quad + 2\lambda^3 \int_0^T \left|v_{2x}(0,t)\right|^2 \left|\varphi_x(0,t)\right|^3 - 6\lambda^3 \iint |v_x|^2 \left(\varphi_x^2 \varphi_{2x}\right)_{2x}. \end{aligned}$$

These terms will be written in $I(v_{2x})$, I_x and $I(v_x)$.

By using the identity $v_{3x} v_{4x} = \frac{1}{2}\partial_x |v_{3x}|^2$ we see that

$$\begin{aligned} I_{3,3} &= 4\lambda \iint v_{3x} v_{4x} \varphi_x = -2\lambda \iint |v_{3x}|^2 \varphi_{xx} \\ &\quad + 2\lambda \int_0^T \left|v_{3x}(1,t)\right|^2 \varphi_x(1,t)\, dt - 2\lambda \int_0^T \left|v_{3x}(0,t)\right|^2 \varphi_x(0,t)\, dt. \end{aligned}$$

We will list the first term into $I(v_{3x})$ and the two trace terms into I_x.

After several integrations by parts in x, we get

$$\begin{aligned} I_{3,4} &= 5\lambda^3 \iint v_{4x} v \varphi_x^2 \varphi_{xx} = -5\lambda^3 \iint v_{3x} v_x \varphi_x^2 \varphi_{xx} - 5\lambda^3 \iint v_{3x} v \left(\varphi_x^2 \varphi_{xx}\right)_x \\ &= 5\lambda^3 \iint |v_{2x}|^2 \varphi_x^2 \varphi_{xx} + 10\lambda^3 \iint v_x v_{2x} \left(\varphi_x^2 \varphi_{xx}\right)_x + 5\lambda^3 \iint v v_{2x} \left(\varphi_x^2 \varphi_{xx}\right)_{xx} \\ &= 5\lambda^3 \iint |v_{2x}|^2 \varphi_x^2 \varphi_{xx} - 10\lambda^3 \iint |v_x|^2 \left(\varphi_x^2 \varphi_{xx}\right)_{xx} + \frac{5}{2}\lambda^3 \iint |v|^2 \left(\varphi_x^2 \varphi_{xx}\right)_{4x}. \end{aligned}$$

The resulting terms are included into $I(v_{2x})$, $I(v_x)$ and $I(v)$ respectively.

We have directly that

$$I_{4,1} = -12\lambda^2 \iint v_x v_t \varphi_x \varphi_{xx} \quad \text{and} \quad I_{4,2} = 48\lambda^5 \iint |v_x|^2 \varphi_x^4 \varphi_{xx}.$$

The term $I_{4,1}$ vanishes when adding with the first term in $I_{1,1}$ and $I_{4,2}$ belongs to the dominant term in $I(v_x)$.

From two integration by parts with respect to x we get

$$I_{4,3} = 48\lambda^3 \iint v_x v_{3x} \varphi_x^2 \varphi_{xx} = -48\lambda^3 \iint |v_{2x}|^2 \varphi_x^2 \varphi_{xx} + 24\lambda^3 \iint |v_x|^2 \left(\varphi_x^2 \varphi_{xx}\right)_{xx},$$

terms listed into $I(v_{2x})$ and $I(v_x)$.

Finally, one integration by parts in x gives us

$$I_{4,4} = 60\lambda^5 \iint v v_x \varphi_x^3 \varphi_{xx}^2 = -30\lambda^5 \iint |v|^2 \left(\varphi_x^3 \varphi_{xx}^2\right)_x,$$

which is part of $I(v)$.

It is not difficult to see that adding up the obtained expressions for terms $I_{i,j}$, we get the identity stated in Lemma 3.1.

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