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CONTROL OF A KORTEWEG-DE VRIES EQUATION: A TUTORIAL

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Abstract. These notes are intended to be a tutorial material revisiting in an almost self-contained way, some control results for the Korteweg-de Vries (KdV) equation posed on a bounded interval. We address the topics of boundary controllability and internal stabilization for this nonlinear control system. Concerning controllability, homogeneous Dirichlet boundary conditions are considered and a control is put on the Neumann boundary condition at the right end-point of the interval. We show the existence of some critical domains for which the linear KdV equation is not controllable. In despite of that, we prove that in these cases the nonlinearity gives the exact controllability. Regarding stabilization, we study the problem where all the boundary conditions are homogeneous. We add an internal damping mechanism in order to force the solutions of the KdV equation to decay exponentially to the origin in L^2-norm.

1. **Introduction.** In 1834 John Scott Russell, a Scottish naval engineer, was observing the Union Canal in Scotland when he unexpectedly witnessed a very special physical phenomenon. He saw a particular wave traveling through this channel without losing its shape or velocity. He was so captivated by this event that he focused his attention on these waves for several years and asked the mathematical community to find a specific mathematical model describing them.

A number of researchers took up Russell's challenge, among them the French mathematician Joseph Boussinesq and the English physicist Lord Rayleigh. In 1895 the Dutch mathematicians Diederik J. Korteweg and his student Gustav de Vries published the article [27] deriving the equation (up to rescaling)

$$y_t + y_{xxx} + 6yy_x = 0, \quad x \in \mathbb{R}, \, t \geq 0,$$

where $y = y(t,x)$ models for a time t the amplitude of the water wave at position x. This equation describes approximately long waves in water of relatively shallow depth. A very good book to understand both physical motivation and deduction of the KdV equation, is the book by Whitham [48].

This nonlinear dispersive partial differential equation, named Korteweg-de Vries equation (often abbreviated as the KdV equation), has the important property of allowing solutions describing the phenomenon discovered by Russell. The study of

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the well-posedness of this equation has motivated a huge development of different tools and techniques. We refer the interested reader to the books [46] by Tao and [30] by Linares and Ponce, where the KdV equation and other nonlinear dispersive partial differential equations are studied.

In these notes, we are interested in control properties of the KdV equation posed on a bounded interval. In this case, as suggested in [5], the extra term y_x should be incorporated in the equation in order to obtain an appropriate model for water waves in a uniform channel when coordinates x is taken with respect to a fixed frame. Thus, the equation considered here is

$$y_t + y_x + y_{xxx} + yy_x = 0, \quad x \in [0, L],\ t \geq 0,$$

for some $L > 0$.

From a mathematical point of view, a control system is a dynamical system on which we can act by means of a control in order to reach some goal. In this paper we consider a control system where the state, at each time, is given by the solution of the KdV equation and where the control is some term in the equation. If the control is a term distributed in a region of the domain (for instance, a source term), one calls that an internal control. On the other hand, if the control is a term on a region of the boundary (for instance, a boundary condition), one calls that a boundary control.

There are different notions which appear in control theory of partial differential equations. One says that a control system is exactly controllable in time T if for any pair of given states, one can find a control steering the system from the first one at time $t = 0$ to the second one at time $t = T$. If one can drive the system as close as one wants to any state, one says that the approximate controllability holds. If a system can be driven, by means of a control, from any state to the origin, one says that the system is null controllable. We can also consider stability properties. A system is said to be asymptotically stable if the solution of the system without control converges as the time goes to infinity to a stationary solution of the partial differential equation. If this convergence holds with a control depending, at time t, on the state at time t only, one says that the system is stabilizable by means of a feedback law. Some good references concerning the control of partial differential equations are the review by Russell [40], and the books by Lions [31] and by Coron [17].

Regarding the KdV equation, the first results about control properties as controllability and stabilization were obtained by Russell and Zhang in [41] and [42] for a system with periodic boundary conditions and with an internal control. In the case of a boundary control, always with periodic boundary conditions, the first reference is [42] by the same authors and [45] by Sun.

These notes are concerned first with the controllability in the non periodic framework. Rosier studies in [36] the controllability of the KdV equation posed on a finite interval $(0, L)$. The homogeneous Dirichlet boundary condition is considered and the control acts on the Neumann data at the right end-point of the interval. He uses the classical approach of considering first the linearized system around the origin and then to come back to the nonlinear system by means of a fixed point theorem. He proves that if this value L does not belong to a set of critical values, the linear system is exactly controllable and the nonlinear one as well. When L is a critical value, he also proves that the linear system is not controllable because of the existence of a finite-dimensional subspace of unreachable states. Later on, in a series of papers [18, 7, 9], the exact controllability of the nonlinear KdV equation has been proven in the case of critical domains. Concerning controllability, these notes can be seen as a compilation of those papers.

The other topic we address here is the stabilization of KdV by considering an internal damping term and homogeneous boundary data. We prove that this internal feedback control forces the solutions of KdV to decay exponentially to zero in L^2-norm. This feedback prevents the existence of solutions whose energy do not decay to zero. Those solutions are linked to the critical domains appeared in the study of the controllability. In the stabilization part, we focus on the results obtained in articles [35, 33].

This paper is intended to be a tutorial material revisiting in an almost self-contained way, the technical results proved in the mentioned papers. For a complete revision on control results for the KdV equation, we recommend the excellent survey [39] by Rosier and Zhang.

At the end of this paper, we discuss some open problems related to the KdV control system considered here.

1.1. **Boundary controllability.** Let $L > 0$ be fixed. Let us consider the following Korteweg-de Vries (KdV) control system with the Dirichlet boundary condition

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = h(t), \end{cases} \tag{1}$$

where the state is $y(t,\cdot) : [0,L] \to \mathbb{R}$ and the control is $h(t) \in \mathbb{R}$. We are concerned with the exact controllability properties of (1). Let us give some definitions.

Definition 1.1. System (1) is exactly controllable if for any $y_0, y_T \in L^2(0,L)$, there exists a control $h \in L^2(0,T)$ such that the solution of (1) with $y(0,\cdot) = y_0$ satisfies $y(T,\cdot) = y_T$.

The classical strategy to study the controllability of (1) is first considering the system linearized around the origin, which is given by

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = h(t), \end{cases} \tag{2}$$

then, to prove that this linear system is exactly controllable and finally to recover this property for the original nonlinear system by means of a fixed-point argument for instance. This strategy does not give the exact controllability but a local version.

Definition 1.2. System (1) is locally exactly controllable if there exists $r > 0$ such that for any $y_0, y_T \in L^2(0,L)$ with

$$\|y_0\|_{L^2(0,L)} \le r, \quad \|y_T\|_{L^2(0,L)} \le r,$$

there exists a control $h \in L^2(0,T)$ such that the solution of (1) with $y(0,\cdot) = y_0$ also satisfies $y(T,\cdot) = y_T$.

Rosier proved that the linearized control system (2) is not controllable if L belongs to a set of critical lengths. More precisely he proved the following.

Theorem 1.3. [36, Theorem 1.2] *Let $T > 0$. The system (2) is exactly controllable if and only if*

$$L \notin \mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2 + k\ell + \ell^2}{3}};\ k,\ell \in \mathbb{N}^* \right\}. \tag{3}$$

Indeed, there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by M, which is unreachable for the linear system. More precisely, for every non zero state $\psi \in M$, for every $h \in L^2(0,T)$ and for every $y \in C([0,T],L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ satisfying (2) and $y(0,\cdot)=0$, one has $y(T,\cdot)\neq\psi$.

For the nonlinear system the local exact controllability holds.

Theorem 1.4. [36, Theorem 1.3] *Let $T>0$ and assume that $L\notin\mathcal{N}$. Then there exists $r>0$ such that, for every $(y_0,y_T)\in L^2(0,L)^2$ with $\|y_0\|_{L^2(0,L)}<r$ and $\|y_T\|_{L^2(0,L)}<r$, there exist $h\in L^2(0,T)$ and*

$$y\in C([0,T],L^2(0,L))\cap L^2(0,T,H^1(0,L))$$

satisfying (1), $y(0,\cdot)=y_0$ and $y(T,\cdot)=y_T$.

If one is allowed to use more that one boundary control input, there is no critical spatial domains and the exact controllability holds for any $L>0$. More precisely, let us consider the nonlinear control system

$$\begin{cases} y_t+y_x+y_{xxx}+yy_x=0, \\ y(t,0)=h_1(t), \quad y(t,L)=h_2(t), \quad y_x(t,L)=h_3(t), \end{cases} \tag{4}$$

where the controls are $h_1(t),h_2(t)$ and $h_3(t)$. As it has been pointed out by Rosier in [36], for every $L>0$ the system (4) with $h_1\equiv 0$ is locally exactly controllable in $L^2(0,L)$ around the origin. Moreover, using all the three control inputs, Zhang proved in [50] that for every $L>0$ the system (4) is exactly controllable in the space $H^s(0,L)$ for any $s\geq 0$ in a neighborhood of a given smooth solution of the KdV equation. As it has been proven in [37, 24], the system with only h_1 as control input ($h_2\equiv h_3\equiv 0$) can be proved to be null-controllable, which means that the system can be driven from any initial data to zero. If we consider any combination of two controls (h_1 and h_2 or h_1 and h_3), then the system is exactly controllable for any domain. See [25].

Coron and Crépeau in [18] have proved local exact controllability of (1) for the critical lengths $L=2k\pi$ with $k\in\mathbb{N}^*$ satisfying

$$\nexists (m,n)\in\mathbb{N}^*\times\mathbb{N}^* \quad \text{with} \quad m^2+mn+n^2=3k^2 \quad \text{and} \quad m\neq n. \tag{5}$$

For these values of L, the subspace M of missed directions is one-dimensional and is generated by the function $(1-\cos(x))$. Their method consists, first, in moving along this direction by performing a power series expansion of the solution and then, in using a fixed point theorem.

Remark 1. *Condition (5) has been communicated to the author by J.-M. Coron and E. Crépeau. They pointed out that if it is not satisfied, then the dimension of the missed directions subspace is higher than one and the proof given in [18] does not work anymore.*

Later on, in [7, 9] the method by Coron and Crépeau is applied to address the case of any critical length. Thus, the final local result is given next.

Theorem 1.5. [18, 7, 9] *For any $L\in\mathcal{N}$ there exists $T_L>0$ such that for any $T>T_L$, there exists $r>0$ such that for every $(y_0,y_T)\in L^2(0,L)^2$ with $\|y_0\|_{L^2(0,L)}<r$ and $\|y_T\|_{L^2(0,L)}<r$, there exist $h\in L^2(0,T)$ and*

$$y\in C([0,T],L^2(0,L))\cap L^2(0,T,H^1(0,L))$$

satisfying (1), $y(0,\cdot)=y_0$ and $y(T,\cdot)=y_T$.

The goal of Section 3 is to explain the proofs of Theorems 1.3, 1.4 and 1.5.

1.2. **Internal stabilization.** From the existence of critical domains, we have the existence of some solutions of the linear KdV equation whose energy does not decay to zero as the time goes to infinity. Thus, it is not clear whenever the solutions of the nonlinear KdV equation go to zero. In order to avoid this phenomena, we add to the equation an internal control term F, possibly localized on a small subdomain of $[0, L]$

$$y_t + y_x + y_{xxx} + yy_x = F.$$

The goal is to design a control which dissipates enough energy to force the decay of the solutions in L^2-norm. We look for a control in the form $F = F(y)$, which is called a feedback law. A feedback control is one so that at time t it does not depend on the initial state but on the state at the same time t. The input (control) depends on the output (the full state or a measure of it) in a closed form.

We consider controls in the form $F(y) = -ay$, where $a \in L^\infty(0, L)$ satisfies

$$\begin{cases} a(x) \geq a_0 > 0, \quad \forall x \in \omega, \\ \text{where } \omega \text{ is nonempty open subset of } (0, L). \end{cases} \tag{6}$$

In this way, we are concerned with the equation

$$\begin{cases} y_t + y_x + y_{xxx} + ay + yy_x = 0, \\ y(t, 0) = y(t, L) = y_x(t, L) = 0, \\ y(0, x) = y_0. \end{cases} \tag{7}$$

As in the study of the controllability, a natural strategy is to consider first the linearized equation around the origin. Thus, we consider

$$\begin{cases} y_t + y_x + y_{xxx} + ay = 0, \\ y(t, 0) = y(t, L) = y_x(t, L) = 0, \\ y(0, x) = y_0, \end{cases} \tag{8}$$

and prove the exponential decay of its solutions in the following result.

Theorem 1.6. [35, Theorem 2.2] *Let $L > 0$ and $a = a(x)$ satisfying (6). There exist $C > 0$ and $\mu > 0$ such that*

$$\|y(t, x)\|_{L^2(0,L)} \leq Ce^{-\mu t}\|y_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of (8) with $y_0 \in L^2(0, L)$.

In the proof of Theorem 1.6, we will see that we can take $a = 0$ if the domain is not critical ($L \notin \mathcal{N}$).

Using a perturbative argument, a local version of this theorem for equation (7) is also given in [35] by adding a smallness condition on the initial data.

Theorem 1.7. [35, Section 3.3] *Let $L > 0$ and $a = a(x)$ satisfying (6). There exist $C, r > 0$ and $\mu > 0$ such that*

$$\|y(t, x)\|_{L^2(0,L)} \leq Ce^{-\mu t}\|y_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of (7) with $\|y_0\|_{L^2(0,L)} \leq r$.

Other alternative approach is dealing directly with the nonlinear system (7) without passing by the linear system (8). Thus, the following semi-global stabilization result can be proven.

Theorem 1.8. [35, 33] *Let $L > 0$, $a = a(x)$ satisfying (6), and $R > 0$. There exist $C = C(R) > 0$ and $\mu = \mu(R) > 0$ such that*

$$\|y(t, x)\|_{L^2(0,L)} \leq Ce^{-\mu t}\|y_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of (7) *with* $\|y_0\|_{L^2(0,L)} \leq R$.

The goal of Section 4 is to explain the proofs of Theorems 1.6, 1.7 and 1.8.

Remark 2. Theorem 1.8 has been first proved in [35] by assuming the extra condition

$$\exists \delta > 0, \quad (0,\delta) \cup (L-\delta, L) \subset \omega \tag{9}$$

which has been removed by Pazoto in [33].

Remark 3. Theorem 1.8 still holds for nonlinearities other than yy_x. For instance, in [38] they consider the nonlinearity $y^p y_x$ with $p < 4$ (generalized KdV equation) and in [29] the nonlinearity $y^4 y_x$ (critical generalized KdV equation). Others feedback laws can be also considered. In [32] they prove Theorem 1.8 by considering the feedback control $F(y) = (-\frac{d^2}{dx^2}y)^{-1}1_\omega$ instead of $F(y) = -ay$. Notice that we have denoted by 1_ω the characteristic function of the subset ω.

Remark 4. In this paper, we focus on internal damping mechanisms. Some boundary feedback laws are built in [10] and [8] by using a Gramian approach [47] and the Backstepping method [28], respectively.

2. Well-posedness.

In this section we state the well-posedness framework for the control systems considered in these notes.

2.1. Linear system.

Looking at the linear control system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = h(t), \\ y(0,\cdot) = y_0, \end{cases} \tag{10}$$

we find the underlying spatial operator defined by

$$D(A) = \{w \in H^3(0,L) / w(0) = w(L) = w'(L) = 0\},$$

$$A : w \in D(A) \subset L^2(0,L) \longmapsto (-w' - w''') \in L^2(0,L).$$

The first step is to consider regular data. Let $y_0 \in D(A)$ and $h \in C^2([0,T])$ with $h(0) = 0$. Consider the function $\psi(t,x) = -\frac{x(L-x)}{L}h(t)$ and note that it is very regular in space and satisfies

$$\psi(t,0) = \psi(t,L) = 0, \quad \psi_x(t,L) = h(t), \quad \psi \in C^2([0,T]; C^\infty[0,L]).$$

Let us define $g_h := (-\psi_t - \psi_x - \psi_{xxx}) \in C^1([0,T]; C^\infty(0,L))$ and $z := (y - \psi)$ that satisfies

$$\begin{cases} z_t = Az + g_h, \\ z(t,0) = z(t,L) = z_x(t,L) = 0, \\ z(0,\cdot) = y_0, \end{cases} \tag{11}$$

It can be proven that operator A is dissipative, which means that

$$\int_0^L wA(w)dx \leq 0, \quad \forall w \in D(A).$$

Its adjoint operator A^*, defined by

$$D(A^*) = \{w \in H^3(0,L) / w(0) = w(L) = w'(0) = 0\},$$

$$A^* : w \in D(A^*) \subset L^2(0,L) \longmapsto (w' + w''') \in L^2(0,L),$$

is also dissipative and therefore A generates a strongly continuous semigroup of contractions and the following result follows (see [34] and [36]).

Proposition 1. *Under previous hypothesis on data y_0 and h, there exists a unique classical solution $z \in C([0,T];D(A)) \cap C^1([0,T];L^2(0,L))$ of (11). Thus, we get the existence of a classical solution $y \in C([0,T];D(A)) \cap C^1([0,T];L^2(0,L))$ to (10). Moreover, this solution y is unique.*

Uniqueness of the solutions for (10) can be proven by using energy estimates obtained for (11). Next, we obtain some useful inequalities in order to state the well-posedness framework with less regular data. Pick up a regular function $q = q(t,x)$. By multiplying (10) by qy and integrating by parts we get after some computations

$$
\begin{aligned}
-\int_0^s \int_0^L (q_t + q_x + q_{xxx})|y|^2\,dxdt + \int_0^L q(s,x)|y(s,x)|^2\,dx - \int_0^L q(0,x)|y(0,x)|^2\,dx \\
+ 3\int_0^s \int_0^L q_x|y_x|^2\,dxdt = \int_0^s q(t,L)|h(t)|^2\,dt - \int_0^s q(t,0)|y_x(t,0)|^2\,dt. \quad (12)
\end{aligned}
$$

By choosing $q = 1$ in (12), we get

$$
\int_0^L |y(s,x)|^2\,dx + \int_0^s |y_x(t,0)|^2\,dt = \int_0^L |y_0(x)|^2\,dx + \int_0^s |h(t)|^2\,dt. \quad (13)
$$

From that we deduce

$$
\max_{s\in[0,T]} \int_0^L |y(s,x)|^2\,dx \le \int_0^L |y_0(x)|^2\,dx + \int_0^T |h(t)|^2\,dt, \quad (14)
$$

which implies that the solution belongs to $C([0,T];L^2(0,L))$ provided that $y_0 \in L^2(0,L)$ and $h \in L^2(0,T)$. Moreover, from (14) we get

$$
\|y\|^2_{L^2(0,T;L^2(0,L))} \le T(\|y_0\|^2_{L^2(0,L)} + \|h\|^2_{L^2(0,T)}), \quad (15)
$$

and from (13) we deduce that

$$
\int_0^T |y_x(t,0)|^2\,dt \le \int_0^L |y_0(x)|^2\,dx + \int_0^T |h(t)|^2\,dt, \quad (16)
$$

which implies that $y_x(t,0) \in L^2(0,T)$ provided that $y_0 \in L^2(0,L)$ and $h \in L^2(0,T)$. This is a hidden regularity effect.

By choosing $q = x$ and $s = T$ in (12), we get

$$
\begin{aligned}
\int_0^T \int_0^L |y|^2\,dx\,dt + \int_0^L x|y_0(x)|^2\,dx + L\int_0^T |h(t)|^2dt \\
= \int_0^L x|y(T,x)|^2\,dx + 3\int_0^T \int_0^L |y_x|^2\,dxdt \quad (17)
\end{aligned}
$$

From (17) we obtain

$$
3\int_0^T \int_0^L |y_x|^2\,dx\,dt \le \int_0^T \int_0^L |y|^2 + L\int_0^L |y_0(x)|^2\,dx + L\int_0^T |h(t)|^2dt \quad (18)
$$

and by using (15), we get

$$
\int_0^T \int_0^L |y_x|^2\,dx\,dt \le \left(\frac{L+T}{3}\right)\left(\int_0^L |y_0(x)|^2\,dx + \int_0^T |h(t)|^2dt\right) \quad (19)
$$

which implies that the solution belongs to $L^2(0,T;H^1(0,L))$ provided that $y_0 \in L^2(0,L)$ and $h \in L^2(0,T)$.

Furthermore, by choosing $q = (T-t)$ in (12), we get

$$T\int_0^L |y_0(x)|^2\,dx \leq T\int_0^T |y_x(t,0)|^2\,dt + \int_0^T\int_0^L |y|^2\,dt. \tag{20}$$

which will be useful later.

We have considered the framework of classical solutions $y \in C([0,T];D(A)) \cap C^1([0,T];L^2(0,L))$ when $y_0 \in D(A)$ and h belongs to $\{h \in C^2([0,L])/h(0)=0\}$. By using the density of $D(A)$ in $L^2(0,L)$, the density of $\{h \in C^2([0,L])/h(0)=0\}$ in $L^2(0,T)$, and inequalities (14), (16) and (19), we can extend the notion of solution for less regular data $y_0 \in L^2(0,L)$ and $h \in L^2(0,T)$. Thus, we obtain what is called a mild solution in the space $C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$.

Proposition 2. [36, Proposition 3.7] *Let $y_0 \in L^2(0,L)$ and $h \in L^2(0,T)$. Then there exists a unique mild solution $y \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ of (10). Moreover, there exists $C>0$ such that the solutions of (10) satisfy*

$$\|y\|_{C([0,T];L^2(0,L))} + \|y\|_{L^2(0,T;H^1(0,L))} \leq C\left(\|y_0\|^2_{L^2(0,L)} + \|h\|^2_{L^2(0,T)}\right)^{1/2}$$

and the extra regularity

$$\|y_x(\cdot,0)\|_{L^2(0,T)} \leq \left(\|y_0\|^2_{L^2(0,L)} + \|h\|^2_{L^2(0,T)}\right)^{1/2}.$$

In order to be able to consider the nonlinear KdV equation we need a well-posedness result with a right hand side.

Proposition 3. [36, Proposition 4.1] *Let $y_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$ and $h \in L^2(0,T)$. Then there exists a unique mild solution $y \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ of*

$$\begin{cases} y_t + y_x + y_{xxx} = f, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = h(t), \\ y(0,\cdot) = y_0. \end{cases} \tag{21}$$

Moreover, there exists $C>0$ such that the solutions of (21) satisfy

$$\|y\|_{C([0,T];L^2(0,L))} + \|y\|_{L^2(0,T;H^1(0,L))} \leq C\left(\|y_0\|^2_{L^2(0,L)} + \|f\|^2_{L^1(0,T;L^2(0,L))} + \|h\|^2_{L^2(0,T)}\right)^{1/2}.$$

Proof. From previous results, we only have to consider the case $y_0 = 0$ and $h = 0$. In addition, the semigroup theory gives that the (unique) mild solution y belongs to $C([0,T];L^2(0,L))$ if the right-hand side belongs to $L^1(0,T;L^2(0,L))$ and there exists a constant $C>0$ such that

$$\|y\|_{C([0,T];L^2(0,L))} \leq C\|f\|_{L^1(0,T;L^2(0,L))}.$$

The only thing we have to prove is that $y \in L^2(0,T;H^1(0,L))$. In fact, there exists a constant $\tilde{C}>0$ such that

$$\|y\|_{L^2(0,T;H^1(0,L))} \leq \tilde{C}\|f\|_{L^1(0,T;L^2(0,L))}.$$

As in (17) we can write

$$2\int_0^T\int_0^L xfy\,dxdt = -\int_0^T\int_0^L |y|^2\,dx\,dt + \int_0^L x|y(T,x)|^2\,dx + 3\int_0^T\int_0^L |y_x|^2\,dxdt$$

and hence

$$\begin{aligned}&3\int_0^T\int_0^L |y_x|^2\,dxdt \le \int_0^T\int_0^L |y|^2\,dx\,dt + 2L\int_0^T\int_0^L fy\,dxdt\\ &\le T\|f\|^2_{L^1(0,T;L^2(0,L))} + 2L\int_0^T \|f\|_{L^2(0,L)}\|y\|_{L^2(0,L)} \le (T+2L)\|f\|^2_{L^1(0,T;L^2(0,L))},\end{aligned}$$

which ends the proof. □

As we have seen, the following space is very important in this regularity framework. We define the space

$$\mathcal{B} := C([0,T],L^2(0,L)) \cap L^2(0,T,H^1(0,L)) \tag{22}$$

endowed with the norm

$$\|y\|_{\mathcal{B}} = \max_{t\in[0,T]}\|y(t)\|_{L^2(0,L)} + \left(\int_0^T \|y(t)\|^2_{H^1(0,L)}dt\right)^{1/2}.$$

2.2. **Nonlinear system.** We want to consider the KdV equation. For that, the first step is to show that the nonlinearity yy_x can be considered as a source term of the linear equation.

Proposition 4. [36, Proposition 4.1] *Let $y \in L^2(0,T;H^1(0,L))$. Then $yy_x \in L^1(0,T;L^2(0,L))$ and the map $y \in L^2(0,T;H^1(0,L)) \longmapsto yy_x \in L^1(0,T;L^2(0,L))$ is continuous.*

Proof. Let $y,z \in L^2(0,T;H^1(0,L))$. Denoting with the constant K the norm of the embedding $H^1(0,L) \hookrightarrow L^\infty(0,L)$, we have

$$\begin{aligned}\|yy_x - zz_x\|_{L^1(0,T;L^2(0,L))} &\le \int_0^T \|(y-z)y_x\|_{L^2(0,L)}dt + \int_0^T \|z(y_x - z_x)\|_{L^2(0,L)}dt\\ &\le \int_0^T \|y-z\|_{L^\infty(0,L)}\|y_x\|_{L^2(0,L)}dt + \int_0^T \|z\|_{L^\infty(0,L)}\|y_x-z_x\|_{L^2(0,L)}dt\\ &\le K\int_0^T \|y-z\|_{H^1(0,L)}\|y\|_{H^1(0,L)}dt + K\int_0^T \|z\|_{H^1(0,L)}\|y-z\|_{H^1(0,L)}dt\\ &\le K(\|y\|_{L^2(0,T;H^1(0,L))} + \|z\|_{L^2(0,T;H^1(0,L))})\|y-z\|_{L^2(0,T;H^1(0,L))}\end{aligned} \tag{23}$$

By taking $z=0$ we see that $yy_x \in L^1(0,T;L^2(0,L))$ provided that y lies in $L^2(0,T;H^1(0,L))$. From (23), we also get that the map

$$y \in L^2(0,T;H^1(0,L)) \longmapsto yy_x \in L^1(0,T;L^2(0,L))$$

is continuous. □

Let us state the well posedness property proved by Coron and Crépeau in [18] for the following nonlinear KdV equation

$$\begin{cases} y_t + y_x + y_x + yy_x = f, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = h(t), \\ y(0,\cdot) = y_0. \end{cases} \tag{24}$$

Proposition 5. [18, Proposition 14] *Let $L > 0$ and $T > 0$. There exist $\varepsilon > 0$ and $C > 0$ such that, for every $f \in L^1(0,T,L^2(0,L))$, $h \in L^2(0,T)$ and $y_0 \in L^2(0,L)$ such that*

$$\|f\|_{L^1(0,T,L^2(0,L))} + \|h\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)} \leq \varepsilon,$$

there exists a unique solution of (24) which satisfies

$$\|y\|_{\mathcal{B}} \leq C(\|f\|_{L^1(0,T,L^2(0,L))} + \|h\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)}). \tag{25}$$

Proof. Let $f \in L^1(0,T,L^2(0,L))$, $h \in L^2(0,T)$ and $y_0 \in L^2(0,L)$ as in the theorem with ε to be chosen later. Given $z \in \mathcal{B}$, we consider the map $M : \mathcal{B} \to \mathcal{B}$ defined by $M(z) = \tilde{y}$ where $\tilde{y}$ is the solution of

$$\begin{cases} \tilde{y}_t + \tilde{y}_x + \tilde{y}_x = f - zz_x, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \tilde{y}_x(t,L) = h(t), \\ \tilde{y}(0,\cdot) = y_0. \end{cases}$$

Clearly $y \in \mathcal{B}$ is a solution of (24) if and only if y is a fixed point of the map M. From Proposition 3 and equation (23) we get the existence of a constant D such that

$$\|M(z)\|_{\mathcal{B}} \leq D\left\{\|f\|_{L^1(0,T;L^2(0,L))} + \|h\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)} + \|z\|_{\mathcal{B}}^2\right\}$$

and

$$\|M(z_1) - M(z_2)\|_{\mathcal{B}} \leq D(\|z_1\|_{\mathcal{B}} + \|z_2\|_{\mathcal{B}})(\|z_1 - z_2\|_{\mathcal{B}})$$

We consider M restricted to the closed ball $\{z \in \mathcal{B}/\ \|z\|_{\mathcal{B}} \leq R\}$ with R to be chosen later. We can write

$$\|M(z)\|_{\mathcal{B}} \leq D\left\{\varepsilon + \|z\|_{\mathcal{B}}^2\right\}$$

and

$$\|M(z_1) - M(z_2)\|_{\mathcal{B}} \leq 2DR(\|z_1 - z_2\|_{\mathcal{B}})$$

If R, ε are small enough so that

$$R < \frac{1}{2D}, \quad \varepsilon < \frac{R}{2D},$$

we can apply the Banach fixed point theorem and prove that a unique fixed point of M exists. ☐

Remark 5. The smallness condition given by ε in Proposition 5 can be removed in order to get a global well-posedness result. See [23, 12].

In addition, we have the following result whose proof we omit.

Proposition 6. [18, Proposition 15] *Let $T > 0$ and let $L > 0$. There exists $C > 0$ such that for every $(y_{01}, y_{02}) \in L^2(0,L)^2$, $(h_1,h_2) \in L^2(0,T)^2$ and $(f_1,f_2) \in L^1(0,T,L^2(0,L))^2$ for which there exist solutions $y_1 = y_1(t,x)$ and $y_2 = y_2(t,x)$ of (24), one has the following estimates:*

$$\int_0^T \int_0^L |y_{1x}(t,x) - y_{2x}(t,x)|^2 dxdt \leq e^{C(1+\|y_1\|^2_{L^2(0,T,H^1(0,L))}+\|y_2\|^2_{L^2(0,T,H^1(0,L))})} \cdot \left(\|h_1 - h_2\|^2_{L^2(0,T)} + \|f_1 - f_2\|^2_{L^1(0,T,L^2(0,L))} + \|y_{01} - y_{02}\|^2_{L^2(0,L)} \right),$$

$$\int_0^L |y_1(t,x) - y_2(t,x)|^2 dx \leq e^{C(1+\|y_1\|^2_{L^2(0,T,H^1(0,L))}+\|y_2\|^2_{L^2(0,T,H^1(0,L))})} \cdot \left(\|h_1 - h_2\|^2_{L^2(0,T)} + \|f_1 - f_2\|^2_{L^1(0,T,L^2(0,L))} + \|y_{01} - y_{02}\|^2_{L^2(0,L)} \right),$$

for every $t \in [0,T]$.

3. Boundary controllability.

3.1. Linear system.

The exact controllability property says that we can steer the linear system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = h(t), \end{cases} \tag{26}$$

from any initial data to any final data. To study that, it is useful to introduce the notion of set of reachable states from y_0, that is defined by

$$\mathcal{R}(y_0) = \{y(T,\cdot)/y \text{ is solution of (26) with } y(0,\cdot) = y_0 \text{ and } h \in L^2(0,T)\}$$

By the linearity of the equation (26), it is easy to see that

$$\mathcal{R}(y_0) = \tilde{y}(T,\cdot) + \mathcal{R}(0),$$

where $\tilde{y}$ is the solution of (26) with no control ($h = 0$). Thus, $\mathcal{R}(y_0) = L^2(0,L)$ if and only if $\mathcal{R}(0) = L^2(0,L)$ and hence it is enough to study the controllability in the case $y_0 = 0$.

Let us introduce the linear operator mapping the control to the final state

$$\Pi : h \in L^2(0,T) \longmapsto y(T,\cdot) \in L^2(0,L).$$

From the well-posedness results, we get that this is a bounded operator. Looking at Definition 1.1 we see that exact controllability of (26) is equivalent to the surjectivity of operator Π. In order to characterize the surjectivity of this operator we will use the following Functional Analysis result linking this property with Π^*, the adjoint operator of Π.

Proposition 7. [6, Théorème II.19] *Let E, F be two Banach spaces. Let $A : D(A) \subset E \to F$ a closed operator with $D(A)$ dense in E. Then, we have:*

- *$A(E)$ is dense in F if and only if A^* is injective.*
- *$A(E) = F$ if and only if there exists a constant $C > 0$ such that*

$$\|v\|_{F^*} \leq C\|A^*(v)\|_{E^*}, \quad \forall v \in D(A^*)$$

Remark 6. This is a sort of rank-nullity theorem in infinite dimension. If E is finite dimensional, then $A(E) = F$ is equivalent to the fact that $A(E)$ is dense in F.

To apply this proposition, we first look for the operator $\Pi^* : L^2(0,L) \to L^2(0,T)$. The symmetry condition is

$$\int_0^L \Pi(h)\psi dx = \int_0^T h\Pi^*(\psi)dt, \quad \forall \psi \in L^2(0,L). \tag{27}$$

Let us multiply (26) by a function $\phi = \phi(t,x)$ and integrate on $(0,T) \times (0,L)$. By choosing ϕ solution of

$$\begin{cases} \phi_t + \phi_x + \phi_{xxx} = 0, \\ \phi(t,0) = \phi(t,L) = \phi_x(t,0) = 0, \\ \phi(T,x) = \psi(x) \end{cases} \tag{28}$$

and after some integrations by parts we get (27) with $\Pi^*(\psi)$ defined by

$$\Pi^* : \psi \in L^2(0,L) \longmapsto \phi_x(\cdot, L) \in L^2(0,T).$$

Thanks to the hidden regularity property proved in Proposition 2, we see that this operator is well defined. Notice that up to the change of variables $t \to (T-t)$ and $x \to (L-x)$, system (28) is the same as (26) with no control. Now, we can state the following.

Proposition 8. *The system (26) is exactly controllable if and only if there exists a constant $C > 0$ such that*

$$\|\phi_x(\cdot, L)\|_{L^2(0,T)} \geq C\|\psi\|_{L^2(0,L)}, \quad \forall \psi \in L^2(0,L). \tag{29}$$

We see that we have translated the controllability problem into an inequality for system (28). This kind of inequality is called an observability result. Proposition 8 is known as the duality between controllability and observability for the considered system.

This approach does not give a way to compute the control driving the system from an initial state to another one. Let us give a more constructive way to prove that the observability inequality (29) implies the exact controllability. This is called the Hilbert Uniqueness Method, abbreviated as HUM and introduced by Lions [31].

Given $y_T \in L^2(0,T)$, we look for a control $h \in L^2(0,T)$ such that $\Pi(h) = y_T$. This is equivalent to the problem

$$\begin{cases} \text{To find } h \in L^2(0,T) \text{ such that} \\ \int_0^L \Pi(h)\psi\, dx = \int_0^L y_T \psi\, dx, \quad \forall \psi \in L^2(0,L) \end{cases}$$

Looking for h in the particular form $h = \Pi^*(\hat{\psi})$, we obtain

$$\begin{cases} \text{To find } h \in L^2(0,T) \text{ such that} \\ \int_0^T \Pi^*(\hat{\psi})\Pi^*(\psi)\, dt = \int_0^L y_T \psi\, dx, \quad \forall \psi \in L^2(0,L) \end{cases} \tag{30}$$

which can be written in a variational version by defining the bounded bilinear form

$$a : (\hat{\psi}, \psi) \in L^2(0,L)^2 \longmapsto \int_0^T \Pi^*(\hat{\psi})\Pi^*(\psi)\, dt \in \mathbb{R}$$

Thanks to (29) the bilinear form is coercive and then the Lax-Milgram theorem can be applied to solve problem (30). Furthermore, the function $\hat{\psi}$ defining the control h, can be found by minimizing in $L^2(0,L)$ the following functional

$$J(\psi) := \frac{1}{2}\int_0^T |\Pi^*(\psi)|^2\, dx - \int_0^L y_T \psi\, dt.$$

Summarizing, we found out $\hat{\psi}$ the argmin of J and then the control is given by $h = \Pi^*(\hat{\psi})$, i.e., we put $\hat{\psi}$ as data in (28) getting the solution $\hat{\phi}$ and finally the control is the trace $h = \hat{\phi}_x(t, L)$.

Now we focus in the proof of (29). By contradiction, we assume that (29) does not hold, i.e.,

$$\forall C > 0, \exists \psi \in L^2(0, L) \text{ such that } \|\phi_x(\cdot, x)\|_{L^2(0,T)} < C\|\psi\|_{L^2(0,L)}.$$

By using this successively with $C = 1/n$, we obtain a sequence $\{\psi^n\}_{n\in\mathbb{N}} \subset L^2(0, L)$ such that $\|\psi^n\|_{L^2(0,L)} = 1$ (if not, we could consider the same sequence but normalized $\{\frac{\psi^n}{\|\psi^n\|_{L^2(0,L)}}\}_{n\in\mathbb{N}}$. This is due to the linearity of the equation) and

$$\|\phi_x^n(\cdot, L)\|_{L^2(0,T)} < \frac{1}{n}. \tag{31}$$

The goal now is to pass to the limit and get a contradiction by finding a nontrivial solution of (28) with the extra condition $\|\phi_x(\cdot, L)\|_{L^2(0,T)} = 0$. Hereafter we will need estimates (19) and (20) for system (28). Let us write them out.

$$\int_0^T \int_0^L |\phi_x|^2 \, dx \, dt \le \left(\frac{L+T}{3}\right) \int_0^L |\psi(x)|^2 \, dx \tag{32}$$

and

$$T \int_0^L |\psi(x)|^2 \, dx \le T \int_0^T |\phi_x(t, L)|^2 \, dt + \int_0^T \int_0^L |\phi|^2 \, dt. \tag{33}$$

We apply (32) to the sequence and get

$$\int_0^T \int_0^L |\phi_x^n|^2 \, dx \, dt \le \left(\frac{L+T}{3}\right) \int_0^L |\psi^n(x)|^2 \, dx = \frac{L+T}{3} \tag{34}$$

and therefore we get that ϕ^n is bounded in $L^2(0, T; H^1(0, L))$. From the equation we see that $\phi_t^n = -\phi_x^n - \phi_{xxx}^n$ is bounded in $L^2(0, T; H^{-2}(0, L))$ and we can apply the Aubin-Lions lemma below to conclude that ϕ^n is relatively compact in $L^2(0, T; L^2(0, L))$. Hence, we can assume that ϕ^n converges in $L^2(0, T; L^2(0, L))$.

Lemma 3.1. *(Aubin-Lions, see* [44, Corollary 4]*) Let $X_0 \subset X \subset X_1$ be three Banach spaces with X_0, X_1 reflexive spaces. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Then $\{h \in L^p(0, T; X_0) \,|\, \dot{h} \in L^q(0, T; X_1)\}$ embeds compactly in $L^p(0, T; X)$ for any $1 < p, q < \infty$.*

Inequality (33) implies that $\{\psi^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore it converges to a function ψ such that $\|\psi\|_{L^2(0,L)} = 1$. The corresponding solution ϕ of (28) satisfies $\phi_x(t, L) = 0$ but it is not the trivial one. We would like to say that the system

$$\begin{cases} \phi_t + \phi_x + \phi_{xxx} = 0, \\ \phi(t, 0) = \phi(t, L) = \phi_x(t, 0) = \phi_x(t, L) = 0, \end{cases} \tag{35}$$

is overdetermined and thus getting a contradiction. This is not always possible. For instance, for the length $L = 2\pi$, we can consider $\phi(t, x) = (1 - \cos(x))$ which clearly is not zero and satisfy (35).

We have to study the solutions of system (35). For that, we first prove that it is enough to look for solutions of the form $\phi(t, x) = e^{\lambda t}\varphi(x)$.

Proposition 9. [36, proof of Lemma 3.4] *There exists a non-trivial solution $\phi \in \mathcal{B}$ of (35) if and only if there exist a complex number λ and a non-trivial solution $\varphi \in H^3(0,L)$ of*

$$\begin{cases} \lambda\varphi + \varphi' + \varphi''' = 0, \\ \varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = 0, \end{cases} \tag{36}$$

Proof. Obviously a non-trivial solution φ of (36) gives a nontrivial solution $\phi(t,x) = e^{\lambda t}\varphi(x)$ of (35). The following argument has been used in [1].

Assume that there exists a nontrivial solution $\phi \in \mathcal{B}$ of (35). Let us denote by $N(0,T)$ the set of states $\psi \in L^2(0,L)$ for which the solution of (35) is nontrivial on $[0,T]$ and define $M(0,T)$ the set of solutions of (35) with $\psi \in N(0,T)$. We have the inclusions $N(0,T) \subset L^2(0,L)$ and $M(0,T) \subset C([0,T];L^2(0,L))$.

We have that $N(0,T)$ is a finite-dimensional vector space because the unit ball

$$\{w \in N(0,T) / \|w\|_{L^2(0,L)} = 1\}$$

is compact. Indeed, given a sequence on this unit ball, we can prove as before that there exists a convergent subsequence. This can be done due to the fact that the elements in $N(0,T)$ are initial data for solutions of (35). Then, we apply the Riesz Theorem [6, Théorème VI.5], which says that every normed vector space whose closed unit ball is compact, has to be finite dimensional.

If $\psi \in N(0,T)$, then there exists $\varepsilon \in (0,T)$ such that $\phi(T-\varepsilon) \in N(-\varepsilon, T-\varepsilon)$ and so $\phi(T-\varepsilon) \in N(0,T)$. Thus

$$\frac{\phi(T) - \phi(T-t)}{t} \in N(0,T), \ \forall t \in (0,\varepsilon) \tag{37}$$

By using the equation we have $\phi \in H^1(-\varepsilon, T; H^{-2}(0,L))$ and so

$$\lim_{\delta \to 0} \frac{\phi(\cdot) - \phi(\cdot - \delta)}{\delta} = \phi' \in L^2(0,T;H^{-2}(0,L))$$

Moreover, from (37), we have $\phi(\cdot) - \phi(\cdot - \delta) \in M(0,T)$, which is closed in $L^2(0,T;H^{-2}(0,L))$ ($M(0,T)$ is finite-dimensional). Thus $\phi' \in M(0,T)$ and consequently $\phi \in C^1([0,T], L^2(0,L))$. Hence we get

$$\phi'(T) = \lim_{t \to 0} \frac{\phi(T) - \phi(T-t)}{t} \in N(0,T)$$

From equation (35) we get $\psi \in D(A^*)$, $A^*(\psi) = \phi'(T)$, $\phi \in C([0,T];L^2(0,L))$ and therefore $\phi_x(t,L) \in C([0,T])$. Moreover $\psi'(L) = \phi_x(0,0) = 0$.

We consider the operator A^* restricted to $\mathbb{C}N(0,T)$ where $\mathbb{C}N(0,T)$ denotes the complexification of $N(0,T)$. As we are assuming that $N(0,T)$ is not $\{0\}$, we get that this operator has at least one eigenvalue. Therefore, there exist $\lambda \in \mathbb{C}$ and a nonzero $\varphi \in D(A)$ satisfying (36).

Thus, starting from a nontrivial solution of (35), we get a nontrivial solution of (36), which concludes the proof of this proposition. ☐

We can now give the proof of Theorem 1.3.

Proof of Theorem 1.3. We will get the controllability of system (26) for all domains $(0,L)$ for which system (36) has only the trivial solution. On the other hand, if system (36) has a nontrivial solution, the inequality (29) will not hold and the system (26) will be uncontrollable.

The solution of (36) has the form $\varphi(x) = \sum_{j=1}^{3} C_j e^{\alpha_j x}$ where α_j are the roots of $x^3 + x + \lambda = 0$. These roots satisfy

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \tag{38}$$
$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 = 1, \tag{39}$$
$$\alpha_1\alpha_2\alpha_3 = -\lambda, \tag{40}$$

and since the four homogeneous boundary conditions, the constants C_j should satisy

$$\begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ e^{\alpha_1 L} & e^{\alpha_2 L} & e^{\alpha_3 L} \\ \alpha_1 e^{\alpha_1 L} & \alpha_2 e^{\alpha_2 L} & \alpha_3 e^{\alpha_3 L} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{41}$$

After some calculations, we get that this system has a nontrivial solution if and only if the roots satisfy the extra condition

$$e^{\alpha_1 L} = e^{\alpha_2 L} = e^{\alpha_3 L}. \tag{42}$$

We then see that, given α_1, the other roots are given by

$$\alpha_2 = \alpha_1 + ik\frac{2\pi}{L}, \quad \alpha_3 = \alpha_2 + i\ell\frac{2\pi}{L}, \quad \text{with } k, \ell \in \mathbb{N}.$$

Using these expressions in (38), (39), (40) we get the roots in terms of k, ℓ

$$\alpha_1 = -i(2k+\ell)\frac{2\pi}{3L}$$

and the formula for the lenghts for which the system (36) has nontrivial solutions

$$L = 2\pi\sqrt{\frac{k^2 + k\ell + \ell^2}{3}}.$$

Moreover, the corresponding eigenvalue is

$$\lambda = -i\frac{8\pi^3}{27L^3}(2k+\ell)(k-\ell)(k+2\ell). \tag{43}$$

This concludes the proof of Theorem 1.3. $\square$

Now, we focus our attention on the domains of critical length. In particular, we describe the space M of unreachable states for the linear control system (26). For each $L \in \mathcal{N}$, there exist a finite number of pairs $\{(k_j, \ell_j)\}_{j=1}^{n} \subset \mathbb{N}^* \times \mathbb{N}^*$ with $k_j \geq \ell_j$ such that

$$L = 2\pi\sqrt{\frac{k_j^2 + k_j\ell_j + \ell_j^2}{3}}. \tag{44}$$

Let us introduce the notation

$$J^> := \{\, j \in \{1, \dots, n\}\,;\, k_j > l_j\}, \; J^= := \{\, j \in \{1, \dots, n\}\,;\, k_j = l_j\}, \; n^> := |J^>|. \tag{45}$$

From the proof of Theorem 1.3, we know that for each $j \in \{1, \dots, n\}$ there exist two non zero real-valued functions $\varphi_1^j = \varphi_1^j(x)$ and $\varphi_2^j = \varphi_2^j(x)$ such that $\varphi^j := \varphi_1^j + i\varphi_2^j$ is a solution of

$$\begin{cases} -ip^j\varphi^j + \varphi^{j\prime} + \varphi^{j\prime\prime\prime} = 0, \\ \varphi^j(0) = \varphi^j(L) = 0, \\ \varphi^{j\prime}(0) = \varphi^{j\prime}(L) = 0, \end{cases} \tag{46}$$

where, for $(k,\ell)\in\mathbb{N}^*\times\mathbb{N}^*$, p^j is defined by

$$p^j := \frac{(2k+\ell)(k-\ell)(2\ell+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}}. \tag{47}$$

Easy computations lead to

$$\begin{aligned} \varphi_1^j &= C\left(\cos(\gamma_1^j x) - \tfrac{\gamma_1^j-\gamma_3^j}{\gamma_2^j-\gamma_3^j}\cos(\gamma_2^j x) + \tfrac{\gamma_1^j-\gamma_2^j}{\gamma_2^j-\gamma_3^j}\cos(\gamma_3^j x)\right),\\ \varphi_2^j &= C\left(\sin(\gamma_1^j x) - \tfrac{\gamma_1^j-\gamma_3^j}{\gamma_2^j-\gamma_3^j}\sin(\gamma_2^j x) + \tfrac{\gamma_1^j-\gamma_2^j}{\gamma_2^j-\gamma_3^j}\sin(\gamma_3^j x)\right), \end{aligned} \tag{48}$$

where C is a constant and $\gamma_m^j = -i\alpha_m^j$ with $m=1,2,3$ and α_m^j three roots of

$$x^3 + x + ip(k_j,\ell_j) = 0.$$

One can easily verify that these roots are given by

$$\gamma_1^j = -\frac{2\pi}{L}\left(\frac{2k_j+\ell_j}{3}\right), \quad \gamma_2^j = \gamma_1^j + \frac{2\pi k_j}{L}, \quad \gamma_3^j = \gamma_2^j + \frac{2\pi\ell_j}{L}. \tag{49}$$

Moreover, by choosing the constant C, we can assume that

$$\|\varphi_1^j\|_{L^2(0,L)} = \|\varphi_2^j\|_{L^2(0,L)} = 1.$$

Lemma 3.2. *With the previous notations, we get*

1. *If $j\in J^>$, then $p^j\neq 0$.*
2. *If $j\in J^=$, then $p^j = 0$.*
3. *If $i\neq j$, then $p^i\neq p^j$.*

Proof. Items 1. and 2. are obvious from (45) and (47). Let $i,j\in J$ such that $p_i=p_j$. Then, $\gamma_k^i=\gamma_k^j$ for $k=1,2,3$. With the definitions of γ_k^j, (49) we obtain $k_i=k_j$, $\ell_i=\ell_j$ and hence $i=j$. □

Remark 7. We can easily notice that $|J^=|\leq 1$.

Thus we can reorganize the indexes such that

$$p_1 > p_2 > \cdots > p_n \geq 0.$$

With this notation, we define,

- for $j\in J^>$, the subspace of $L^2(0,L)$
$$M_j := \{\lambda_1\varphi_1^j + \lambda_2\varphi_2^j;\ \lambda_1,\lambda_2\in\mathbb{R}\} = \langle\varphi_1^j,\varphi_2^j\rangle,$$
- for $j\in J^=$, the subspace of $L^2(0,L)$
$$M_j := \{\lambda(1-\cos x);\ \lambda\in\mathbb{R}\} = \langle 1-\cos(x)\rangle.$$

Then, one can define the following subspaces of $L^2(0,L)$

$$M := \bigoplus_{j=1}^{n} M_j \quad\text{and}\quad H := M^\perp.$$

Note that either

$$\bigcup_{j=1}^{n^>}\{\varphi_1^j,\varphi_2^j\} \quad (\text{if } L\neq 2\pi k \text{ for any } k)$$

or

$$\{1-\cos(x)\}\bigcup_{j=1}^{n^>}\{\varphi_1^j,\varphi_2^j\} \quad (\text{if } L = 2\pi k \text{ for some } k)$$

is an orthogonal basis from M.

Remark 8. If $p^j = 0$ for some $j \in \{1, \ldots, n\}$, then $\varphi_1^j = \varphi_2^j = 1 - \cos(x)$. It occurs when $k_j = \ell_j$, i.e., if $L = 2\pi k_j$. If k_j satisfies the condition (5), then the space M is one-dimensional. This is the case treated in [18]. It corresponds for example to the length $L = 2\pi$.

If $p^j \neq 0$, it is easy to see that $\varphi_1^j \perp \varphi_2^j$. Moreover, for distinct $j_1, j_2 \in \{1, \ldots, n\}$, $\varphi_m^{j_1} \perp \varphi_s^{j_2}$ for $m, s = 1, 2$. Let us give some examples. The pair $(2, 1)$ defines a critical length for which the space M is two-dimensional. The pair $(11, 8)$ defines a critical length for which the space M is four-dimensional since the pairs $(11, 8)$ and $(16, 1)$ define the same critical length.

At this point, we can state the following controllability result in H, which follows directly from what we have done.

Theorem 3.3. [36, Propositions 3.3 and 3.9] *Let $T > 0$. For every $(y_0, y_T) \in H \times H$, there exist $h \in L^2(0, T)$ and $y \in \mathcal{B}$ satisfying (26), $y(0, \cdot) = y_0$ and $y(T, \cdot) = y_T$.*

Next section will be devoted to study the controllability of the nonlinear system. To do that, it is important to know how the linear system evolves in a critical case.

Let us define the set $\mathcal{N}'$ by

$$\mathcal{N}' := \left\{ 2\pi\sqrt{\frac{k^2 + k\ell + \ell^2}{3}}; (k, \ell) \in \mathbb{N}^* \times \mathbb{N}^* \text{ satisfying } k > \ell, \text{ and (51)} \right\}, \tag{50}$$

$$\forall m, n \in \mathbb{N}^* \backslash \{k\}, \; k^2 + k\ell + \ell^2 \neq m^2 + mn + n^2. \tag{51}$$

It is easy to see that $\mathcal{N}'$ is the set of critical lengths for which the space of unreachable states is two-dimensional. Indeed, let $L \in \mathcal{N}'$, from (51) there exists a unique pair $(k_1, \ell_1) := (k, \ell)$ satisfying (44) and since $k_1 > \ell_1$, $p^1 > 0$ and therefore the functions φ_1^1, φ_2^1 are orthogonal.

Following the proof of Proposition 8.3 in [17], it is possible to see that $\mathcal{N}'$ contains an infinite number of elements. Let $q \geq 1$ be an integer satisfying

$$\forall m, n \in \mathbb{N}^* \backslash \{q\}, \quad m^2 + mn + n^2 \neq 7q^2. \tag{52}$$

Let us consider the critical length L_q defined by the pair $(2q, q)$, that is

$$L_q := 2\pi\sqrt{\frac{(2q)^2 + 2q^2 + q^2}{3}} = 2\pi q\sqrt{\frac{7}{3}}.$$

From (52), it is easy to see that $L_q \in \mathcal{N}'$. One can verify that (52) holds for $q = 1, 2, 3$ and therefore $L_1, L_2, L_3 \in \mathcal{N}'$. Moreover, the following lemma, whose proof is omitted, says that the set $\mathcal{N}'$ contains an infinite number of lengths L_q.

Lemma 3.4. [7, Lemma 2.5] *There are infinitely many positive integers q satisfying* (52).

We consider $L \in \mathcal{N}'$. From (51), for each $L \in \mathcal{N}'$ we can define a unique

$$p := \frac{(2k + \ell)(k - \ell)(2\ell + k)}{3\sqrt{3}(k^2 + k\ell + \ell^2)^{3/2}}.$$

The space M is then defined by

$$M := \langle \varphi_1, \varphi_2 \rangle = \{\alpha\varphi_1 + \beta\varphi_2 \,;\, \alpha, \beta \in \mathbb{R}\}$$

where φ_1 and φ_2 are given by

$$\begin{aligned} \varphi_1 &= \left(\cos(\gamma_1 x) - \tfrac{\gamma_1-\gamma_3}{\gamma_2-\gamma_3}\cos(\gamma_2 x) + \tfrac{\gamma_1-\gamma_2}{\gamma_2-\gamma_3}\cos(\gamma_3 x)\right), \\ \varphi_2 &= \left(\sin(\gamma_1 x) - \tfrac{\gamma_1-\gamma_3}{\gamma_2-\gamma_3}\sin(\gamma_2 x) + \tfrac{\gamma_1-\gamma_2}{\gamma_2-\gamma_3}\sin(\gamma_3 x)\right), \end{aligned} \tag{53}$$

with $\gamma_m = -i\alpha_m$, and α_m are the roots of $x^3 + x + ip = 0$. From (46) we also have that φ_1 and φ_2 satisfy

$$\begin{cases} \varphi_1' + \varphi_1''' = -p\,\varphi_2, \\ \varphi_1(0) = \varphi_1(L) = 0, \\ \varphi_1'(0) = \varphi_1'(L) = 0, \end{cases} \tag{54}$$

and

$$\begin{cases} \varphi_2' + \varphi_2''' = p\,\varphi_1, \\ \varphi_2(0) = \varphi_2(L) = 0, \\ \varphi_2'(0) = \varphi_2'(L) = 0. \end{cases} \tag{55}$$

Now, we investigate the evolution of the projection on the subspace M of a solution of (26). Let us consider $(y,h) \in \mathcal{B} \times L^2(0,T)$ satisfying (26). Let us multiply (54) by y and integrate on $[0,L]$. Using integrations by parts we get

$$\frac{d}{dt}\left(\int_0^L y(t,x)\varphi_1(x)dx\right) = -p\int_0^L y(t,x)\varphi_2(x)dx. \tag{56}$$

Similarly, multiplying (55) by y, we get

$$\frac{d}{dt}\left(\int_0^L y(t,x)\varphi_2(x)dx\right) = p\int_0^L y(t,x)\varphi_1(x)dx. \tag{57}$$

Hence, from (56) and (57), we obtain

$$\int_0^L y(t,x)\varphi_1(x)\,dx = \int_0^L y(0,x)(\cos(p\,t)\varphi_1(x) - \sin(p\,t)\varphi_2(x))\,dx, \tag{58}$$

$$\int_0^L y(t,x)\varphi_2(x)\,dx = \int_0^L y(0,x)(\sin(p\,t)\varphi_1(x) + \cos(p\,t)\varphi_2(x))\,dx. \tag{59}$$

From (58) and (59), we see that the projection on M of $y(t,\cdot)$, denoted $P_M(y(t,\cdot))$, only turns in this two-dimensional subspace and therefore conserves its $L^2(0,L)$ norm. The period of this rotation is $2\pi/p$. Furthermore, we see that if the initial condition $y(0,\cdot)$ lies in H, so does the solution for every time t. Combining this rotation with Theorem 3.3, we obtain the following proposition.

Proposition 10. *Let $y_0, y_1 \in L^2(0,L)$ be such that*

$$\|P_M(y_0)\|_{L^2(0,L)} = \|P_M(y_1)\|_{L^2(0,L)}.$$

Then, there exists $t^* \leq \frac{2\pi}{p}$ and $u \in L^2(0,t^*)$ such that the solution y of (26) with $y(0,\cdot) = y_0$, satisfies $y(t^*,\cdot) = y_1$.

Proof. Let $y_M = y_M(t,x)$ be the solution of (26) with $y_M(0,\cdot) = P_M(y_0)$ and without control ($h \equiv 0$). We know that there exists a time $0 < t^* \leq \frac{2\pi}{p}$ such that $y_M(t^*,\cdot) = P_M(y_1)$. On the other hand, from Theorem 3.3 there exists a control $h_H \in L^2(0,t^*)$ such that the corresponding solution $y_H = y_H(t,x)$ of (26) satisfies

$$y_H(0,\cdot) = P_H(y_0) \in H \quad \text{and} \quad y_H(t^*,\cdot) = P_H(y_1).$$

Then $y(t,x) := y_H(t,x) + y_M(t,x)$ satisfies (26) with $h = h_H$, $y(0,\cdot) = y_0$ and $y(t^*,\cdot) = y_1$, which ends the proof of this proposition. □

Our proof of (29) is by contadiction, which do not allow to know the observability constant. Let us give a direct proof of (29). The drawback is this proof does not work for any couple (L,T).

Proposition 11. *If T, L satisfy*

$$\frac{L^3}{3T\pi^2} + \frac{L^2}{3\pi^2} < 1, \tag{60}$$

then inequality (29) holds.

Proof. From (33) we obtain

$$\|\psi\|^2_{L^2(0,L)} \leq \|\phi_x(\cdot,L)\|^2_{L^2(0,T)} + \frac{1}{T}\|\phi\|^2_{L^2((0,T)\times(0,L))} \tag{61}$$

By using Poincaré's inequality in (61), we get

$$\int_0^L |\psi(x)|^2\,dx \leq \frac{L^2}{\pi^2 T}\int_0^T\int_0^L |\phi_x|^2\,dxdt + \int_0^T |\phi_x(t,L)|^2\,dt \tag{62}$$

From (32) we can write

$$\int_0^L |\psi(x)|^2\,dx \leq \frac{L^2}{\pi^2 T}\left(\frac{L+T}{3}\right)\int_0^L |\psi|^2\,dx + \int_0^T |\phi_x(t,L)|^2\,dt \tag{63}$$

and therefore if (60) holds, we obtain the observability inequality (29) with constant

$$C = \frac{3T\pi^2}{3T\pi^2 - L^3 - TL^2}. \tag{64}$$

□

Remark 9. Condition (60) can be satisfied only for $L < \pi\sqrt{3}$ and a time of control T large enough

$$T > \frac{L^3}{3\pi^2 - L^2}.$$

We see in (60) that if $L > \pi\sqrt{3}$ we can not obtain the explicit observability in this way. Notice that the first critical value is $L = 2\pi$.

3.2. **Nonlinear system on a noncritical interval.** In this section, we study the local controllability property of system

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = h(t), \end{cases} \tag{65}$$

when the domain is not critical ($L \notin \mathcal{N}$).

Proof of Theorem 1.4. Let $L \notin \mathcal{N}$. Let $y_0, y_T \in L^2(0,L)$ be such that $\|y_0\|_{L^2(0,L)} \leq r$ and $\|y_T\|_{L^2(0,L)} \leq r$ with $r > 0$ to be chosen later on.

We define the map

$$\Pi : y \in \mathcal{B} \longmapsto y^1 + y^2 + y^3 \in \mathcal{B}$$

where y^1, y^2 are the solutions of

$$\begin{cases} y^1_t + y^1_x + y^1_{xxx} = 0, \\ y^1(t,0) = y^1(t,L) = y^1_x(t,L) = 0, \\ y^1(0,x) = y_0(x), \end{cases} \tag{66}$$

$$\begin{cases} y^2_t + y^2_x + y^2_{xxx} = -yy_x, \\ y^2(t,0) = y^2(t,L) = y^2_x(t,L) = 0, \\ y^2(0,x) = 0, \end{cases} \tag{67}$$

and y^3 is the solution of

$$\begin{cases} y^3_t + y^3_x + y^3_{xxx} = 0, \\ y^3(t,0) = y^3(t,L) = 0, \\ y^3_x(t,L) = h(t), \\ y^3(0,x) = 0, \end{cases} \tag{68}$$

with h the control such that $y^3(T,\cdot) = y_T - y^1(T,\cdot) - y^2(T,\cdot)$. This control exists because of Theorem 1.3. It is important to notice that the control operator $y_T \longmapsto h$ mapping the final state to the control driving the linear system to that state, is continuous.

In order to prove Theorem 1.4, we have to find a fixed point to the map Π. We will apply the Banach fixed-point Theorem.

Let $B_R = \{y \in L^2(0,T;H^1(0,L)) / \, \|y\|_{L^2(0,T;H^1(0,L))} \leq R\}$ with R to be chosen later. By choosing R, r small enough, we can prove that

$$\Pi(B_R) \subset B_R$$

and

$$\exists C \in (0,1), \forall y,z \in B_R, \quad \|\Pi(y) - \Pi(z)\|_{\mathcal{B}} \leq C\|y - z\|_{L^2(0,T;H^1(0,L))}$$

and therefore the Banach fixed-point theorem applies. Indeed, from previous computation and the continuity of the control operator, we get

$$\begin{aligned} \|\Pi(y)\|_{\mathcal{B}} &\leq C_1\|y_0\|_{L^2(0,L)} + C_2\|y_T\|_{L^2(0,L)} + C_3\|yy_x\|_{L^1(0,T;L^2(0,L))} \\ &\leq C_1\|y_0\|_{L^2(0,L)} + C_2\|y_T\|_{L^2(0,L)} + C_3\|y\|^2_{\mathcal{B}} \leq (C_1 + C_2)r + C_3R^2 \end{aligned}$$

where hereafter C_j denote positive constants. We get thus the first condition: $(C_1 + C_2)r + C_3R^2 \leq R$. Moreover we get

$$\|\Pi(y) - \Pi(z)\|_{\mathcal{B}} \leq 2C_4R\|y - z\|_{\mathcal{B}}$$

that impose the second condition: $2C_4R < 1$. These conditions are satisfied for instance if we choose r, R such that

$$R < \min\Big\{\frac{1}{2C_4}, \frac{1}{2C_3}\Big\}, \quad r < \frac{R}{2(C_1 + C_2)}.$$

That ends the proof of Theorem 1.4. □

3.3. **Nonlinear system on a critical interval.** In this section, we consider a critical domain ($L \in \mathcal{N}$) and we want to prove Theorem 1.5. The method applied is a classical approach to study the local controllability of a finite-dimensional control system and it has been applied in [18] to prove the local exact controllability around the origin of the control system (65) for some critical domains for which the space of unreachable states is one-dimensional. First, we apply this method to deal with the case $dim(M) = 2$. Next, we explain what happens in the case $dim(M) = 1$ and in the general case.

3.3.1. *M is two-dimensional.* Let us first explain the general idea of the method. Let $y = y(t,x)$ be a solution of (65) with control $h = h(t)$. We consider a power series expansion of (y,h) with the same scaling on the state and on the control

$$y = \varepsilon\alpha + \varepsilon^2\beta + \varepsilon^3\gamma + \dots$$
$$h = \varepsilon u + \varepsilon^2 v + \varepsilon^3 w + \dots$$

In this way, we see that the nonlinear term is given by

$$yy_x = \varepsilon^2\alpha\alpha_x + \varepsilon^3\alpha\beta_x + \varepsilon^3\beta\alpha_x + (\text{higher order terms})$$

and therefore, for a small ε, we have the expansion of second order $y \approx \varepsilon\alpha + \varepsilon^2\beta$, where α and β are given by

$$\begin{cases} \alpha_t + \alpha_x + \alpha_{xxx} = 0, \\ \alpha(t,0) = \alpha(t,L) = 0, \\ \alpha_x(t,L) = u(t), \\ \alpha(0,\cdot) = 0 \end{cases} \tag{69}$$

and

$$\begin{cases} \beta_t + \beta_x + \beta_{xxx} = -\alpha\alpha_x, \\ \beta(t,0) = \beta(t,L) = 0, \\ \beta_x(t,L) = v(t), \\ \beta(0,\cdot) = 0. \end{cases} \tag{70}$$

The strategy consists first, in proving that the expansion to the second order of $y = y(t,x)$, i.e., $\varepsilon\alpha + \varepsilon^2\beta$, can reach all the missed directions and then, in using a fixed point argument to prove that it is sufficient to get Theorem 1.5.

Let us see that we can "enter" into the subspace M. More precisely, the result we prove is the following one.

Proposition 12. *Let $T > 0$. There exists $(u,v) \in L^2(0,T)^2$ such that if $\alpha = \alpha(t,x)$ and $\beta = \beta(t,x)$ are the solutions of* (69) *and* (70)*, respectively, then*

$$\alpha(T,\cdot) = 0 \quad \textit{and} \quad \beta(T,\cdot) \in M\backslash\{0\}.$$

Proof. In order to study the trajectory $\beta = \beta(t,x)$, we set $\beta = \beta^u + \beta^v$ where $\beta^u = \beta^u(t,x)$ and $\beta^v = \beta^v(t,x)$ are the solutions of

$$\begin{cases} \beta^u_t + \beta^u_x + \beta^u_{xxx} = -\alpha\alpha_x, \\ \beta^u(t,0) = \beta^u(t,L) = 0, \\ \beta^u_x(t,L) = 0, \\ \beta^u(0,\cdot) = 0, \end{cases} \tag{71}$$

and

$$\begin{cases} \beta^v_t + \beta^v_x + \beta^v_{xxx} = 0, \\ \beta^v(t,0) = \beta^v(t,L) = 0, \\ \beta^v_x(t,L) = v(t), \\ \beta^v(0,\cdot) = 0. \end{cases} \tag{72}$$

If $u \in L^2(0,T)$ is given, by Theorem 3.3 one can find $v \in L^2(0,T)$ such that

$$\beta^v(T,\cdot) = -P_H(\beta^u(T,\cdot))$$

and thus $\beta(T,\cdot) = P_M(\beta^u(T,\cdot))$. From this fact, one sees that the proof of Proposition 12 can be reduced to prove

$$\exists u \in L^2(0,T) \quad \text{such that} \quad \alpha(T,\cdot) = 0 \quad \text{and} \quad P_M(\beta^u(T,\cdot)) \neq 0. \tag{73}$$

Let $u \in L^2(0,T)$. Let us multiply (71) by φ_1 and integrate the resulting equality on $[0,L]$. Then, using integration by parts, (54), boundary and initial conditions in (71), one gets

$$\frac{d}{dt}\left(\int_0^L \beta^u(t,x)\varphi_1(x)dx\right) = -p\int_0^L \beta^u(t,x)\varphi_2(x)dx + \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_1'(x)dx.$$

In a similar way, if we now multiply (71) by φ_2, we get

$$\frac{d}{dt}\left(\int_0^L \beta^u(t,x)\varphi_2(x)dx\right) = p\int_0^L \beta^u(t,x)\varphi_1(x)dx + \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_2'(x)dx.$$

If we call

$$\eta_k(t) := \int_0^L \beta^u(t,x)\varphi_k(x)dx \quad \text{for } k = 1,2,$$

we can write the system

$$\begin{cases} \begin{pmatrix}\dot\eta_1(t)\\ \dot\eta_2(t)\end{pmatrix} = \begin{pmatrix}0 & -p\\ p & 0\end{pmatrix}\begin{pmatrix}\eta_1(t)\\ \eta_2(t)\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_1'(x)dx\\ \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_2'(x)dx\end{pmatrix} \\ \eta_1(0) = 0, \quad \eta_2(0) = 0. \end{cases} \tag{74}$$

The solution of (74) is given by

$$\begin{pmatrix}\eta_1(t)\\ \eta_2(t)\end{pmatrix} = \begin{pmatrix}\cos(p\,t) & -\sin(p\,t)\\ \sin(p\,t) & \cos(p\,t)\end{pmatrix}\begin{pmatrix}I_1(t)\\ I_2(t)\end{pmatrix}$$

where

$$I_1(t) := \frac{1}{2}\int_0^t\int_0^L \alpha^2(s,x)(\cos(ps)\varphi_1'(x) + \sin(ps)\varphi_2'(x))dx\,ds,$$
$$I_2(t) := \frac{1}{2}\int_0^t\int_0^L \alpha^2(s,x)(-\sin(ps)\varphi_1'(x) + \cos(ps)\varphi_2'(x))dx\,ds.$$

If we work with complex numbers calling $\varphi := \varphi_1 + i\varphi_2$, we get

$$\eta_1(t) + i\eta_2(t) = \frac{1}{2}e^{ip\,t}\int_0^t\int_0^L e^{-ips}\alpha^2(s,x)\varphi'(x)dx\,ds.$$

Now, let us assume that (73) fails to be true, i.e., let us suppose that

$$\forall\, u \in L^2(0,T), \quad \eta_1(T) = \eta_2(T) = 0 \quad \text{or} \quad \alpha(T,\cdot) \neq 0. \tag{75}$$

If we define

$$U_{ad} := \left\{u \in L^2(0,T)\,;\ \text{the solution } \alpha \text{ of (69) satisfies } \alpha(T,\cdot) = 0\right\},$$

then condition (75) implies that

$$\forall\, u \in U_{ad}, \qquad \int_0^T\int_0^L e^{-ips}\alpha^2(s,x)\varphi'(x)dx\,ds = 0. \tag{76}$$

Let $\alpha_1 = \alpha_1(t,x)$ and $\alpha_2 = \alpha_2(t,x)$ be two solutions of (69) such that

$$\alpha_1(T,\cdot) = \alpha_2(T,\cdot) = 0.$$

Now, for $(\rho_1, \rho_2) \in \mathbb{R}^2$, let $\alpha := \rho_1\alpha_1 + \rho_2\alpha_2$ and $u := \alpha_x(\cdot, L)$. By linearity, we see that $\alpha = \alpha(t, x)$ is a solution of (69) and $u \in U_{ad}$. Consequently, (76) implies that, for every $(\rho_1, \rho_2) \in \mathbb{R}^2$,

$$\rho_1^2 \int_0^T \int_0^L e^{-ips}\alpha_1^2(s,x)\varphi'(x)dx\,ds + 2\rho_1\rho_2 \int_0^T \int_0^L e^{-ips}\alpha_1(s,x)\alpha_2(s)\varphi'(x)dx\,ds$$
$$+ \rho_2^2 \int_0^T \int_0^L e^{-ips}\alpha_2^2(s,x)\varphi'(x)dx\,ds = 0.$$

Looking at the coefficient of $\rho_1\rho_2$, we get

$$\int_0^T \int_0^L e^{-ips}\alpha_1(s,x)\alpha_2(s,x)\varphi'(x)dx\,ds = 0. \tag{77}$$

Let t_1, t_2 be such that $0 < t_1 < t_2 < T$. We choose the trajectories $\alpha_1 = \alpha_1(t, x)$ and $\alpha_2 = \alpha_2(t, x)$ such that

$$\alpha_2 \text{ is not identically equal to } 0, \tag{78}$$

$$\alpha_2(t,x)|_{([0,t_1]\cup[t_2,T])\times[0,L]} = 0 \quad \text{and} \quad \alpha_1(t,x)|_{[t_1,t_2]\times[0,L]} = Re(e^{\lambda t}y_\lambda(x)), \tag{79}$$

where $\lambda \in \mathbb{C}\backslash\{\pm ip\}$ and $y_\lambda = y_\lambda(x)$ is a complex-valued function which satisfies

$$\begin{cases} \lambda y_\lambda + y_\lambda' + y_\lambda''' = 0, \\ y_\lambda(0) = y_\lambda(L) = 0. \end{cases} \tag{80}$$

If $\lambda \neq \pm ip$, one can see that $Re(y_\lambda), Im(y_\lambda) \in H$ and then by Theorem 3.3 there exists such a trajectory $\alpha_1 = \alpha_1(t, x)$.

Let us introduce the operator $\tilde{A}w = -w' - w'''$ on the domain $D(\tilde{A}) \subset L^2(0, L)$ defined by

$$D(\tilde{A}) := \left\{w \in H^3(0,L);\, w(0) = w(L) = 0, w'(0) = w'(L)\right\}.$$

It is not difficult to see that $i\tilde{A}$ is a self-adjoint operator on $L^2(0, L)$ with compact resolvent. Hence, the spectrum $\sigma(\tilde{A})$ of $\tilde{A}$ consists only of eigenvalues. Furthermore, the spectrum is a discrete subset of $i\mathbb{R}$.

If we take λ such that $(-ip+\lambda) \notin \sigma(\tilde{A})$, the operator $(\tilde{A}-(-ip+\lambda)I)$ is invertible, and thus, there exists a unique complex-valued function $\phi_\lambda = \phi_\lambda(x)$ solution of

$$\begin{cases} (-ip+\lambda)\phi_\lambda + \phi_\lambda' + \phi_\lambda''' = y_\lambda\varphi', \\ \phi_\lambda(0) = \phi_\lambda(L) = 0, \\ \phi_\lambda'(0) = \phi_\lambda'(L). \end{cases} \tag{81}$$

We multiply (81) by $\alpha_2(t, x)e^{(-ip+\lambda)t}$, integrate on $[0, L]$ and use integrations by parts together with (69), boundary and initial conditions in (81) to get

$$e^{-ip\,t} \int_0^L e^{\lambda t}y_\lambda\alpha_2(t,x)\varphi'(x)dx =$$
$$\frac{d}{dt}\left(\int_0^L e^{(-ip+\lambda)t}\phi_\lambda(x)\alpha_2(t,x)dx\right) - e^{(-ip+\lambda)t}\phi_\lambda'(L)\alpha_{2x}(t,x)\Big|_{x=0}^{L}.$$

Then, let us integrate this equality on $[0,T]$ and use the fact that $\alpha_2(0,\cdot)=0$ and $\alpha_2(T,\cdot)=0$. We obtain

$$\int_0^T\int_0^L e^{-ip\,t}e^{\lambda t}y_\lambda\alpha_2(t,x)\varphi'(x)dx\,dt = \\ -\phi_\lambda'(L)\int_0^T e^{(-ip+\lambda)t}\left(\alpha_{2x}(t,L)-\alpha_{2x}(t,0)\right)dt. \quad (82)$$

On the other hand, by (77) and (79), it follows that

$$\int_0^T\int_0^L e^{-ip\,t}Re(e^{\lambda t}y_\lambda)\alpha_2(t,x)\varphi'(x)dx\,dt=0, \quad (83)$$

and, since one can also take a trajectory $\tilde{\alpha}_1=\tilde{\alpha}_1(t,x)$ such that

$$\tilde{\alpha}_1(t,x)|_{[t_1,t_2]\times[0,L]}=Im(e^{\lambda t}y_\lambda(x)),$$

one deduces from (77) that

$$\int_0^T\int_0^L e^{-ip\,t}Im(e^{\lambda t}y_\lambda)\alpha_2(t,x)\varphi'(x)dx\,dt=0. \quad (84)$$

Therefore, from (83) and (84), one gets

$$\int_0^T\int_0^L e^{-ip\,t}e^{\lambda t}y_\lambda\alpha_2(t,x)\varphi'(x)dx\,dt=0$$

and consequently from (82), for every $\lambda\neq\pm ip$ such that $(-ip+\lambda)\notin\sigma(\tilde{A})$, one has

$$\phi_\lambda'(L)\int_0^T e^{(-ip+\lambda)t}\left(\alpha_{2x}(t,L)-\alpha_{2x}(t,0)\right)dt=0. \quad (85)$$

Let $a\in\mathbb{R}\backslash[-1/\sqrt{3},1/\sqrt{3}]$. We take $\lambda=2ai(4a^2-1)$. Let

$$y_\lambda(x)=Ce^{(-\sqrt{3a^2-1}-ai)x}+(1-C)e^{(\sqrt{3a^2-1}-ai)x}-e^{2aix}, \quad (86)$$

where

$$C=\frac{e^{2aiL}-e^{(\sqrt{3a^2-1}-ai)L}}{e^{(-\sqrt{3a^2-1}-ai)L}-e^{(\sqrt{3a^2-1}-ai)L}}.$$

One easily checks that such a $y_\lambda=y_\lambda(x)$ satisfies (80) and $y_\lambda\neq 0$. Let us define

$$\Sigma:=\left\{a\in\mathbb{R}\backslash[-1/\sqrt{3},1/\sqrt{3}]\,;\,\lambda\notin\sigma(\tilde{A}),(\lambda-ip)\notin\sigma(\tilde{A})\right\},$$

where $\lambda=2ai(4a^2-1)$. Then the function $S:\Sigma\to\mathbb{C}$, $S(a)=\phi_\lambda'(L)$ is continuous. Now we use the fact that S is not identically equal to the function 0 (the proof of this statement will be given in Appendix A). Then, there exist $\widehat{a}\in\Sigma$ and $\varepsilon>0$ such that for every $a\in\Sigma$ with $|a-\widehat{a}|<\varepsilon$, $S(a)\neq 0$. From (85) one gets

$$\forall\, a\in\Sigma,\quad |a-\widehat{a}|<\varepsilon,\quad \int_0^T e^{(-p+2a(4a^2-1))i\,t}\left(\alpha_{2x}(t,L)-\alpha_{2x}(t,0)\right)dt=0$$

and since the function $\beta\in\mathbb{C}\mapsto\int_0^T e^{\beta t}\left(\alpha_{2x}(t,L)-\alpha_{2x}(t,0)\right)dt\in\mathbb{C}$ is holomorphic, it follows that

$$\forall\,\beta\in\mathbb{C},\quad \int_0^T e^{\beta t}\big(\alpha_{2x}(t,L)-\alpha_{2x}(t,0)\big)dt=0,$$

which implies that $\alpha_{2x}(t,0) - \alpha_{2x}(t,L) = 0$ for every t. In summary, one has that $\alpha_2 = \alpha_2(t,x)$ satisfies

$$\begin{cases} \alpha_{2t} + \alpha_{2x} + \alpha_{2xxx} = 0, \\ \alpha_2(t,0) = \alpha_2(t,L) = 0, \\ \alpha_{2x}(t,0) = \alpha_{2x}(t,L), \\ \alpha_2(0,\cdot) = 0, \\ \alpha_2(T,\cdot) = 0. \end{cases} \tag{87}$$

If we multiply (87) by α_2, integrate on $[0,L]$ and use integration by parts together with the boundary conditions, we obtain that

$$\frac{d}{dt}\int_0^L |\alpha_2(t,x)|^2 dx = 0,$$

which, together with $\alpha_2(0,\cdot) = 0$, implies that

$$\alpha_2(t,x) = 0 \quad \forall\, x \in [0,L],\, \forall\, t \in [0,T]. \tag{88}$$

But this is in contradiction with (78). Thus, we have proved (73) and therefore Proposition 12. □

From now on, for each $T_c > 0$, we denote by $(u_c, v_c) \in L^2(0,T)^2$ the controls given by Proposition 12 and by (α_c, β_c) the corresponding trajectories. Let us define $\tilde{\varphi}_1 := \beta_c(T_c,\cdot)$. Let us notice that by scaling the controls, we can assume that $\|\tilde{\varphi}_1\|_{L^2(0,L)} = 1$. We will prove now that in any time $T > \pi/p$, we can reach all the states lying in M.

Proposition 13. *Let $T > \pi/p$. Let $\psi \in M$. There exists $(u,v) \in L^2(0,T)^2$ such that if $\alpha = \alpha(t,x)$ and $\beta = \beta(t,x)$ are the solutions of (69) and (70), then*

$$\alpha(T,\cdot) = 0 \quad \textit{and} \quad \beta(T,\cdot) = \psi.$$

Proof. Let $\hat{T} > 0$ be such that $T = (\pi/p) + \hat{T}$. Let T_c be such that $0 < T_c < \hat{T}$. Let $T_a := T - T_c$. If we take in (69) and (70) the controls

$$(u^1, v^1)(t) = \begin{cases} (0,0) & \text{if } t \in (0,T_a), \\ (u_c(t-T_a), v_c(t-T_a)) & \text{if } t \in (T_a, T), \end{cases}$$

we obtain that $\beta^1(T,\cdot) = \tilde{\varphi}_1$, where $\beta^1 = \beta^1(t,x)$ is the corresponding solution of (70). Now, we use the rotation showed in (58) and (59) in order to reach other states lying in M. Let us define $\tilde{\varphi}_2 := \beta^2(T,\cdot)$, where $\beta^2 = \beta^2(t,x)$ is defined by the controls

$$(u^2, v^2)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a - \frac{\pi}{2p}), \\ (u_c(t - T_a + \frac{\pi}{2p}), v_c(t - T_a + \frac{\pi}{2p})) & \text{if } t \in (T_a - \frac{\pi}{2p}, T - \frac{\pi}{2p}), \\ (0,0) & \text{if } t \in (T - \frac{\pi}{2p}, T). \end{cases}$$

In a similar way, the controls

$$(u^3, v^3)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a - \frac{\pi}{p}), \\ (u_c(t - T_a + \frac{\pi}{p}), v_c(t - T_a + \frac{\pi}{p})) & \text{if } t \in (T_a - \frac{\pi}{p}, T - \frac{\pi}{p}), \\ (0,0) & \text{if } t \in (T - \frac{\pi}{p}, T), \end{cases}$$

allow us to define $\tilde{\varphi}_3 := \beta^3(T,\cdot)$. Notice that $\tilde{\varphi}_3 = -\tilde{\varphi}_1$.

Let T_θ be such that $0 < T_\theta < \min\{\pi/(2p), \hat{T} - T_c\}$ and let $T_b := (\pi/p) + T_\theta$. Let us define $\tilde{\varphi}_4 := \beta^4(T, \cdot)$, where $\beta^4 = \beta^4(t, x)$ is the solution of (70) with

$$(u^4, v^4)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a - T_b), \\ (u_c(t - T_a + T_b), v_c(t - T_a + T_b)) & \text{if } t \in (T_a - T_b, T - T_b), \\ (0,0) & \text{if } t \in (T - T_b, T). \end{cases}$$

We have thus proved that we can reach the missed directions $\{\tilde{\varphi}_k\}_{k=1}^4$. Let us now define the cones

$$\begin{aligned} M_1 &:= \{d_1\tilde{\varphi}_1 + d_2\tilde{\varphi}_2; d_1 > 0, d_2 \geq 0\}, \\ M_2 &:= \{d_1\tilde{\varphi}_2 + d_2\tilde{\varphi}_3; d_1 > 0, d_2 \geq 0\}, \\ M_3 &:= \{d_1\tilde{\varphi}_3 + d_2\tilde{\varphi}_4; d_1 > 0, d_2 \geq 0\}, \\ M_4 &:= \{d_1\tilde{\varphi}_4 + d_2\tilde{\varphi}_1; d_1 > 0, d_2 \geq 0\}. \end{aligned}$$

By construction of these cones, one has that $M = \bigcup_{k=1}^{4} M_k$.

Remark 10. *It is easy to see that if one chooses T_c, T_θ such that $T_c < T_\theta$, then the supports of the trajectories $\alpha^k = \alpha^k(t, x)$ for $k = 1, \dots, 4$ are disjoint.*

For each $w = (w_1, w_2) \in \mathbb{R}^2$, let us define

$$\rho_w := \sqrt{w_1^2 + w_2^2} \quad \text{and} \quad z_w := (w_1\varphi_1 + w_2\varphi_2)/\rho_w \in M.$$

We have that $z_w \in M_i$ for some $i \in \{1, \dots, 4\}$ and hence there exist $d_{1w} > 0$ and $d_{2w} \geq 0$ such that $z_w = d_{1w}\tilde{\varphi}_i + d_{2w}\tilde{\varphi}_{i+1}$. If we take the control

$$(u_w, v_w) = (d_{1w}^{1/2}u^i + d_{2w}^{1/2}u^{i+1}, d_{1w}v^i + d_{2w}v^{i+1})$$

and use the fact that the trajectories α^k for $k = 1, \dots, 4$ are disjoints, then we see that the corresponding solution $\beta_w = \beta_w(t, x)$ of (70) satisfies $\beta_w(T, \cdot) = z_w$.

Finally, let $\psi \in M$. With $R := \|\psi\|_{L^2(0,L)}$ we can write $\psi = Rz_w$ for a $(w_1, w_2) \in \mathbb{R}^2$ such that $w_1^2 + w_2^2 = 1$. It is easy to see that the control $(u, v) = (R^{1/2}u_w, Rv_w)$ allows us to reach the state ψ and so the proof of this proposition is ended. □

Remark 11. *The proof of Proposition 13 is the only part which needs a time large enough. Hence, Theorem 1.5 holds for $T_L := \pi/p$.*

Let us denote, for $D > 0$ and $R > 0$,

$$B_R^D := \Big\{\xi \in L^2(0, D)\,;\, \|\xi\|_{L^2(0,D)} \leq R\Big\},$$

and recall that for each $w = (w_1, w_2) \in \mathbb{R}^2$, we write

$$\rho_w := \sqrt{w_1^2 + w_2^2}, \quad \text{and } z_w := (w_1\varphi_1 + w_2\varphi_2)/\rho_w.$$

We also write $(u_w, v_w) \in L^2(0, T)$ the controls defined in Proposition 13 in order to drive the solutions $\beta_w = \beta_w(t, x)$ from zero at $t = 0$ to z_w at $t = T$.

From the work done for the linear system, we know that for each $y_0 \in L^2(0, L)$ there exists a continuous linear affine map (it is a consequence of applying the HUM method [31] to prove Theorem 3.3)

$$\Gamma_0 : \Psi \in H \subset L^2(0, L) \longmapsto \Gamma_0(\Psi) \in L^2(0, T),$$

such that the solution of

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = \Gamma_0(\Psi), \\ y(0,\cdot) = P_H(y_0), \end{cases}$$

satisfies $y(T,\cdot) = \Psi$. Moreover, there exist constants $D_1, D_2 > 0$ such that

$$\forall y_0 \in L^2(0,L),\ \forall \Psi \in H \quad \|\Gamma_0(\Psi)\|_{L^2(0,T)} \leq D_1(\|\Psi\|_{L^2(0,L)} + \|y_0\|_{L^2(0,L)}), \tag{89}$$

$$\forall y_0 \in L^2(0,L),\ \forall \Psi_1, \Psi_2 \in H \quad \|\Gamma_0(\Psi_1) - \Gamma_0(\Psi_2)\|_{L^2(0,T)} \leq D_2 \|\Psi_1 - \Psi_2\|_{L^2(0,L)}. \tag{90}$$

Let $y_0 \in L^2(0,L)$ be such that $\|y_0\|_{L^2(0,L)} < r$, $r > 0$ to be chosen later. Let us define the functions G and F

$$\begin{aligned} G : L^2(0,L) &\longrightarrow \quad L^2(0,T), \\ z = P_H(z) + w_1\varphi_1 + w_2\varphi_2 &\mapsto \quad G(z) = \Gamma_0(P_H(z)) + \rho_w^{1/2} u_w + \rho_w v_w, \end{aligned}$$

$$\begin{aligned} F : B^T_{\varepsilon_1} \cap L^2(0,T) &\longrightarrow \quad L^2(0,L), \\ h &\longmapsto \quad F(h) = y(T,\cdot), \end{aligned}$$

where $y = y(t,x)$ is the solution of

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = h(t), \\ y(0,\cdot) = y_0, \end{cases} \tag{91}$$

and ε_1 is small enough so that the function F is well defined. It holds if $\varepsilon_1 + r < \varepsilon$ where ε is given by Proposition 5. The map F is even continuous according to Proposition 6. Let $y_T \in L^2(0,L)$ be such that $\|y_T\| < r$. Let Λ_{y_0,y_T} denote the map

$$\begin{aligned} \Lambda_{y_0,y_T} : B^L_{\varepsilon_2} \cap L^2(0,L) &\longrightarrow \quad L^2(0,L), \\ z &\longmapsto \quad \Lambda_{y_0,y_T}(z) = z + y_T - F \circ G(z), \end{aligned}$$

where ε_2 is small enough so that Λ_{y_0,y_T} is well defined (ε_2 exists according to Proposition 5 and to the continuity of G).

Let us notice that if we find a fixed point $\tilde{z} \in L^2(0,L)$ of the map Λ_{y_0,y_T}, then we will have $F \circ G(\tilde{z}) = y_T$ and this means that the control $h := G(\tilde{z}) \in L^2(0,T)$ drives the solution of (91) from y_0 at $t = 0$ to y_T at $t = T$.

Let us assert the following technical result which will be needed to study the map Λ_{y_0,y_T}.

Lemma 3.5. *There exist $\varepsilon_3 > 0$ and $C_3 > 0$ such that, for every $z, y_0 \in B^L_{\varepsilon_3}$, the following estimate holds.*

$$\|z - F(G(z))\|_{L^2(0,L)} \leq C_3(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}).$$

Proof. Let $z, y_0 \in L^2(0,L)$. Let $w = (w_1, w_2) \in \mathbb{R}^2$ be such that $z = P_H(z) + \rho_w z_w$. Let $y = y(t,x)$ be a solution of

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = G(z), \\ y(0,\cdot) = y_0. \end{cases} \tag{92}$$

From (89) and since $\rho_w \leq \|z\|_{L^2(0,L)}$, one deduces that if $\|z\|_{L^2(0,L)}$ is small enough, then there exists a constant D_3 such that

$$\|G(z)\|_{L^2(0,T)} \leq D_3(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}). \tag{93}$$

By using (25) and (93), one can find $\varepsilon_2, C_2 > 0$ such that for every $z, y_0 \in B^L_{\varepsilon_2}$ the unique solution of (92) satisfies

$$\|y\|_{\mathcal{B}} \leq C_2(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}). \tag{94}$$

Let $\tilde{y} = \tilde{y}(t,x)$, $\alpha_w = \alpha_w(t,x)$, $\beta_w = \beta_w(t,x)$ and $\beta^0 = \beta^0(t,x)$ be the solutions of

$$\begin{cases} \tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = 0, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \tilde{y}_x(t,L) = \Gamma_0(P_H(z)), \\ \tilde{y}(0,\cdot) = P_H(y_0), \end{cases}$$

$$\begin{cases} \alpha_{wt} + \alpha_{wx} + \alpha_{wxxx} = 0, \\ \alpha_w(t,0) = \alpha_w(t,L) = 0, \\ \alpha_{wx}(t,L) = u_w(t), \\ \alpha_w(0,\cdot) = 0, \end{cases}$$

$$\begin{cases} \beta_{wt} + \beta_{wx} + \beta_{wxxx} = -\alpha_w\alpha_{wx}, \\ \beta_w(t,0) = \beta_w(t,L) = 0, \\ \beta_{wx}(t,L) = v_w(t), \\ \beta_w(0,\cdot) = 0, \end{cases}$$

$$\begin{cases} \beta^0_t + \beta^0_x + \beta^0_{xxx} = 0, \\ \beta^0(t,0) = \beta^0(t,L) = 0, \\ \beta^0_x(t,L) = 0, \\ \beta^0(0,\cdot) = P_M(y_0). \end{cases}$$

Let us define

$$\phi := y - \tilde{y} - \rho_w^{1/2}\alpha_w - \rho_w\beta_w - \beta^0.$$

We have that $\phi = \phi(t,x)$ satisfies

$$\begin{cases} \phi_t + \phi_x + \phi_{xxx} + \phi\phi_x = -(\phi a)_x - b, \\ \phi(t,0) = \phi(t,L) = 0, \\ \phi_x(t,L) = 0, \\ \phi(0) = 0, \end{cases}$$

where
$$a := \tilde{y} + \rho_w^{1/2}\alpha_w + \rho_w\beta_w + \beta^0,$$

$$\begin{aligned} b := \quad & \tilde{y}\tilde{y}_x + (\tilde{y}(\rho_w^{1/2}\alpha_w + \rho_w\beta_w + \beta^0))_x + \rho_w^{3/2}(\alpha_w\beta_w)_x \\ & + \rho_w^2\beta_w\beta_{wx} + \rho_w^{1/2}(\alpha_w\beta^0)_x + \rho_w(\beta_w\beta^0)_x + \beta^0\beta^0_x. \end{aligned}$$

It is easy to see that there exists $C_4 > 0$ such that for every $z, y_0 \in B^L_{\varepsilon_2}$

$$\|\phi\|_{\mathcal{B}} \leq C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}), \tag{95}$$

$$\|a\|_{\mathcal{B}} \leq C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}), \tag{96}$$

$$\|b\|_{L^1(0,T,L^2(0,L))} \leq C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}). \tag{97}$$

One can also prove that there exists $C_5 > 0$ such that for every $f, g \in \mathcal{B}$

$$\|(fg)_x\|_{L^1(0,T,L^2(0,L))} \leq C_5 \|f\|_{\mathcal{B}} \|g\|_{\mathcal{B}}. \tag{98}$$

By Proposition 6, (97) and (98), there exists $C_6 > 0$ such that

$$\|\phi\|_{\mathcal{B}}^2 \leq C_6(\|\phi\|_{\mathcal{B}}^2 \|a\|_{\mathcal{B}}^2 + \|y_0\|_{L^2(0,L)}^2 + \|z\|_{L^2(0,L)}^3),$$

which, together with (95) and (96), implies the existence of ε_3 and C_7 such that for every $z, y_0 \in B_{\varepsilon_3}^L$

$$\|\phi\|_{\mathcal{B}} \leq C_7(\|y_0\|_{L^2(0,L)} + \|z\|_{L^2(0,L)}^{3/2}). \tag{99}$$

Finally, from (99) one obtains with $C_3 := C_7 + 1$

$$\begin{aligned}
\|z - F\circ G(z)\|_{L^2(0,L)} &\leq \|z - F\circ G(z) - \beta^0(T)\|_{L^2(0,L)} + \|\beta^0(T)\|_{L^2(0,L)} \\
&= \|\phi(T)\|_{L^2(0,L)} + \|\beta^0(T)\|_{L^2(0,L)} \\
&\leq C_7(\|y_0\|_{L^2(0,L)} + \|z\|_{L^2(0,L)}^{3/2}) + \|y_0\|_{L^2(0,L)} \\
&\leq C_3(\|y_0\|_{L^2(0,L)} + \|z\|_{L^2(0,L)}^{3/2}),
\end{aligned}$$

which ends the proof of Lemma 3.5. $\square$

For $w = (w_1, w_2) \in \mathbb{R}^2$ fixed, let us study the map $\Pi := P_H \circ \Lambda_{y_0,y_T}(\cdot + \rho_w z_w)$ on the subspace H (recall that $\rho_w z_w = w_1\varphi_1 + w_2\varphi_2$).

$$\begin{aligned}
\Pi : H &\longrightarrow H, \\
\Psi &\longmapsto \Pi(\Psi) = \Psi + P_H(y_T) - P_H(F\circ G(\Psi + \rho_w z_w)).
\end{aligned}$$

In order to prove the existence of a fixed point of the map Π, we will apply the Banach fixed point theorem to the restriction of Π to the closed ball $B_R^L \cap H$ with $R > 0$ small enough. By using Lemma 3.5 we see that

$$\begin{aligned}
\|\Pi(\Psi)\|_{L^2(0,L)} &\leq \|y_T\|_{L^2(0,L)} + \|\Psi + \rho_w z_w - F\circ G(\Psi + \rho_w z_w)\|_{L^2(0,L)} \\
&\leq \|y_T\|_{L^2(0,L)} + C_3(\|y_0\|_{L^2(0,L)} + \|\Psi + \rho_w z_w\|_{L^2(0,L)}^{3/2}) \\
&\leq C_3'(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} + \rho_w^{3/2}) + C_3\|\Psi\|_{L^2(0,L)}^{3/2} \\
&\leq C_3'(2r + \rho_w^{3/2}) + C_3\|\Psi\|_{L^2(0,L)}^{3/2},
\end{aligned}$$

where $C_3' := C_3 + 1$. Hence, if we choose R such that $R^{3/2} \leq \frac{R}{2C_3}$ and r, ρ_w such that

$$C_3'(2r + \rho_w^{3/2}) \leq \frac{R}{2},$$

then it follows that

$$\|\Pi(\Psi)\|_{L^2(0,L)} \leq R \quad \text{and so} \quad \Pi(B_R^L \cap H) \subset (B_R^L \cap H).$$

It remains to prove that the map Π is a contraction. Let $\Psi_1, \Psi_2 \in B_R^L \cap H$. Let $y = y(t,x)$, $q = q(t,x)$, $\tilde{y} = \tilde{y}(t,x)$ and $\tilde{q} = \tilde{q}(t,x)$ be the solutions of the following problems

$$\begin{cases}
y_t + y_x + y_{xxx} + yy_x = 0, \\
y(t,0) = y(t,L) = 0, \\
y_x(t,L) = G(\Psi_1 + \rho_w z_w), \\
y(0,\cdot) = y_0,
\end{cases}$$

$$\begin{cases}
q_t + q_x + q_{xxx} + qq_x = 0, \\
q(t,0) = q(t,L) = 0, \\
q_x(t,L) = G(\Psi_2 + \rho_w z_w), \\
q(0,\cdot) = y_0,
\end{cases}$$

$$\begin{cases} \tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = 0, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \tilde{y}_x(t,L) = \Gamma_0(\Psi_1), \\ \tilde{y}(0,\cdot) = P_H(y_0), \end{cases}$$

$$\begin{cases} \tilde{q}_t + \tilde{q}_x + \tilde{q}_{xxx} = 0, \\ \tilde{q}(t,0) = \tilde{q}(t,L) = 0, \\ \tilde{q}_x(t,L) = \Gamma_0(\Psi_2), \\ \tilde{q}(0,\cdot) = P_H(y_0). \end{cases}$$

Let us define $\phi := y - \tilde{y}$, $\psi := q - \tilde{q}$ and $\gamma := \phi - \psi$. One sees that γ satisfies

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \gamma\gamma_x = -(\gamma a)_x - b, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \gamma_x(t,L) = 0, \\ \gamma(0) = 0, \end{cases} \tag{100}$$

where

$$a := \tilde{y} + \psi \quad \text{and} \quad b := \left(q(\tilde{y} - \tilde{q})\right)_x + (\tilde{y} - \tilde{q})(\tilde{y} - \tilde{q})_x.$$

It is easy to see that there exists a constant $C_8 > 0$ such that

$$\|b\|_{L^1(0,T,L^2(0,L))} \le C_8 \left(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\right) \|\tilde{y} - \tilde{q}\|_{\mathcal{B}}, \tag{101}$$

$$\|(a\gamma)_x\|_{L^1(0,T,L^2(0,L))} \le C_8 \left(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\right) \|\gamma\|_{\mathcal{B}}. \tag{102}$$

By using Proposition 6, (101) and (102) we get the existence of $C_9 > 0$ such that

$$\|\gamma\|_{\mathcal{B}}^2 \le C_9(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}})^2(\|\tilde{y} - \tilde{q}\|_{\mathcal{B}}^2 + \|\gamma\|_{\mathcal{B}}^2). \tag{103}$$

In addition, since $w := \tilde{y} - \tilde{q}$ satisfies the following linear equation

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \\ w(t,0) = w(t,L) = 0, \\ w_x(t,L) = \Gamma_0(\Psi_1) - \Gamma_0(\Psi_2), \\ w(0,\cdot) = 0, \end{cases}$$

there exists $C_{10} > 0$ such that

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \le C_{10} \|\Gamma_0(\Psi_1) - \Gamma_0(\Psi_2)\|_{L^2(0,T)}$$

and so, from (90), one gets

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \le C_{10} D_2 \|\Psi_1 - \Psi_2\|_{L^2(0,L)}. \tag{104}$$

Moreover, it is easy to see that there exists a constant $C_{11} > 0$ such that

$$\|q\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} \le C_{11}(\|y_0\|_{L^2(0,L)} + \|\Psi_1\|_{L^2(0,L)} + \|\Psi_2\|_{L^2(0,L)} + \rho_w^{1/2}). \tag{105}$$

Thus, using (103), (104) and (105) we see that if R, ρ_w, r are small enough, it follows that

$$\|\gamma\|_{\mathcal{B}} \le \frac{1}{2} \|\Psi_1 - \Psi_2\|_{L^2(0,L)}.$$

Therefore, we have

$$\begin{aligned} &\|\Pi(\Psi_1) - \Pi(\Psi_2)\|_{L^2(0,L)} \le \\ &\qquad \|\Psi_1 - F \circ G(\Psi_1 + \rho_w z_w) - \Psi_2 + F \circ G(\Psi_2 + \rho_w z_w)\|_{L^2(0,L)} \\ &\qquad\qquad = \|\gamma(T)\|_{L^2(0,L)} \le \|\gamma\|_{\mathcal{B}} \le \frac{1}{2} \|\Psi_1 - \Psi_2\|_{L^2(0,L)}, \end{aligned}$$

which implies the existence of a unique fixed point $\Psi(y_0, y_T, w_1, w_2) \in B_R^L \cap H$ of the map $\Pi|_{B_R^L \cap H}$. Moreover, follows the more precise proposition.

Proposition 14. *There exist $R_0 > 0$, $D > 1$, such that for every $0 < R < R_0$, for every $y_0, y_T \in B^L_{R/D}$, $(w_1, w_2) \in \mathbb{R}^2$ with $\rho_w < R/D$, there exists a unique $\Psi(y_0, y_T, w_1, w_2) \in B^L_R \cap H$ fixed point of the map $\Pi|_{B^L_R \cap H}$.*

We now apply the Brouwer fixed point theorem to the restriction of the map

$$\begin{aligned} \tau : M &\longrightarrow \quad M, \\ w_1\varphi_1 + w_2\varphi_2 &\mapsto \quad P_M(\rho_w z_w + y_T - F \circ G(\rho_w z_w + \Psi(y_0, y_T, w_1, w_2))), \end{aligned}$$

to the closed ball $B^L_{\hat{R}} \cap M$ with $\hat{R}$ small enough. Using Lemma 3.5, the continuity (in a neighborhood of $0 \in (L^2(0,L))^2 \times \mathbb{R}^2$) of the map $(y_0, y_T, w_1, w_2) \longmapsto \Psi(y_0, y_T, w_1, w_2)$ and choosing r small enough, we get the existence of a radius $\hat{R} > 0$ such that $\tau(B^L_{\hat{R}} \cap M) \subset B^L_{\hat{R}} \cap M$. This inclusion and the continuity of the map τ allow us to apply the Brouwer fixed point theorem. Therefore, there exists $(\tilde{w}_1, \tilde{w}_2) \in \mathbb{R}^2$ with $\tilde{w}_1^2 + \tilde{w}_2^2 \leq \hat{R}^2$ such that $\tilde{\Psi} := \Psi(y_0, y_T, \tilde{w}_1, \tilde{w}_2)$ satisfies

$$P_M(y_T - F \circ G(\tilde{\Psi} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2)) = 0. \tag{106}$$

Using the fact that

$$\Pi(\tilde{\Psi}) = P_H(\tilde{\Psi} + y_T - F \circ G(\tilde{\Psi} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2)) = \tilde{\Psi},$$

we obtain

$$P_H(y_T - F \circ G(\tilde{\Psi} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2)) = 0,$$

which together with (106), implies that

$$y_T = F \circ G(\tilde{\Psi} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2),$$

which ends the proof of Theorem 1.5 when M is two-dimensional.

3.3.2. *M is one-dimensional.* This case, which was in fact the first critical case solved (see [18]) requires a higher-order expansion. Indeed, it was proven in [18, Appendix B] that a second-order approximation is not good enough to get into the set $M := < 1 - \cos(x) >$. Let us give an idea of the proof of Theorem 1.5 in this case. We perform a third-order expansion $y \approx \varepsilon\alpha + \varepsilon^2\beta + \varepsilon^3\gamma$, with α solution of (69), β solution of (70) and γ solution of

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} = -(\alpha\beta)_x, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \gamma_x(t,L) = w(t), \\ \gamma(0,\cdot) = 0. \end{cases} \tag{107}$$

One first proves the following.

Proposition 15. *([18]) Let $T > 0$. There exists $(u, v, w) \in L^2(0,T)^3$ such that if α, β, γ are the solutions of (69), (70) and (107), respectively, then*

$$\alpha(T,\cdot) = 0, \quad \beta(T,\cdot) = 0 \quad \textit{and} \quad \gamma(T,\cdot) \in M\backslash\{0\}.$$

Proof. It is done by contradiction in a similar way to the proof of Proposition 12. □

Using this proposition we get γ such that $\gamma(T,\cdot) = \mu(1 - \cos(x))$ for some $\mu \in \mathbb{R}$. If (u, v, w) are the corresponding controls, then the new controls

$$\tilde{u} = \lambda u, \quad \tilde{v} = \lambda^2 v, \quad \tilde{w} = \lambda^3 w$$

are such that the new trajectories are $\tilde{\alpha} = \lambda\alpha$, $\tilde{\beta} = \lambda^2\beta$ and $\tilde{\gamma} = \lambda^3\gamma$. Thus,

$$\tilde{\alpha}(T,\cdot) = 0, \quad \tilde{\beta}(T,\cdot) = 0 \quad \textit{and} \quad \tilde{\gamma}(T,\cdot) = \mu\lambda^3(1 - \cos(x)).$$

By choosing $\lambda \in \mathbb{R}$ in an appropriate way we can reach with $\tilde{\gamma}$ any state lying in M.

Once this is known, we can proceed as in the two-dimensional case and use a fixed point argument to get Theorem 1.5. This argument is explained again in next section in the general case.

Let us notice that in this case there is no condition on the control time T.

3.3.3. *General case.* Here and in the sequel, we denote by L a critical length such that $\dim M > 2$ and by P_A the orthogonal projection on a subspace A in $L^2(0,L)$.

The first point is that for any $j \in J^>$, we can *enter* into the two-dimensional subspace M_j. The strategy is the same as in previous cases. We consider a power series expansion of (y,h) with the same scaling on the state y and on the control h. One has the following result that can be proved in the same way as before.

Proposition 16. *Let $T > 0$. For every $i = 1,\dots,n^>$, there exists $(u_i,v_i) \in L^2(0,T)^2$ such that if $\alpha_i = \alpha_i(t,x)$ and $\beta_i = \beta_i(t,x)$ are the solutions of*

$$\begin{cases} \alpha_{it} + \alpha_{ix} + \alpha_{ixxx} = 0, \\ \alpha_i(t,0) = \alpha_i(t,L) = 0, \\ \alpha_{ix}(t,L) = u_i(t), \\ \alpha_i(0,.) = 0, \end{cases} \tag{108}$$

and

$$\begin{cases} \beta_{it} + \beta_{ix} + \beta_{ixxx} = -\alpha_i\alpha_{ix}, \\ \beta_i(t,0) = \beta_i(t,L) = 0, \\ \beta_{ix}(t,L) = v_i(t), \\ \beta_i(0,.) = 0, \end{cases} \tag{109}$$

then

$$\alpha_i(T,.) = 0,\ P_H(\beta_i(T,.)) = 0 \text{ and } P_{M_i}(\beta_i(T,.)) \neq 0.$$

Let us denote, for $j = 1,\dots,n^>$,

$$\phi_i^j := P_{M_j}(\beta_i(T,.)).$$

From Proposition 16, $\phi_i^i \neq 0$. Consequently, using scaling on the controls, we can assume that $\|\phi_i^i\|_{L^2(0,L)} = 1$. Notice that the previous proposition says nothing about ϕ_i^j for $j \neq i$.

Now, we shall prove that we can reach all the states lying in the subspace

$$M^> := \bigoplus_{i \in J^>} M_i,$$

in any time $T > T^>$, where

$$T^> := \pi \sum_{i=1}^{n^>} (n^> + 1 - i)\frac{1}{p_i}.$$

In order to do that, we will strongly use the fact that if there is no control (i.e., $h = 0$) and if the initial condition lies in M_j for $j \in J^>$ (i.e., $y_0 \in M_j$), then the solution y of the linear KdV equation only turns in the two-dimensional subspace M_j with an angular velocity equal to p_j (defined in (47)) and conserves its L^2-norm. More precisely, we have the following result.

Lemma 3.6. *Let $j \in J^>$. Let $y_0 \in M_j$. Let $\lambda \geq 0$ and $\delta \in [0, 2\pi)$ be such that*

$$y_0 = \lambda \cos(\delta)\varphi_1^j + \lambda \sin(\delta)\varphi_2^j. \tag{110}$$

Then the solution of

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,\cdot) = y_0 \end{cases} \tag{111}$$

is given by

$$y(t,x) = \lambda \cos(p_j t + \delta)\varphi_1^j + \lambda \sin(p_j t + \delta)\varphi_2^j. \tag{112}$$

For the sake of brevity we introduce, for $j \in J^>$, $\theta \in \mathbb{R}$ and $y_0 \in M_j$ reading as (110), the notation

$$R^j(y_0, \theta) := \lambda \cos(\theta + \delta)\varphi_1^j + \lambda \sin(\theta + \delta)\varphi_2^j, \tag{113}$$

i.e., $R^j(\cdot, \theta)$ represents a rotation of an angle θ in the subspace M_j. Thus, the solution of (111) can be written as

$$y(t,x) = R^j(y_0, p_j t).$$

Proposition 17. *Let $T > T^>$. Let $\psi \in M^>$. There exists $(u_\psi, v_\psi) \in L^2(0,T)^2$ such that if $\alpha_\psi = \alpha_\psi(t,x)$ and $\beta_\psi = \beta_\psi(t,x)$ are the solutions of (108) and (109), then*

$$\alpha_\psi(T,.) = 0,\ \beta_\psi(T,.) = \psi.$$

Proof. First at all, let us notice that if $L = 2k\pi$ for some $k \in \mathbb{N}^*$, then $M_n = \langle 1 - \cos x \rangle$ and *a priori* $P_{M_n}(\beta_\psi(T,\cdot))$ may be non-null. However, we know from [18, Corollary 19] that a second order expansion is not sufficient to enter into the subspace M_n and therefore $P_{M_n}\beta_\psi(T.\cdot) = 0$. That is the reason for which we do not care about the projection on M_n of second-order trajectories.

The case $n^> = 1$ has already been studied. Let us consider the case $n^> = 2$, i.e., where we have 2 subspaces, M_1 and M_2 associated to (k_1, l_1) and (k_2, l_2) with $p_1 > p_2 > 0$ (for instance, $L = 2\pi\sqrt{91}$ leads to the couples $(k_1, l_1) = (16, 1)$ and $(k_2, l_2) = (11, 8)$).

Let $T > \dfrac{2\pi}{p_1} + \dfrac{\pi}{p_2}$. Let T_1 be such that

$$T_1 > \frac{\pi}{p_1} \quad \text{and} \quad T - T_1 > \frac{\pi}{p_1} + \frac{\pi}{p_2}.$$

Let $T_\theta > 0$ and $T_c > 0$ be such that

$$T_c < T_\theta, \quad T_c < \frac{\pi}{p_1},$$

$$T_c + T_\theta < \min\left(T - T_1 - \frac{\pi}{p_1} - \frac{\pi}{p_2}, \frac{\pi}{p_2} - \frac{\pi}{p_1}, T_1 - \frac{\pi}{p_1}\right).$$

Thanks to Proposition 16, there exist two pairs of controls, (u_1, v_1) and (u_2, v_2) in $L^2(0, T_c)$ such that the respective solutions of (108) and (109), (α_1, β_1) and (α_2, β_2), satisfy

$$P_{M_1}(\beta_1(T_c,\cdot)) \neq 0, \quad \text{and } P_{M_2}(\beta_2(T_c,\cdot)) \neq 0.$$

With the notations introduced before,

$$\begin{cases} (\phi_1^1, \phi_1^2) = (P_{M_1}(\beta_1(T,\cdot)), P_{M_2}(\beta_1(T,\cdot))), \\ (\phi_2^1, \phi_2^2) = (P_{M_1}(\beta_2(T,\cdot)), P_{M_2}(\beta_2(T,\cdot))). \end{cases}$$

We now use the rotation phenomena explained before and Proposition 16 to reach a basis for the missed directions lying in $M^{>}$. For the sake of clarity in our control strategy, we define for a time t_1,the following control in $L^2(0,T)$.

$$(U_{t_1},V_{t_1})(t) := \begin{cases} (0,0) & \text{if } t\in(0,t_1),\\ (u_1(t-t_1),v_1(t-t_1)) & \text{if } t\in(t_1,t_1+T_c),\\ (0,0) & \text{if } t\in(t_1+T_c,T). \end{cases}$$

This control becomes active at time $t=t_1$, between $t=t_1$ and $t=t_2$, it drives the system to enter into the space M_1 and after $t=t_2$, it becomes inactive, producing a rotation in M_1.

Now, we define the controls

$$\begin{array}{ll} (u_1^1,v_1^1) := (U_{t_1},V_{t_1}) & \text{with } t_1 = T-T_c,\\ (u_1^2,v_1^2) := (U_{t_1},V_{t_1}) & \text{with } t_1 = T-T_c-\frac{\pi}{2p_1},\\ (u_1^3,v_1^3) := (U_{t_1},V_{t_1}) & \text{with } t_1 = T-T_c-\frac{\pi}{p_1},\\ (u_1^4,v_1^4) := (U_{t_1},V_{t_1}) & \text{with } t_1 = T-T_c-\frac{\pi}{p_1}-T_\theta. \end{array}$$

Let $\alpha_1^j,\beta_1^j\in B$ be the solutions of (108) and (109) with controls u_1^j and v_1^j for $j=1,\dots,4$ and let us denote

$$\psi_1^j := P_{M_1}\beta_1^j(T,\cdot) \quad\text{and}\quad \tilde\psi_2^j := P_{M_2}\beta_1^j(T,\cdot).$$

It is easy to see that

$$\begin{array}{ll} \psi_1^1=\phi_1^1, & \tilde\psi_2^1=\phi_1^2,\\ \psi_1^2=R^1(\phi_1^1,\frac{\pi}{2}), & \tilde\psi_2^2=R^2(\phi_1^2,\frac{p_2\pi}{2p_1}),\\ \psi_1^3=R^1(\phi_1^1,\pi)=-\phi_1^1, & \tilde\psi_2^3=R^2(\phi_1^2,\frac{p_2\pi}{p_1}),\\ \psi_1^4=R^1(-\phi_1^1,p_1T_\theta), & \tilde\psi_2^4=R^2(\phi_1^2,p_2(T_\theta+\frac{\pi}{p_1})). \end{array}$$

Thus, we have constructed some controls allowing to reach the missed states

$$\psi_1^1+\tilde\psi_2^1,\quad \psi_1^2+\tilde\psi_2^2,\quad \psi_1^3+\tilde\psi_2^3,\quad\text{and}\quad \psi_1^4+\tilde\psi_2^4.$$

Now, we define for a time t_2, the following control in $L^2(0,T)$

$$(U^{t_2},V^{t_2})(t) = \begin{cases} (0,0) & \text{if } t\in(0,t_2),\\ (u_1(t-t_2),v_1(t-t_2)) & \text{if } t\in(t_2,t_2+T_c),\\ (0,0) & \text{if } t\in(t_2+T_c,t_2+\frac{\pi}{p_1}),\\ (u_1(t-t_2-\frac{\pi}{p_1}),v_1(t-t_2-\frac{\pi}{p_1})) & \text{if } t\in(t_2+\frac{\pi}{p_1},t_2+\frac{\pi}{p_1}+T_c),\\ (0,0) & \text{if } t\in(t_2+\frac{\pi}{p_1}+T_c,T), \end{cases}$$

which is the superposition of two controls of type (U_{t_1},V_{t_1})

$$(U^{t_2},V^{t_2})(t) = (U_{t_2+\frac{\pi}{p_1}},V_{t_2+\frac{\pi}{p_1}}) + (U_{t_2},V_{t_2}).$$

This fact means that the solution corresponding to the controls (U^{t_2},V^{t_2}) is the addition of two trajectories which enter into M and then turn during different times.

We define the following controls in $L^2(0,T)$.

$$\begin{array}{ll} (u_1^1,v_1^1) = (U^{t_2},V^{t_2}) & \text{with } t_2 = T-T_1-\frac{\pi}{p_1}-T_c,\\ (u_2^1,v_2^1) = (U^{t_2},V^{t_2}) & \text{with } t_2 = T-T_1-\frac{\pi}{p_1}-T_c-T_\theta,\\ (u_3^1,v_3^1) = (U^{t_2},V^{t_2}) & \text{with } t_2 = T-T_1-\frac{\pi}{p_1}-\frac{\pi}{p_2}-T_c,\\ (u_4^1,v_4^1) = (U^{t_2},V^{t_2}) & \text{with } t_2 = T-T_1-\frac{\pi}{p_1}-\frac{\pi}{p_2}-T_c-T_\theta. \end{array}$$

Let $\alpha_2^j, \beta_2^j \in B$ be the solutions of (108) and (109) with controls u_2^j and v_2^j for $j = 1, \ldots, 4$ and let us denote

$$\psi_2^j := P_{M_2}\beta_2^j(T, \cdot)$$

Here, it is very important to note that, by construction and since $p_1 > p_2$, one has

$$P_{M_1}\beta_2^1(T, \cdot) = 0 \quad \text{and} \quad \psi_2^1 = R^2(\phi_1^2, p_2 T_1) + R^2(\phi_1^2, p_2(T_1 + \pi/p_1)) \neq 0$$

Thus, we have constructed some controls allowing to reach the following missed states

$$\psi_2^1, \quad \psi_2^2, \quad \psi_2^3, \quad \text{and} \quad \psi_2^4,$$

where

$$\begin{aligned} \psi_2^2 &= R^2(\psi_2^1, p_2 T_\theta), \\ \psi_2^3 &= R^2(\psi_2^1, \pi) = -\psi_2^1, \\ \psi_2^4 &= R^2(-\psi_2^2, p_2 T_\theta). \end{aligned}$$

Furthermore, we have for $k = 1, 2$

$$M_k = \bigcup_{j=1}^{4} M_k^j \tag{114}$$

where

$$\begin{aligned} M_k^1 &:= \{d_k^1 \psi_k^1 + d_k^2 \psi_k^2; d_k^1 > 0, d_k^2 \geq 0\}, \\ M_k^2 &:= \{d_k^1 \psi_k^2 + d_k^2 \psi_k^3; d_k^1 > 0, d_k^2 \geq 0\}, \\ M_k^3 &:= \{d_k^1 \psi_k^3 + d_k^2 \psi_k^4; d_k^1 > 0, d_k^2 \geq 0\}, \\ M_k^4 &:= \{d_k^1 \psi_k^4 + d_k^2 \psi_k^1; d_k^1 > 0, d_k^2 \geq 0\}. \end{aligned}$$

Let $\psi \in M^{>}$. From (114), we know that $P_{M_1}(\psi) \in M_1^i$ for some $i \in \{1, \ldots, 4\}$. Hence, there exist $d_1^1 > 0, d_1^2 \geq 0$, such that

$$\psi = d_1^1 \psi_1^i + d_1^2 \psi_1^{i+1} + P_{M_2}(\psi).$$

Let us write ψ as follows

$$\psi = d_1^1 \psi_1^i + d_1^2 \psi_1^{i+1} + d_1^1 \tilde{\psi}_2^i + d_1^2 \tilde{\psi}_2^{i+1} + \left(P_{M_2}(\psi) - d_1^1 \tilde{\psi}_2^i - d_1^2 \tilde{\psi}_2^{i+1}\right).$$

Since the states $\tilde{\psi}_2^i, \tilde{\psi}_2^{i+1}$ lie in M_2, there exists $j \in \{1, \ldots, 4\}$ such that

$$P_{M_2}(\psi) - d_1^1 \tilde{\psi}_2^i - d_1^2 \tilde{\psi}_2^{i+1} \in M_2^j$$

and therefore there exist $d_2^1 > 0, d_2^2 \geq 0$ such that

$$\psi = d_1^1(\psi_1^i + \tilde{\psi}_2^i) + d_1^2(\psi_1^{i+1} + \tilde{\psi}_2^{i+1}) + d_2^1 \psi_2^j + d_2^2 \psi_2^{j+1}.$$

Thus, we have decomposed ψ in terms of reachables directions for the second-order expansion. Now, we take the controls u_ψ, v_ψ defined by

$$u_\psi = \sqrt{d_1^1} u_1^i + \sqrt{d_1^2} u_1^{i+1} + \sqrt{d_2^1} u_2^j + \sqrt{d_2^2} u_2^{j+1},$$

$$v_\psi = d_1^1 v_1^i + d_1^2 v_1^{i+1} + d_2^1 v_2^j + d_2^2 v_2^{j+1},$$

and $\alpha_\psi, \beta_\psi \in \mathcal{B}$ the corresponding solutions of (108) and (109), respectively. Here, it is important to note that, with the choices of T, T_1, T_c and T_θ, the supports of the trajectories α_k^j for $k = 1, 2$ and $j = 1, \dots, 4$ are disjoint and that all these trajectories go from 0 at $t = 0$ to 0 at $t = T$, i.e.,

$$\alpha_k^j(0, \cdot) = \alpha_k^j(T, \cdot) = 0.$$

Thus, it is not difficult to verify that

$$\alpha_\psi(T, \cdot) = 0 \quad \text{and} \quad \beta_\psi(T, \cdot) = \psi$$

which ends the proof in the case $n^> = 2$. The previous method can be easily adapted to the case where $n^> > 2$. In order to construct the controls needed in the general case, our method requires a time of control T greater than $T^>$. □

We assume in this section that $L = 2k\pi$ for some $k \in \mathbb{N}\backslash\{0\}$. Let us recall that in this case we have

$$M_n = \langle 1 - \cos x \rangle \quad \text{and} \quad n^> = n - 1. \tag{115}$$

The proof of the following result follows Proposition 15. See [18, Proposition 8].

Proposition 18. *Let $T_c > 0$. There exists (u, v, w) in $L^2(0, T_c)^3$ such that, if α, β, γ are the mild solutions of*

$$\begin{cases} \alpha_t + \alpha_x + \alpha_{xxx} = 0, \\ \alpha(t, 0) = \alpha(t, L) = 0, \\ \alpha_x(t, L) = u(t), \\ \alpha(0, .) = 0, \end{cases} \tag{116}$$

$$\begin{cases} \beta_t + \beta_x + \beta_{xxx} = -\alpha\alpha_x, \\ \beta(t, 0) = \beta(t, L) = 0, \\ \beta_x(t, L) = v(t), \\ \beta(0, .) = 0, \end{cases} \tag{117}$$

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} = -(\alpha\beta)_x, \\ \gamma(t, 0) = \gamma(t, L) = 0, \\ \gamma_x(t, L) = w(t), \\ \gamma(0, .) = 0, \end{cases} \tag{118}$$

then

$$\alpha(T_c, .) = 0,\ \beta(T_c, .) = 0 \text{ and } \gamma(T_c, .) = (1 - \cos x) + \sum_{i=1}^{n^>} P_{M_i}(\gamma(T_c, .)).$$

The idea to vanish the projections of $\gamma(T_c, \cdot)$ on M_i, and thus to reach the direction $(1 - \cos(x))$, is the same as before, that is, to use the rotation phenomena given in Lemma 3.6. In addition, we use the fact that the function $(1 - \cos x)$ satisfies

$$\begin{cases} y_x + y_{xxx} = 0, \\ y(0) = y(2k\pi) = y_x(2k\pi) = 0. \end{cases}$$

The case $n = 1$ has already been considered. We deal with the case $n = 2$ (for example, $L = 14\pi$ leads to the couples $(k_1, l_1) = (11, 2)$ and $(k_2, l_2) = (7, 7)$).

Let us define the following control lying in $L^2(0, T)^3$, where $T > \pi/p_1$. (Here, we omit the time translation needed for the controls u, v and w which are defined in $(0, T_c)$)

$$(u_+, v_+, w_+)(t) = \begin{cases} (0,0,0) & \text{if } t \in (0, T - T_c - \frac{\pi}{p_1}), \\ (u,v,w) & \text{if } t \in (T - T_c - \frac{\pi}{p_1}, T - \frac{\pi}{p_1}), \\ (0,0,0) & \text{if } t \in (T - \frac{\pi}{p_1}, T - T_c), \\ (u,v,w) & \text{if } t \in (T - T_c, T). \end{cases}$$

By defining $\alpha_+, \beta_+, \gamma_+ \in \mathcal{B}$ as the solutions of (116) with control u_+, (117) with control v_+ and (118) with control w_+ respectively, it is not difficult to see that

$$\alpha_+(T,.) = 0,\ \beta_+(T,.) = 0,\ \gamma_+(T,.) = 2(1 - \cos x). \tag{119}$$

Now, if we consider the control $(u_-, v_-, w_-) := (-u_+, v_+, -w_+)$ we get

$$\alpha_-(T,.) = 0,\ \beta_-(T,.) = 0,\ \gamma_-(T,.) = -2(1 - \cos x), \tag{120}$$

where obviously $\alpha_-, \beta_-, \gamma_- \in \mathcal{B}$ are the solutions of (116), (117) and (118) with controls u_-, v_- and w_- respectively. Thus we can reach all directions in M_2 in a time $T > \dfrac{\pi}{p_1}$.

We can easily deduce the same result in the case $n > 2$. We just have to construct a control that vanishes the components in the other missed subspaces M_j, $j \in J^>$. In order to do that, a time of control T, with

$$T > T^n := \pi \sum_{i=1}^{n-1} \frac{1}{p_i}, \tag{121}$$

is sufficient.

Let us apply a fixed point argument in order to prove Theorem 1.5 in the general case. If $L \neq 2k\pi$, then we can use the same proof as in the two-dimensional case and get the controllability with $T_L = T^>$. Thus the only interesting case we detail here is when $L = 2k\pi$ and $\dim M > 2$. We have to combine second and third order approximations.

Recall that for $L \in N$, we have n pairs (k_j, l_j) describing L. We have introduced some important notations

$$J^> := \{j; k_j > l_j\}, \quad n^> := |J^>|, \quad M^> := \bigoplus_{j=1}^{n^>} M_j.$$

We consider the case where $n^> = (n-1)$ and consequently where $M_n = \langle 1 - \cos x \rangle$. Thus we can write any $z \in L^2(0,L)$ as

$$z = P_H(z) + \rho_z \psi_z + d_z(1 - \cos x), \tag{122}$$

where

$$\rho_z := \|P_{M^>}(z)\|_{L^2(0,L)}, \quad \rho_z \psi_z := P_{M^>}(z), \quad \text{and} \quad d_z(1 - \cos x) = P_{M_n}(z).$$

Let us also denote, for $D > 0$ and $R > 0$,

$$B_R^D := \Big\{ \xi \in L^2(0,D)\,;\, \|\xi\|_{L^2(0,D)} \le R \Big\}.$$

From previous sections we have the existence of the controls $u_\pm$, $v_\pm$, $w_\pm \in L^2(0,T^n)$ and for every $\psi \in M^>$, the controls $u_\psi, v_\psi \in L^2(0,T^>)$. As we shall see later, we need that the corresponding trajectories of first order $\alpha_\pm$ and α_ψ are disjoint and therefore for every $z \in L^2(0,L)$ written as (122), and for every T satisfying

$$T > T_L := T^n + T^>,$$

we define the following controls lying in $L^2(0,T)$

$$(\tilde{u},\tilde{v},\tilde{w})(t) = \begin{cases} (0,0,0) & \text{if } t \in (0, T-T_L), \\ (u_{\text{sign}(d_z)}, v_{\text{sign}(d_z)}, w_{\text{sign}(d_z)})|_{(t-T+T_L)} & \text{if } t \in (T-T_L, T-T^>), \\ (0,0,0) & \text{if } t \in (T-T^>, T) \end{cases}$$

and

$$(\hat{u},\hat{v})(t) := \begin{cases} (0,0) & \text{if } t \in (0, T-T^>), \\ (u_{\psi_z}, v_{\psi_z})|_{(t-T+T^>)} & \text{if } t \in (T-T^>, T), \end{cases}$$

where we use the notation

$$\text{sign}(d_z) = \begin{cases} + & \text{if } d_z \geq 0, \\ - & \text{if } d_z < 0. \end{cases} \tag{123}$$

Let $y_0 \in L^2(0,L)$ be such that $\|y_0\|_{L^2(0,L)} < r$, where $r > 0$ has to be chosen later. Using (122), we define the functions G and F by

$$\begin{aligned} G : L^2(0,L) &\longrightarrow L^2(0,T), \\ z &\longmapsto \Gamma_0(P_H(z)) + \rho_z^{\frac{1}{2}}\hat{u} + \rho_z\hat{v} + |d_z|^{\frac{1}{3}}\tilde{u} + |d_z|^{\frac{2}{3}}\tilde{v} + |d_z|\tilde{w}, \end{aligned}$$

$$\begin{aligned} F : B^T_{\varepsilon_1} \cap L^2(0,T) &\longrightarrow L^2(0,L), \\ h &\longmapsto F(h) := y(T,\cdot), \end{aligned}$$

where $y = y(t,x)$ is the solution of

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = h(t), \\ y(0,\cdot) = y_0, \end{cases} \tag{124}$$

and ε_1 is small enough so that the function F is well defined.

Let $y_T \in L^2(0,L)$ be such that $\|y_T\| < r$. Let Λ_{y_0,y_T} denote the map

$$\begin{aligned} \Lambda_{y_0,y_T} : B^L_{\varepsilon_2} \cap L^2(0,L) &\longrightarrow L^2(0,L), \\ z &\longmapsto \Lambda_{y_0,y_T}(z) := z + y_T - F \circ G(z), \end{aligned}$$

where ε_2 is small enough so that Λ_{y_0,y_T} is well defined (see Proposition 5).

Let us remark that if we find a fixed point $\tilde{z} \in L^2(0,L)$ of the map Λ_{y_0,y_T}, then we will have

$$F \circ G(\tilde{z}) = y_T$$

which means that the control

$$h := G(\tilde{z}) \in L^2(0,T)$$

drives the solution of (124) from y_0 at $t = 0$ to y_T at $t = T$. In the following sections, we prove that such a fixed point exists.

Let us assert the following technical result which will be needed to study the map Λ_{y_0,y_T}.

Lemma 3.7. *There exist $\varepsilon_3 > 0$ and $C_1 > 0$ such that, for every $z, y_0 \in B^L_{\varepsilon_3}$, the following estimate holds*

$$\|z - F \circ G(z)\|_{L^2(0,L)} \leq C_1(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}).$$

Proof. Let $z, y_0 \in L^2(0,L)$. Let $y = y(t,x)$ be the solution of (92).

From (89) and the fact that $\rho_z \le \|z\|_{L^2(0,L)}$, one deduces that if $\|z\|_{L^2(0,L)}$ is smaller than 1 (and therefore $\|z\|_{L^2(0,L)} \le \|z\|^{1/2}_{L^2(0,L)}$), then there exists a constant C_2 such that

$$\|G(z)\|_{L^2(0,T)} \le C_2(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}). \tag{125}$$

Thus, one can find $\varepsilon_4, C_3 > 0$ such that for every $z, y_0 \in B^L_{\varepsilon_4}$, the unique solution of (92) satisfies

$$\|y\|_{\mathcal{B}} \le C_3(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}). \tag{126}$$

Let $\tilde{y}$, $\hat{\alpha}$, $\hat{\beta}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\hat{y}$ be the solutions of

$$\begin{cases} \tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = 0, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \tilde{y}_x(t,L) = \Gamma_0(P_H(z)), \\ \tilde{y}(0,\cdot) = P_H(y_0), \end{cases} \tag{127}$$

$$\begin{cases} \hat{\alpha}_t + \hat{\alpha}_x + \hat{\alpha}_{xxx} = 0, \\ \hat{\alpha}(t,0) = \hat{\alpha}(t,L) = 0, \\ \hat{\alpha}_x(t,L) = \hat{u}(t), \\ \hat{\alpha}(0,\cdot) = 0, \end{cases} \tag{128}$$

$$\begin{cases} \hat{\beta}_t + \hat{\beta}_x + \hat{\beta}_{xxx} = -\hat{\alpha}\hat{\alpha}_x, \\ \hat{\beta}(t,0) = \hat{\beta}(t,L) = 0, \\ \hat{\beta}_x(t,L) = \hat{v}(t), \\ \hat{\beta}(0,\cdot) = 0, \end{cases} \tag{129}$$

$$\begin{cases} \tilde{\alpha}_t + \tilde{\alpha}_x + \tilde{\alpha}_{xxx} = 0, \\ \tilde{\alpha}(t,0) = \tilde{\alpha}(t,L) = 0, \\ \tilde{\alpha}_x(t,L) = \tilde{u}(t), \\ \tilde{\alpha}(0,\cdot) = 0, \end{cases} \tag{130}$$

$$\begin{cases} \tilde{\beta}_t + \tilde{\beta}_x + \tilde{\beta}_{xxx} = -\tilde{\alpha}\tilde{\alpha}_x, \\ \tilde{\beta}(t,0) = \tilde{\beta}(t,L) = 0, \\ \tilde{\beta}_x(t,L) = \tilde{v}(t), \\ \tilde{\beta}(0,\cdot) = 0, \end{cases} \tag{131}$$

$$\begin{cases} \tilde{\gamma}_t + \tilde{\gamma}_x + \tilde{\gamma}_{xxx} = -(\tilde{\alpha}\tilde{\beta})_x, \\ \tilde{\gamma}(t,0) = \tilde{\gamma}(t,L) = 0, \\ \tilde{\gamma}_x(t,L) = \tilde{w}(t), \\ \tilde{\gamma}(0,\cdot) = 0, \end{cases} \tag{132}$$

$$\begin{cases} \hat{y}_t + \hat{y}_x + \hat{y}_{xxx} = 0, \\ \hat{y}(t,0) = \hat{y}(t,L) = 0, \\ \hat{y}_x(t,L) = 0, \\ \hat{y}(0,\cdot) = P_M(y_0). \end{cases} \tag{133}$$

Let us define

$$\phi := y - \tilde{y} - \rho_z^{1/2}\hat{\alpha} - \rho_z\hat{\beta} - |d_z|^{1/3}\tilde{\alpha} - |d_z|^{2/3}\tilde{\beta} - |d_z|\tilde{\gamma} - \hat{y}.$$

Then $\phi = \phi(t,x)$ satisfies

$$\begin{cases} \phi_t + \phi_x + \phi_{xxx} + \phi\phi_x = -(\phi a)_x - b, \\ \phi(t,0) = \phi(t,L) = 0, \\ \phi_x(t,L) = 0, \\ \phi(0,\cdot) = 0, \end{cases} \tag{134}$$

where $\qquad a := y - \phi,$

$$\begin{aligned} b := \ & \tilde{y}\tilde{y}_x + \hat{y}\hat{y}_x + \rho_z^2\hat{\beta}\hat{\beta}_x + \rho_z^{3/2}(\hat{\alpha}\hat{\beta})_x + |d_z|^{4/3}\tilde{\beta}\tilde{\beta}_x + |d_z|^{5/3}(\tilde{\beta}\tilde{\gamma})_x \\ & + |d_z|^{4/3}(\tilde{\alpha}\tilde{\gamma})_x + |d_z|^2\tilde{\gamma}\tilde{\gamma}_x \\ & + \Big(\tilde{y}(\rho_z^{1/2}\hat{\alpha} + \rho_z\hat{\beta} + |d_z|^{1/3}\tilde{\alpha} + |d_z|^{2/3}\tilde{\beta} + |d_z|\tilde{\gamma} + \hat{y})\Big)_x \\ & + \Big((\rho_z^{1/2}\hat{\alpha} + \rho_z\hat{\beta})(|d_z|^{1/3}\tilde{\alpha} + |d_z|^{2/3}\tilde{\beta} + |d_z|\tilde{\gamma} + \hat{y})\Big)_x \\ & + \Big(\hat{y}(|d_z|^{1/3}\tilde{\alpha} + |d_z|^{2/3}\tilde{\beta} + |d_z|\tilde{\gamma})\Big)_x. \end{aligned}$$

Here, in order to use equation (134) we need some estimates on its right-hand side.

Lemma 3.8. *There exists $C_4 > 0$ such that for every $z, y_0 \in B^L_{\varepsilon_4}$,*

$$\|\phi\|_{\mathcal{B}} \le C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}), \tag{135}$$

$$\|a\|_{\mathcal{B}} \le C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}), \tag{136}$$

$$\|b\|_{L^1(0,T,L^2(0,L))} \le C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}). \tag{137}$$

Proof of Lemma 3.8.

$$\begin{aligned} \|\phi\|_{\mathcal{B}} \le \ & \|y\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \rho_z^{1/2}\|\hat{\alpha}\|_{\mathcal{B}} + \rho_z\|\hat{\beta}\|_{\mathcal{B}} + \\ & \qquad |d_z|^{1/3}\|\tilde{\alpha}\|_{\mathcal{B}} + |d_z|^{2/3}\|\tilde{\beta}\|_{\mathcal{B}} + |d_z|\|\tilde{\gamma}\|_{\mathcal{B}} + \|\hat{y}\|_B \\ \le \ & C(\|G(z)\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)}) + C(\|\Gamma_0(P_H(z))\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)}) \\ & + C\rho_z^{1/2}\|\hat{u}\|_{L^2(0,T)} + C\rho_z(\|\hat{v}\|_{L^2(0,T)} + \|\hat{\alpha}\hat{\alpha}_x\|_{L^1(0,T,L^2(0,L))}) \\ & + C|d_z|^{1/3}\|\tilde{u}\|_{L^2(0,T)} + C|d_z|^{2/3}(\|\tilde{v}\|_{L^2(0,T)} + \|\tilde{\alpha}\tilde{\alpha}_x\|_{L^1(0,T,L^2(0,L))}) \\ & + C|d_z|(\|\tilde{w}\|_{L^2(0,T)} + \|(\tilde{\alpha}\tilde{\beta})_x\|_{L^1(0,T,L^2(0,L))}) + C\|P_M(y_0)\|_{L^2(0,L)}. \end{aligned}$$

By noticing that if $z = P_H(z) + \rho_z\psi_z + d_z(1-\cos(x))$, then

$$\|z\|^2_{L^2(0,L)} = \|P_H(z)\|^2_{L^2(0,L)} + \rho_z^2 + d_z^2\|1-\cos(x)\|^2_{L^2(0,L)},$$

and using (125) and (98), one gets (135). Estimate (136) follows from (135) and the definition of the function a. To prove (137), one uses (98) being very careful with the powers which appear. For instance, looking at the function b, one finds the term $(\rho_z^{1/2}\hat{\alpha}|d_z|^{1/3}\tilde{\alpha})$ which apparently is not bounded by $C_4\|z\|^{4/3}_{L^2(0,L)}$ for $z \in B^L_1$. This is the reason for which one takes the trajectories $\tilde{\alpha}$ and $\hat{\alpha}$ disjoint. □

Thus, from (134) one obtains the existence of $C_6 > 0$ such that

$$\|\phi\|^2_{\mathcal{B}} \le C_6(\|\phi\|^2_{\mathcal{B}}\|a\|^2_{\mathcal{B}} + \|y_0\|^2_{L^2(0,L)} + \|z\|^{8/3}_{L^2(0,L)}),$$

i.e., one has

$$\|\phi\|^2_{\mathcal{B}}(1 - C_6\|a\|^2_{\mathcal{B}}) \le C_6(\|y_0\|^2_{L^2(0,L)} + \|z\|^{8/3}_{L^2(0,L)}),$$

which, together with (96), implies the existence of ε_5 and C_7 such that for every $z, y_0 \in B^L_{\varepsilon_5}$

$$\|\phi\|_{\mathcal{B}} \leq C_7(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}). \tag{138}$$

Finally, from (138) one obtains

$$\begin{array}{rcl}\|z - F\circ G(z)\|_{L^2(0,L)} & \leq & \|z - F\circ G(z) + \hat{y}(T,\cdot)\|_{L^2(0,L)} + \|-\hat{y}(T,\cdot)\|_{L^2(0,L)} \\ & = & \|\phi(T,\cdot)\|_{L^2(0,L)} + \|\hat{y}(0,\cdot)\|_{L^2(0,L)} \\ & \leq & \|\phi\|_{\mathcal{B}} + \|y_0\|_{L^2(0,L)} \\ & \leq & C_7(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}) + \|y_0\|_{L^2(0,L)} \\ & \leq & (C_7+1)(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}),\end{array}$$

which ends the proof of Lemma 3.7 with $C_1 := C_7 + 1$ and $\varepsilon_3 := \varepsilon_5$. □

We proceed now with the fixed point argument on the space H. For $w = (w^1_1, w^2_1, \ldots, w^1_{n-1}, w^2_{n-1}, w_n) \in \mathbb{R}^{2n-1}$ fixed, let us denote

$$\Psi_w := w_n(1-\cos x) + \sum_{j=1}^{n-1}(w^1_j\varphi^1_j + w^2_j\varphi^2_j), \tag{139}$$

where the functions φ^i_j for $i = 1, 2$, $j = 1, \ldots, n-1$ are given in (48). Let us study the map

$$\Pi := P_H \circ \Lambda_{y_0,y_T}(\cdot + \Psi_w)$$

on the subspace H, i.e.,

$$\begin{array}{rcl}\Pi : H & \longrightarrow & H, \\ \Psi & \longmapsto & \Pi(\Psi) = \Psi + P_H(y_T) - P_H(F\circ G(\Psi + \Psi_w)).\end{array}$$

In order to prove the existence of a fixed point of the map Π, we will apply the Banach fixed point theorem to the restriction of Π to the closed ball $B^L_R \cap H$ with $R > 0$ small enough. Using Lemma 3.7 we see that

$$\begin{array}{rcl}\|\Pi(\Psi)\|_{L^2(0,L)} & \leq & \|y_T\|_{L^2(0,L)} + \|\Psi + \Psi_w - F\circ G(\Psi+\Psi_w)\|_{L^2(0,L)} \\ & \leq & \|y_T\|_{L^2(0,L)} + C_1(\|y_0\|_{L^2(0,L)} + \|\Psi+\Psi_w\|^{4/3}_{L^2(0,L)}) \\ & \leq & (C_1+1)(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} + |w|^{4/3}) + C_1\|\Psi\|^{4/3}_{L^2(0,L)} \\ & \leq & (C_1+1)(2r + |w|^{4/3}) + C_1\|\Psi\|^{4/3}_{L^2(0,L)}.\end{array}$$

Hence, if we choose R, r and w such that

$$R^{4/3} \leq \frac{R}{2C_1} \quad \text{and} \quad (2r + |w|^{4/3}) \leq \frac{R}{2(C_1+1)},$$

then it follows that

$$\|\Pi(\Psi)\|_{L^2(0,L)} \leq R \quad \text{and so} \quad \Pi(B^L_R \cap H) \subset (B^L_R \cap H).$$

It remains to prove that the map Π is a contraction. Let $\Psi_1, \Psi_2 \in B^L_R \cap H$. Let $y = y(t,x)$, $q = q(t,x)$, $\tilde{y} = \tilde{y}(t,x)$ and $\tilde{q} = \tilde{q}(t,x)$ be the solutions of the following problems

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = G(\Psi_1 + \Psi_w), \\ y(0,\cdot) = y_0, \end{cases}$$

$$\begin{cases} q_t + q_x + q_{xxx} + qq_x = 0, \\ q(t,0) = q(t,L) = 0, \\ q_x(t,L) = G(\Psi_2 + \Psi_w), \\ q(0,\cdot) = y_0, \end{cases}$$

$$\begin{cases} \tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = 0, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \tilde{y}_x(t,L) = \Gamma_0(\Psi_1), \\ \tilde{y}(0,\cdot) = P_H(y_0), \end{cases}$$

$$\begin{cases} \tilde{q}_t + \tilde{q}_x + \tilde{q}_{xxx} = 0, \\ \tilde{q}(t,0) = \tilde{q}(t,L) = 0, \\ \tilde{q}_x(t,L) = \Gamma_0(\Psi_2), \\ \tilde{q}(0,\cdot) = P_H(y_0). \end{cases}$$

Let us define $\phi := y - \tilde{y}$, $\psi := q - \tilde{q}$ and $\gamma := \phi - \psi$. One sees that γ satisfies

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \gamma\gamma_x = -(\gamma a)_x - b, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \gamma_x(t,L) = 0, \\ \gamma(0,\cdot) = 0, \end{cases} \tag{140}$$

where

$$a := \tilde{y} + \psi \quad \text{and} \quad b := \left(q(\tilde{y} - \tilde{q})\right)_x + (\tilde{y} - \tilde{q})(\tilde{y} - \tilde{q})_x.$$

Recall that from (101), (102),(103), there exist constants C_8, C_9 such that

$$\|b\|_{L^1(0,T,L^2(0,L))} \leq C_8 \left(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\right) \|\tilde{y} - \tilde{q}\|_{\mathcal{B}}, \tag{141}$$

$$\|(a\gamma)_x\|_{L^1(0,T,L^2(0,L))} \leq C_8 \left(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\right) \|\gamma\|_{\mathcal{B}}. \tag{142}$$

$$\|\gamma\|_{\mathcal{B}}^2 \leq C_9 (\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}})^2 (\|\tilde{y} - \tilde{q}\|_{\mathcal{B}}^2 + \|\gamma\|_{\mathcal{B}}^2). \tag{143}$$

In addition, since $z := \tilde{y} - \tilde{q}$ satisfies the following linear equation

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \\ z(t,0) = z(t,L) = 0, \\ z_x(t,L) = \Gamma_0(\Psi_1) - \Gamma_0(\Psi_2), \\ z(0,\cdot) = 0, \end{cases}$$

there exists $C_{10} > 0$ such that

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \leq C_{10} \|\Gamma_0(\Psi_1) - \Gamma_0(\Psi_2)\|_{L^2(0,T)}$$

and so, from (90), one gets

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \leq C_{10} D_2 \|\Psi_1 - \Psi_2\|_{L^2(0,L)}. \tag{144}$$

Moreover, it is easy to see that there exists a constant $C_{11} > 0$ such that

$$\|q\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} \leq C_{11} (\|y_0\|_{L^2(0,L)} + \|\Psi_1\|_{L^2(0,L)} + \|\Psi_2\|_{L^2(0,L)} + |w|^{1/3}). \tag{145}$$

Thus, using (143), (144) and (145) we see that if $R, |w|, r$ are small enough, it follows that

$$\|\gamma\|_{\mathcal{B}} \leq \frac{1}{2} \|\Psi_1 - \Psi_2\|_{L^2(0,L)}.$$

Therefore, we have

$$\begin{aligned}\|\Pi(\Psi_1) - \Pi(\Psi_2)\|_{L^2(0,L)} \leq \\ \|\Psi_1 - F\circ G(\Psi_1+\Psi_w) - \Psi_2 + F\circ G(\Psi_2+\Psi_w)\|_{L^2(0,L)} \\ = \|\gamma(T)\|_{L^2(0,L)} \leq \|\gamma\|_{\mathcal{B}} \leq \frac{1}{2}\|\Psi_1-\Psi_2\|_{L^2(0,L)},\end{aligned}$$

which implies the existence of a unique fixed point $\Psi(y_0, y_T, w) \in B^L_R \cap H$ of the map $\Pi|_{B^L_R \cap H}$.

Let us look for a fixed point in M. We now apply the Brouwer fixed point theorem to the restriction of the map

$$\begin{aligned}\tau : M &\longrightarrow M,\\ \Psi_w &\mapsto P_M(\Psi_w + y_T - F\circ G(\Psi_w + \Psi(y_0, y_T, w))),\end{aligned}$$

to the closed ball $B^L_{\hat{R}} \cap M$ with $\hat{R}$ small enough.

The controls $\hat{u}, \hat{v}, \tilde{u}, \tilde{v}$ and $\tilde{w}$ can be chosen in such a way so that the function G is continuous. Thus, it is easy to see that the map $(y_0, y_T, w) \longmapsto \Psi(y_0, y_T, w)$ is also continuous in a neighborhood of $0 \in L^2(0,L)^2 \times \mathbb{R}^{2n-1}$. Using this continuity, Lemma 3.7, and choosing r small enough, we get the existence of a radius $\hat{R} > 0$ such that $\tau(B^L_{\hat{R}} \cap M) \subset B^L_{\hat{R}} \cap M$. This inclusion and the continuity of the map τ allow us to apply the Brouwer fixed point theorem. Therefore, there exists $\tilde{w} \in \mathbb{R}^{2n-1}$ with $|\tilde{w}| \leq \hat{R}$ such that $\tilde{\Psi} := \Psi(y_0, y_T, \tilde{w})$ satisfies

$$P_M(y_T - F\circ G(\tilde{\Psi} + \Psi_{\tilde{w}})) = 0. \tag{146}$$

Using the fact that

$$\Pi(\tilde{\Psi}) = P_H(\tilde{\Psi} + y_T - F\circ G(\tilde{\Psi} + \Psi_{\tilde{w}})) = \tilde{\Psi},$$

we obtain

$$P_H(y_T - F\circ G(\tilde{\Psi} + \Psi_{\tilde{w}})) = 0,$$

which together with (146), implies that

$$y_T = F\circ G(\tilde{\Psi} + \Psi_{\tilde{w}}),$$

which ends the proof of Theorem 1.5.

4. **Internal stabilization.** This section is devoted to the proof of Theorems 1.6, 1.7 and 1.8. We define for each time the energy of the solution of our KdV equation as its L^2-norm

$$E(t) = \int_0^L |y(t,x)|^2\,dx.$$

We are interested in the long-time behavior of the energy $E(t)$. More precisely we want to prove the exponential decay of $E(t)$ as t goes to infinity. First, we will prove the stability for the linear system on a noncritical domain. Second, we deal with the linear case on a critical domain. Finally, for the nonlinear system we will get a local result and a semi-global result by applying two different approaches.

4.1. **Linear system on a noncritical interval.** We consider the linear system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = y_0, \end{cases} \tag{147}$$

for a noncritical domain $L \notin \mathcal{N}$. The observability inequality (29) holds, which can be written for the direct linear system as follows

$$\forall T > 0, \exists C > 0, \forall y_0 \in L^2(0,L), \quad \|y_x(\cdot,0)\|_{L^2(0,T)} \geq C\|y_0\|_{L^2(0,L)}. \tag{148}$$

By performing integration by parts in the equation

$$\int_0^L (y_t + y_x + y_{xxx})y \, dx = 0$$

we get

$$\frac{d}{ds}\int_0^L |y(s,x)|^2 \, dx = -|y_x(s,0)|^2 \leq 0 \tag{149}$$

and then by integrating on $(0,1)$ and using (148) with $T = 1$ we obtain the existence of C such that

$$\int_0^L |y(1,x)|^2 \, dx - \int_0^L |y_0(x)|^2 \, dx = -\int_0^1 |y_x(s,0)|^2 \, ds \leq -\frac{1}{C^2}\int_0^L |y_0(x)|^2 \, dx,$$

that implies

$$\int_0^L |y(1,x)|^2 \, dx \leq \frac{C^2-1}{C^2}\int_0^L |y_0(x)|^2 \, dx.$$

Of course we also have

$$\int_0^L |y(t+1,x)|^2 \, dx \leq \frac{C^2-1}{C^2}\int_0^L |y(t,x)|^2 \, dx, \tag{150}$$

which gives the exponential decay to the origin of the solutions. Indeed, let $k \leq t \leq k+1$. From (149), (150) and denoting $\gamma := \frac{C^2-1}{C^2} < 1$, we have

$$\begin{aligned} E(t) \leq E(k) \leq \gamma E(k-1) \leq \gamma^2 E(k-2) \leq \cdots \leq \gamma^k E(0) \\ = \frac{\gamma^{k+1}}{\gamma}E(0) = \frac{1}{\gamma}e^{(k+1)\ln(\gamma)}E(0) \leq \frac{1}{\gamma}e^{-t|\ln(\gamma)|}E(0) \end{aligned}$$

which ends the proof of Thereom 1.6 with $a = 0$ in a noncritical domain.

4.2. **Linear system on a critical interval.** If there is no damping ($a = 0$), we know from the controllability analysis that in a critical domain, there are solutions of the linear system which do not decay to zero. In this way a dissipative mechanism is needed in this case. We consider an internal damping given by the term $a(x)y$ in

$$\begin{cases} y_t + y_x + y_{xxx} + ay = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = y_0, \end{cases} \tag{151}$$

with $a \in L^\infty(0,L)$ possibly localized in a small subdomain of $(0,L)$:

$$\begin{cases} a(x) \geq a_0 > 0, \quad \forall x \in \omega, \\ \text{where } \omega \text{ is nonempty open subset of } (0,L). \end{cases} \tag{152}$$

Proof of Theorem 1.6. By performing integration by parts in the equation

$$\int_0^L (y_t + y_x + y_{xxx} + ay)y\,dx = 0$$

we get

$$\frac{d}{ds}\int_0^L |y(s,x)|^2\,dx = -|y_x(s,0)|^2 - \int_0^L a(x)|y(s,x)|^2\,dx \le 0 \tag{153}$$

and then by integrating on $(0,t)$ we obtain

$$\int_0^L |y(t,x)|^2\,dx - \int_0^L |y_0(x)|^2\,dx = -\int_0^t |y_x(s,0)|^2\,ds - \int_0^t\int_0^L a(x)|y(s,x)|^2\,dxds$$

The same proof as before runs if we are able to prove that $\forall T>0, \exists C>0$ such that

$$\forall y_0 \in L^2(0,L),\ \|y_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|y(s,x)|^2\,dxdt \ge C^2\|y_0\|^2_{L^2(0,L)}. \tag{154}$$

From direct computations (as in (20) and considering the term ay), we obtain

$$\|y_0\|^2_{L^2(0,L)} \le \frac{1}{T}\|y\|^2_{L^2(0,T;L^2(0,L))} + \|y_x(\cdot,0)\|^2_{L^2(0,T)} + 2\int_0^T\int_0^L a(x)|y(t,x)|^2\,dxdt, \tag{155}$$

and therefore we will be done if we prove that there exists a constant $K>0$ such that

$$K\|y\|^2_{L^2(0,T;L^2(0,L))} \le \|y_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|y(t,x)|^2\,dxdt. \tag{156}$$

As we did in the proof of the inequality observability we proceed by contradiction. By assuming that (156) does not hold, we build a sequence of initial data $\{y_0^n\}_{n\in\mathbb{N}} \subset L^2(0,L)$ such that $\|y_0^n\|_{L^2(0,L)} = 1$ and the corresponding solutions of (151) satisfies

$$\|y_x^n(\cdot,0)\|_{L^2(0,L)} \to 0, \quad \int_0^T\int_0^L a(x)|y^n(t,x)|^2\,dxdt \to 0, \quad \text{as } n\to\infty$$

As previously, we can pass to the limit and get a nontrivial solution $y \in \mathcal{B}$ of (151) satisfying

$$y_x(\cdot,0) = 0, \quad \int_0^T\int_0^L a(x)|y(t,x)|^2\,dxdt = 0, \tag{157}$$

which implies that $ay = 0$ and therefore the limit y is solution of

$$y_t + y_x + y_{xxx} = 0. \tag{158}$$

In order to get a contradiction, we have to use the property (152) of $a = a(x)$. Recall that ω is a nonempty open subset of $(0,L)$ as small as we want. From (157) and (152) we get that the limit solution is zero in ω, i.e.,

$$y(t,x) = 0, \quad \forall x\in\omega,\ \forall t\in(0,T).$$

As the equation (158) is linear, latter condition implies the solution y is zero everywhere because of the Holmgrem's Uniqueness Theorem. This is a contradiction and consequently we have proved Theorem 1.6. $\square$

Remark 12. An ad-hoc Carleman estimate can also be used in order to quantify this unique continuation property. In the context of KdV equations on bounded domains, this kind of estimates have already been obtained. In [37, 24] some one-parameter Carleman estimates are proved to get some boundary observability inequalities. In [2] a two-parameter Carleman estimate is obtained to solve an inverse problem with boundary measurements. The latter can be easily adapted to obtain the unique continuation property.

4.3. **Nonlinear system.** First, we prove Theorem 1.7, i.e., the exponential decay of small amplitude solutions. This is basically a linear result deduced from applying Theorem 1.6 to system

$$\begin{cases} y_t + y_x + y_{xxx} + ay + yy_x = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = y_0. \end{cases} \tag{159}$$

Proof of Theorem 1.7. Consider $\|y_0\|_{L^2(0,L)} \leq r$ with r to be chosen later. The solution y of (159) can be written as $y = y^1 + y^2$ where y^1 is the solution of

$$\begin{cases} y^1_t + y^1_x + y^1_{xxx} + ay^1 = 0, \\ y^1(t,0) = y^1(t,L) = y^1_x(t,L) = 0, \\ y^1(0,x) = y_0 \end{cases} \tag{160}$$

and y^2 is the solution of

$$\begin{cases} y^2_t + y^2_x + y^2_{xxx} + ay^2 = -yy_x, \\ y^2(t,0) = y^2(t,L) = y^2_x(t,L) = 0, \\ y^2(0,x) = 0. \end{cases} \tag{161}$$

Thus, from (150) and the energy estimates for linear systems, we have

$$\begin{aligned} \|y(t,\cdot)\|_{L^2(0,L)} &\leq \|y^1(t,\cdot)\|_{L^2(0,L)} + \|y^2(t,\cdot)\|_{L^2(0,L)} \\ &\leq \gamma\|y_0\|_{L^2(0,L)} + C\|yy_x\|_{L^1(0,T;L^2(0,L))} \leq \gamma\|y_0\|_{L^2(0,L)} + C\|y\|^2_{L^2(0,T;H^1(0,L))} \end{aligned} \tag{162}$$

with $\gamma < 1$. Of course we need somewhere a nonlinear estimate and it is here in order to deal with the last term in the previous inequality. Let us multiply equation (159) by xy and integrate to obtain

$$\begin{aligned} 3\int_0^T\int_0^L |y_x|^2 dxdt + \int_0^L x|y(T,\cdot)|^2 dx + 2\int_0^T\int_0^L xa|y|^2 dxdt \\ = \int_0^T\int_0^L |y|^2 dxdt + \int_0^L x|y_0|^2 dx - 2\int_0^T\int_0^L xy_x|y|^2 dxdt. \end{aligned} \tag{163}$$

Using

$$3\int_0^T\int_0^L xy_x|y|^2 dxdt = -\int_0^T\int_0^L |y|^3 dxdt$$

into (163) we get

$$\|y\|^2_{L^2(0,T;H^1(0,L))} \leq \frac{(3T+L)}{3}\|y_0\|_{L^2(0,L)} + \frac{2}{9}\int_0^T\int_0^L |y|^3 dxdt. \tag{164}$$

As $y \in L^2(0,T;H^1(0,L))$ and $H^1(0,L)$ embeds into $C([0,L])$, we have

$$\int_0^T\int_0^L |y|^3 dxdt \leq \int_0^T \|y\|_{L^\infty(0,L)} \int_0^L |y|^2 dxdt \leq C\int_0^T \|y\|_{H^1(0,L)} \int_0^L |y|^2 dxdt$$
$$\leq C\|y\|^2_{L^\infty(0,T;L^2(0,L))} \int_0^T \|y\|_{H^1(0,L)} dt \leq CT^{1/2}\|y_0\|^2_{L^2(0,L)}\|y\|_{L^2(0,T;H^1(0,L))}.$$

Thanks to this and (164) we obtain

$$\|y\|^2_{L^2(0,T;H^1(0,L))} \leq \frac{(8T+2L)}{3}\|y_0\|^2_{L^2(0,L)} + \frac{TC}{27}\|y_0\|^4_{L^2(0,L)} \tag{165}$$

which combined with (162) gives the existence of $C > 0$ such that

$$\|y(t,\cdot)\|_{L^2(0,L)} \leq \|y_0\|_{L^2(0,L)} \left\{\gamma + C\|y_0\|_{L^2(0,L)} + C\|y_0\|^3_{L^2(0,L)}\right\}. \tag{166}$$

Given $\varepsilon > 0$ small enough such that $(\gamma+\varepsilon) < 1$, we can take r small enough so that $r + r^3 < \frac{\varepsilon}{C}$, in order to have

$$\|y(t,\cdot)\|_{L^2(0,L)} \leq (\gamma+\varepsilon)\|y_0\|_{L^2(0,L)}$$

The rest of the proof runs as before thanks to the fact that $(\gamma+\varepsilon) < 1$. Thus, we end the proof of Theorem 1.7. □

Remark 13. In previous result, constants C and μ can be chosen as close as we want to the corresponding constants for the linear result given by Theorem 1.6. Of course, the smaller ε is chosen, the smaller is the radius r defining the set of initial data for which the exponential decay rate is valid.

Now, we focus on the proof of Theorem 1.8. Here, we will deal directly with the nonlinear equation in order to get the semi-global result. There arise two main difficulties. On one hand there are nonlinear terms in the equation (159) which are hard to deal with in order to pass to the limit as in the previous argument. On the other hand, the nonlinear character of (159) prevents the use of Holmgrem's Theorem. Instead of that result we will have to use a nonlinear unique continuation result. We will use a result by Saut and Scheurer, which can be written as follows.

Theorem 4.1. *([43, Theorem 4.2]) Let $u \in L^2(0,T;H^3(0,L))$ be a solution of*

$$u_t + u_x + u_{xxx} + uu_x = 0$$

such that

$$u(t,x) = 0, \quad \forall t \in (t_1,t_2), \forall x \in \omega$$

with ω an open nonempty subset of $(0,L)$. Then

$$u(t,x) = 0, \quad \forall t \in (t_1,t_2), \forall x \in (0,L).$$

In order to apply this result, we have to prove that the limit solution in our contradiction argument is more regular, at least in $L^2(0,T;H^3(0,L))$. See also [49, 22, 26].

Proof of Theorem 1.8. Let us notice that $\int_0^L y^2 y_x dx = 0$ and therefore the same computations done in Section 4.2 say that if we integrate by parts in the equation

$$\int_0^L (y_t + y_x + y_{xxx} + ay + yy_x) y \, dx = 0,$$

we get

$$\int_0^L |y(t,x)|^2\,dx - \int_0^L |y_0(x)|^2\,dx = \\ -\int_0^t |y_x(s,0)|^2\,ds - \int_0^t\int_0^L a(x)|y(s,x)|^2\,dxds. \quad (167)$$

The same proof as before runs if we are able to prove that $\forall\, T,R>0$, $\exists\, K(T,R)>0$ such that

$$\|y_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|y(t,x)|^2\,dxdt \geq K\|y_0\|^2_{L^2(0,L)} \quad (168)$$

for any initial data satisfying $\|y_0\|_{L^2(0,L)} \leq R$. We can easily see that it is enough to prove that for any $T,R>0$ there exists a positive constant $K=K(T,R)$ such that

$$K\|y\|^2_{L^2(0,T;L^2(0,L))} \leq \|y_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|y(t,x)|^2\,dxdt \quad (169)$$

for solutions of the nonlinear system (159) with $\|y_0\|_{L^2(0,L)} \leq R$. Indeed, integrating in time (167) we get

$$T\int_0^L |y_0(x)|^2\,dx \leq \\ \int_0^T\int_0^L |y(t,x)|^2\,dxdt + T\int_0^T |y_x(t,0)|^2\,dt + T\int_0^T\int_0^L a(x)|y(t,x)|^2\,dxdt,$$

which together (169) imply (168).

We assume that (169) does not hold. By choosing $K=1/n$, we built a sequence $\{y^n\}_{n\in\mathbb{N}} \subset \mathcal{B}$ solving (159) with $\|y^n(0,\cdot)\|_{L^2(0,L)} \leq R$ and such that

$$\lim_{n\to\infty} \frac{\|y^n\|^2_{L^2(0,T;L^2(0,L))}}{\|y^n_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|y^n(t,x)|^2\,dxdt} = \infty.$$

We define $\lambda_n := \|y^n\|_{L^2(0,T;L^2(0,L))}$ and $v^n := \frac{y^n}{\lambda_n}$. We get that v_n satisfies

$$\begin{cases} v^n_t + v^n_x + v^n_{xxx} + av^n + \lambda_n v^n v^n_x = 0, \\ v^n(t,0) = v^n(t,L) = v^n_x(t,L) = 0, \\ \|v_n\|_{L^2(0,T;L^2(0,L))} = 1 \end{cases} \quad (170)$$

and

$$\|v^n_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|v^n(t,x)|^2\,dxdt \to 0 \quad (171)$$

as $n\to\infty$.

Notice that (155) still holds for solutions of the nonlinear equation thanks to the fact

$$\int_0^L y^2 y_x dx = 0.$$

Using (155) we see that $\{v^n(0,\cdot)\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,L)$. From (165), $\{v^n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^1(0,L))$. Thus, we can prove that $\{v^n v^n_x\}_{n\in\mathbb{N}}$ is a subset of $L^2(0,T;L^1(0,L))$. In fact

$$\|v^n v^n_x\|_{L^2(0,T;L^1(0,L))} \leq \|v^n\|_{C([0,T];L^2(0,L))}\|v^n\|_{L^2(0,T;H^1(0,L))}.$$

All this is used with the equation (170) to say that $\{v_t^n\}_{n\in\mathbb{N}}$ is bounded in the space $L^2(0,T;H^1(0,L))$ and consequently (see Lemma 3.1) a subsequence of $\{v_n\}_{n\in\mathbb{N}}$, also denoted by $\{v_n\}_{n\in\mathbb{N}}$ converges strongly in $L^2(0,T;L^2(0,L))$ to a limit v with $\|v\|_{L^2(0,T;L^2(0,L))}=1$. Furthermore,

$$\|v_x(\cdot,0)\|^2_{L^2(0,T)}+\int_0^T\int_0^L a(x)|v(t,x)|^2\,dxdt \\ \leq \liminf_{n\to\infty}\left\{\|v_x^n(\cdot,0)\|^2_{L^2(0,T)}+\int_0^T\int_0^L a(x)|v^n(t,x)|^2\,dxdt\right\}=0$$

and therefore

$$v(t,x)=0,\quad \forall x\in\omega,\forall t\in(0,T),\quad \text{and } v_x(t,0)=0\quad \forall t\in(0,T). \tag{172}$$

Since $\|y^n(0,\cdot)\|_{L^2(0,L)}\leq R$ we can extract from $\{\lambda_n\}_{n\in\mathbb{N}}$ a convergent subsequence, still denoted by $\{\lambda_n\}_{n\in\mathbb{N}}$, such that $\lambda_n\to\lambda$ with $\lambda\geq 0$. Thus, the limit function v satisfies

$$\begin{cases} v_t+v_x+v_{xxx}+\lambda vv_x=0,\\ v(t,0)=v(t,L)=v_x(t,L)=0,\\ \|v(0,\cdot)\|_{L^2(0,L)}\leq R,\quad \|v\|_{L^2(0,T;L^2(0,L))}=1,\end{cases} \tag{173}$$

with either $\lambda=0$ or $\lambda>0$.

We have now two possibilities:

- If $\lambda=0$, then v satisfies the linear equation and condition (172). Thus, we can apply Holmgrem's Theorem to get that the solution v is the trivial one and get a contradiction.
- If $\lambda>0$, then v satisfies the nonlinear equation and condition (172). Moreover, we can prove that $v\in L^2(0,T;H^3(0,L))$. Indeed, let us consider $u:=v_t$, which satisfies

  $$\begin{cases} u_t+u_x+u_{xxx}+\lambda v_xu+\lambda vu_x=0,\\ u(t,0)=u(t,L)=u_x(t,L)=0,\end{cases} \tag{174}$$

  with

  $$u(0,\cdot)=-v'(0,\cdot)-v'''(0,\cdot)-\lambda v(0,\cdot)v'(0,\cdot)\in H^{-3}(0,L)$$

  and

  $$u(t,x)=0,\quad \forall x\in\omega,\forall t\in(0,T),\quad \text{and } u_x(t,0)=0\quad \forall t\in(0,T).$$

  From Lemma A.2 (in the Appendix) we get $u(0,\cdot)\in L^2(0,L)$ and so $u=v_t\in\mathcal{B}$. In this way, $v_{xxx}=(-v_t-v_x-\lambda vv_x)\in L^2(0,T;L^2(0,L))$ and therefore $v\in L^2(0,T;H^3(0,L))$. We have used that $v,v_t\in L^2(0,T;H^1(0,L))$ and hence $v\in C([0,T];H^1(0,L))$ in order to prove that $vv_x\in L^2(0,T;L^2(0,L))$.

  Finally we can conclude that the solution v is the trivial one by applying Theorem 4.1 and consequently we obtain a contradiction.

We have seen that any of two possibilities ($\lambda=0$ or $\lambda>0$) gives a contradiction, which ends the proof of Theorem 1.8. □

Remark 14. The semi-global character of this result comes from the fact that even if we are able to chose any radius R for the initial data, the decay rate μ depends on R. In the proof, we see that we were able to pass to the limit because the initial data were bounded.

5. **Some open problems.** In this section we state some open problems concerning controllability and stabilization for the Korteweg-de Vries control system.

Open Problem 1. In Theorem 1.5, we get the local controllability for (65) provided that the time of control is large enough. Is this condition on the time really necessary?

This is an interesting open problem since one knows that even if the speed of propagation of the Korteweg-de Vries equation is infinite, it may exist a minimal time of control. This is for example the case of a nonlinear control system for the Schrödinger equation studied by Beauchard and Coron in [4]. They proved the local controllability of this system along the ground state trajectory for a large time and Coron proved in [16] and [17, Theorem 9.8] that this local controllability does not hold in small time, even if the Schrödinger equation has an infinite speed of propagation.

Open Problem 2. What is the minimal regularity of the solution $u = u(t, x)$ in Thereom 4.1?

In the proof of Theorem 1.7 we use the Unique Continuation Principle for the nonlinear KdV equation given by Theorem 4.1. To do that, we first prove that the solution is regular enough. With a result requiring a less regular solution, this step may not be necessary.

Open Problem 3. Let L be a critical length. Let $y_0 \in L^2(0, L)$ and y the solution of

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t, 0) = y(t, L) = y_x(t, L) = 0, \\ y(0, \cdot) = y_0. \end{cases}$$

Does the solution y decay to zero as t goes to infinity?

In other words, the nonlinearity gives us the stability in the critical cases as it does concerning controllability? In order to answer this question, a really nonlinear method is needed because with a first-order approximation one obtains the linear system which has some solution conserving its L^2-norm. On the other hand, it is not clear that the power series expansion method works. It strongly needs the controls to be able to use higher-order approximations.

Open Problem 4. Is the system

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t, 0) = 0,\ y(t, L) = h(t),\ y_x(t, L) = 0, \end{cases}$$

exactly controllable?

This equation was studied in [25] where they proved that the linearized system around the origin is exactly controllable if and only if L does not belong to a set of critical values, which is different to the one presented here. Does the nonlinearity give the controllabilty? In this case, one difficulty is that there is no an explicit expression for the critical lengths.

Open Problem 5. Is the system

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t, 0) = 0,\ y_x(t, L) = h(t),\ y_{xx}(t, L) = 0, \end{cases}$$

exactly controllable?

As in the previous open problem, the linear system is exactly controllable if and only if L does not belong to a set of critical values, which is different from those already mentioned. See [11] where the noncritical cases are solved as well as the cases where two or three boundary controls are considered. To address this problem and the previous one, a possible approach would be to prove the exact controllability of the nonlinear equation around nontrivial stationary solutions (as proved by Crépeau in [20, 21] in the case of homogeneous Dirichlet boundary conditions), and then to apply the method introduced in [15] (see also [3, 4]), that is the return method (see [13, 14]) together with quasi-static deformations (see also [19]). With such a method, one should obtain the exact controllability for a large time. However, it seems that the minimal time required with this approach is far from being optimal.

Open Problem 6. Is the system

$$\left\{\begin{array}{l} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) - y_x(t,0) = h(t), \end{array}\right.$$

well-posed in $L^2(0,L)$ or $H^1(0,L)$?

This problem is linked with the boundary stabilization. There are some feedback laws $h = F(y)$ stabilizing the linearized system around the origin (see [10]), but for these boundary conditions there is no Kato smoothing effect allowing us to deal with the nonlinearity.

Appendix A. Proofs of some technical lemmas.

Lemma A.1. *The function S is not identically equal to 0.*

Proof. Let $a \in \Sigma$ and $\lambda = 2ai(4a^2 - 1)$. Let $\mu \in \mathbb{C}$ and let $y_\mu = y_\mu(x)$ be a solution of

$$\left\{\begin{array}{l} \mu y_\mu + y_\mu' + y_\mu''' = 0, \\ y_\mu(0) = y_\mu(L) = 0. \end{array}\right.$$

We multiply (81) by y_μ and integrate by parts on $[0,L]$. Thus, we get

$$(\lambda - ip + \mu)\int_0^L \phi_\lambda y_\mu dx - \phi_\lambda'(L)\big(y_\mu'(L) - y_\mu'(0)\big) = \int_0^L y_\lambda \varphi' y_\mu dx. \tag{175}$$

From now on, we set $\mu = \mu(a) := -\lambda + ip$. With this choice we obtain from (175)

$$-S(a)\big(y_\mu'(L) - y_\mu'(0)\big) = \int_0^L y_\lambda \varphi' y_\mu dx.$$

Therefore, if we prove that the function

$$a \in \Sigma \longrightarrow J(a) := \int_0^L y_\lambda \varphi' y_\mu dx \in \mathbb{C},$$

is not identically equal to 0, the proof of this lemma is ended. Let $b \in \mathbb{R}$ be such that $\mu = 2bi(4b^2 - 1)$. We take the function y_μ given by

$$y_\mu(x) = De^{(-\sqrt{3b^2-1}-bi)x} + (1-D)e^{(\sqrt{3b^2-1}-bi)x} - e^{2bix}, \tag{176}$$

where

$$D = \frac{e^{2biL} - e^{(\sqrt{3b^2-1}-bi)L}}{e^{(-\sqrt{3b^2-1}-bi)L} - e^{(\sqrt{3b^2-1}-bi)L}}.$$

In the next computations, we use the fact that $e^{i\gamma_1 L} = e^{i\gamma_2 L} = e^{i\gamma_3 L}$ (see (49)) and the following formula

$$\int_0^L e^{(v+iw)x}\varphi' = \frac{\big(1+\gamma_1^2 - 2p/\gamma_1\big)\big(1-e^{(v+iw+i\gamma_1)L}\big)(vi-w)}{(vi-w)^3-(vi-w)+p} \tag{177}$$

which holds if $v+iw \neq -i\gamma_m$ for $m=1,2,3$.

We want to show that as $a\to\infty$, the following expression diverges, which is in contradiction with the fact that $J(a)\equiv 0$

$$R(a) := \frac{(e^{(-\sqrt{3a^2-1}-ai)L} - e^{(\sqrt{3a^2-1}-ai)L})(e^{(-\sqrt{3b^2-1}-bi)L} - e^{(\sqrt{3b^2-1}-bi)L})}{1+\gamma_1^2-2p/\gamma_1} J(a).$$

In fact, by using (177), one computes explicitly $J(a)$ and thus one sees that as a tends to infinity, the dominant term of $R(a)$ is given by

$$\begin{aligned}
Z(a) := e^{(\sqrt{3a^2-1}+\sqrt{3b^2-1})L}\Big\{ &\frac{(e^{(-ai-bi)L} - e^{(ai+bi+\gamma_1 i)L})(-2a-2b)}{(-2a-2b)^3-(-2a-2b)+p} \\
&+ \frac{e^{(-ai-bi)L}(-i\sqrt{3a^2-1}-i\sqrt{3b^2-1}+a+b)}{(-i\sqrt{3a^2-1}-i\sqrt{3b^2-1}+a+b)^3-(-i\sqrt{3a^2-1}-i\sqrt{3b^2-1}+a+b)+p} \\
&- \frac{e^{(ai+bi+\gamma_1 i)L}(i\sqrt{3a^2-1}+i\sqrt{3b^2-1}+a+b)}{(i\sqrt{3a^2-1}+i\sqrt{3b^2-1}+a+b)^3-(i\sqrt{3a^2-1}+i\sqrt{3b^2-1}+a+b)+p} \\
&+ \frac{e^{(ai+bi+\gamma_1 i)L}(i\sqrt{3a^2-1}+a-2b)}{(i\sqrt{3a^2-1}+a-2b)^3-(i\sqrt{3a^2-1}+a-2b)+p} \\
&- \frac{e^{(-ai-bi)L}(-i\sqrt{3b^2-1}-2a+b)}{(-i\sqrt{3b^2-1}-2a+b)^3-(-i\sqrt{3b^2-1}-2a+b)+p} \\
&+ \frac{e^{(ai+bi+\gamma_1 i)L}(i\sqrt{3b^2-1}-2a+b)}{(i\sqrt{3b^2-1}-2a+b)^3-(i\sqrt{3b^2-1}-2a+b)+p} \\
&- \frac{e^{(-ai-bi)L}(-i\sqrt{3a^2-1}+a-2b)}{(-i\sqrt{3a^2-1}+a-2b)^3-(-i\sqrt{3a^2-1}+a-2b)+p}\Big\}
\end{aligned}$$

Using that as $a\to\infty$, $b\to-\infty$ and $a+b\sim -p/(24a^2)$, we obtain the following asymptotical expression for the right hand factor of $Z(a)$,

$$\frac{-(e^{\frac{p}{24a^2}iL} - e^{-\frac{p}{24a^2}iL+i\gamma_1 L})}{12a^2} \sim \begin{cases} -\frac{(1-e^{i\gamma_1 L})}{12a^2} & \text{if } e^{i\gamma_1 L}\neq 1, \\ -\frac{ipL}{144a^4} & \text{if } e^{i\gamma_1 L} = 1. \end{cases}$$

One can see that in both cases $Z(a)$ diverges as $a\to\infty$ and therefore $R(a)$ does, which implies that $J(a)$ is not identically equal to 0. It ends the proof of this lemma. $\square$

Lemma A.2. *(*[33, Lemma 3.2]*) There exists a constant $C = C(T,R)$ such that*

$$\|u_x(\cdot,0)\|^2_{L^2(0,T)} + \|u(0,\cdot)\|^2_{H^{-3}(0,L)} \geq C\|u(0,\cdot)\|^2_{L^2(0,L)}.$$

holds for any solution u of (174) *with v solution of* (173).

Proof. By multiplying (174) by $(T-t)u$, we can deduce

$$\|u(0,\cdot)\|^2_{L^2(0,L)} \leq \frac{1}{T}\|u\|^2_{L^2(0,T;L^2(0,L))} + \|u_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L v_x|u(t,x)|^2\,dxdt$$

Using previous estimates and

$$\int_0^T \int_0^L v_x |u(t,x)|^2 \, dxdt \le \int_0^T \|v_x\|_{L^2(0,L)} \|u\|^2_{L^4(0,L)}$$
$$\le \|v\|_{L^2(0,T;H^1(0,L))} \|u\|^2_{L^4(0,T;L^4(0,L))}$$

we get a constant $C = C(T,R)$ such that

$$\|u(0,\cdot)\|^2_{L^2(0,L)} \le C\|u\|^2_{L^4(0,T;L^4(0,L))} + \|u_x(\cdot,0)\|^2_{L^2(0,T)} \tag{178}$$

We see that in order to prove this Lemma it is enough to prove

$$\|u\|^2_{L^4(0,T;L^4(0,L))} \le C\left\{\|u_x(\cdot,0)\|^2_{L^2(0,T)} + \|u(0,\cdot)\|^2_{H^{-3}(0,L)}\right\}$$

We argue by contradiction. Then, there exists a sequence $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{B}$ solving (174) with $\|u^n(0,\cdot)\|_{L^2(0,L)} \le R$ and such that

$$\lim_{n\to\infty} \frac{\|u^n\|^2_{L^4(0,T;L^4(0,L))}}{\|u^n_x(\cdot,0)\|^2_{L^2(0,T)} + \|u^n_0\|^2_{H^{-3}(0,L)}} = \infty.$$

We define $\lambda_n := \|u^n\|_{L^4(0,T;L^4(0,L))}$ and $w^n := \frac{u^n}{\lambda_n}$. We get that w_n satisfies

$$\begin{cases} w^n_t + w^n_x + w^n_{xxx} + \lambda_n (v w^n)_x = 0, \\ w^n(t,0) = w^n(t,L) = w^n_x(t,L) = 0, \\ \|w_n\|_{L^4(0,T;L^4(0,L))} = 1 \end{cases} \tag{179}$$

and

$$\|w^n_x(\cdot,0)\|^2_{L^2(0,T)} + \|w^n(0,\cdot)\|^2_{H^{-3}(0,L)} \to 0 \tag{180}$$

as $n \to \infty$.

We can check that w^n is bounded in $L^2(0,T;H^1(0,L))$ and that

$$\|(uw^n)_x\|_{L^2(0,T;L^1(0,L))}$$
$$\le \|w^n\|_{L^\infty(0,T;L^2(0,L))}\|u\|_{L^2(0,T;H^1(0,L))} + \|u\|_{L^\infty(0,T;L^2(0,L))}\|w^n\|_{L^2(0,T;H^1(0,L))}$$

By using the equation, we see that $\{w^n_t\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^{-2}(0,L))$. We claim that $\{w^n\}_{n\in\mathbb{N}}$ is bounded in $L^4(0,T;H^{\frac{5}{6}}(0,L))$. Indeed, as $\{w^n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^1(0,L))$ and in $L^\infty(0,T;L^2(0,L))$, and consequently in $L^q(0,T;L^2(0,L))$ for any $q > 0$, we can interpolate between $L^5(0,T;L^2(0,L))$ and $L^2(0,T;H^1(0,L))$ to get that $\{w^n\}_{n\in\mathbb{N}}$ is bounded in $L^4(0,T;H^{\frac{5}{6}}(0,L))$.

Thanks to Lemma 3.1 and the fact that the embedding $H^{\frac{5}{6}}(0,L) \hookrightarrow L^4(0,L)$ is compact, we get that

$$w^n \to w, \quad \text{strongly in } L^4(0,T;L^4(0,L))$$

with $\|w\|_{L^4(0,T;L^4(0,L))} = 1$. In addition

$$\|w_x(\cdot,0)\|^2_{L^2(0,T)} + \|w(0,\cdot)\|_{H^{-3}(0,L)}$$
$$\le \liminf_{n\to\infty}\left\{\|w^n_x(\cdot,0)\|^2_{L^2(0,T)} + \|w^n(0,\cdot)\|_{H^{-3}(0,L)}\right\} = 0$$

As w is the solution of (174) with intial data $w(0,\cdot) = 0$, then w must be zero and we get a contradiction, which ends the proof of this lemma. □

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