ESAIM: COCV 23 (2017) 165–194
DOI: 10.1051/cocv/2015044

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ON THE CONTROL OF THE LINEAR KURAMOTO−SIVASHINSKY EQUATION *

Eduardo Cerpa[1], Patricio Guzmán[1] and Alberto Mercado[1]

Abstract. In this paper we study the null controllability property of the linear Kuramoto−Sivashinsky equation by means of either boundary or internal controls. In the Dirichlet boundary case, we use the moment theory to prove that the null controllability property holds with only one boundary control if and only if the anti-diffusion parameter of the equation does not belong to a critical set of parameters. Regarding the Neumann boundary case, we prove that the null controllability property does not hold with only one boundary control. However, it does always hold when either two boundary controls or an internal control are considered. The proof of the latter is based on the controllability-observability duality and a suitable Carleman estimate.

Mathematics Subject Classification. 35K35, 93B05.

Received September 3, 2014. Revised June 8, 2015. Accepted September 10, 2015.

1. Introduction

The Kuramoto−Sivashinsky equation is given by

$$z_t + z_{xxxx} + \lambda z_{xx} + z z_x = 0, \tag{1.1}$$

where $\lambda > 0$ is known as the anti-diffusion parameter. This equation was derived independently by Kuramoto and Tsuzuki, in [14, 15], as a model for phase turbulence in reaction-diffusion systems, and by Sivashinsky, in [17, 20], as a model for the physical phenomenon of plane flame propagation. The role of λ is to add some instabilities to the model and this occurs with $\lambda > 0$.

In this paper we study the control properties of the linear Kuramoto−Sivashinsky equation (consider (1.1) without the nonlinear term zz_x), posed with Dirichlet or Neumann boundary conditions, by means of either boundary or internal controls. The physical interpretation of these actuators is related to heat flux or fuel supply if flame front propagation is considered. We first focus on the following boundary control systems.

Keywords and phrases. Kuramoto−Sivashinky equation, parabolic equation, boundary control, internal control, null controllability, moment theory, Carleman estimates.

* *This work has been partially supported by Fondecyt 1140741 (E. Cerpa, P. Guzmán), PIIC UTFSM 2014 (P. Guzmán), Fondecyt 1120610 (A. Mercado), Anillo ACT 1106 and Basal Project FB0008-AC3E, UTFSM.*

[1] Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile.
`eduardo.cerpa@usm.cl; patricio.guzman@alumnos.usm.cl; alberto.mercado@usm.cl`

© EDP Sciences, SMAI 2016

Dirichlet Case – Boundary Control:

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = 0,\ (t,x) \in (0,T)\times(0,L), \\ z(t,0) = u_1(t),\ z(t,L) = 0,\ t\in(0,T), \\ z_x(t,0) = u_2(t),\ z_x(t,L) = 0,\ t\in(0,T), \\ z(0,x) = z_0(x),\ x\in(0,L). \end{cases} \tag{1.2}$$

Neumann Case – Boundary Control:

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = 0,\ (t,x) \in (0,T)\times(0,L), \\ z_{xx}(t,0) = u_1(t),\ z_{xx}(t,L) = 0,\ t\in(0,T), \\ z_{xxx}(t,0) = u_2(t),\ z_{xxx}(t,L) = 0,\ t\in(0,T), \\ z(0,x) = z_0(x),\ x\in(0,L). \end{cases} \tag{1.3}$$

Given the parabolic character of these equations, the appropriate control notion to study is the null controllability, which is defined as follows. The above equations are said to be null controllable in time $T > 0$ if, given any initial state z_0, there exist controls (u_1, u_2) such that the corresponding solution $z = z(t,x)$ satisfies $z(T,\cdot) = 0$.

In the literature, there are already some control results for these equations. In fact, by using four boundary controls, in [13], the robust control has been addressed for (1.2). In [16], the stability and stabilizability issues have been studied for (1.2) and (1.3). The question of using less controls in (1.2) has been raised in [2], where the case $u_1 = 0$ is considered. Indeed, by using the control u_2 only, the null controllability of (1.2) is proven to hold if and only if the anti-diffusion parameter, $\lambda > 0$, does not belong to the set of critical parameters

$$\mathcal{C} := \left\{ \frac{(j^2+k^2)\pi^2}{L^2} \ /\ (j,k) \in \mathbb{N}^2 \text{ with the same parity and } j < k \right\}. \tag{1.4}$$

In this paper we complete the study in [2] by proving the following result.

Theorem 1.1. *Consider the case $u_2 = 0$ and suppose that*

$$\lambda \notin \mathcal{G} := \mathcal{C} \cup \left\{ \frac{4l^2\pi^2}{L^2} \ /\ l \in \mathbb{N} \right\}. \tag{1.5}$$

Then, for every $z_0 \in L^2(0,L)$, there exists $u_1 \in H^1(0,T)$ such that the unique solution $z \in C([0,T]; L^2(0,L))$ of equation (1.2) satisfies $z(T,\cdot) = 0$ in $L^2(0,L)$. Moreover, if $\lambda \in \mathcal{G}$, then equation (1.2) is not null controllable in time $T > 0$ in $L^2(0,L)$.

Therefore, in the Dirichlet case, the null controllability property holds with only one control provided that $\lambda > 0$, the anti-diffusion parameter, does not belong to the corresponding set of critical parameters (1.4) or (1.5). The situation improves with two controls, where $\lambda > 0$ does not play any role in the null controllability property. In fact, this property holds with two controls and has already been shown in [2,3] by using the moment theory and a suitable Carleman estimate respectively.

The proof of Theorem 1.1 makes use of the fact that the operator

$$A_D : y \in H^4 \cap H_0^2(0,L) \subset L^2(0,L) \to -y'''' - \lambda y'' \in L^2(0,L), \tag{1.6}$$

which is the underlying spatial operator in (1.2), has a compact resolvent and is self-adjoint. Therefore, it has a discrete spectrum consisting only in real eigenvalues, $\{\sigma_k\}_{k\in\mathbb{N}}$, and its corresponding eigenfunctions, $\{\phi_k\}_{k\in\mathbb{N}}$, form an orthonormal basis of $L^2(0,L)$. This allows us to transform the null controllability problem into a problem of moments (see Lem. 2.5), which is solved by using the moment theory developed by Fattorini and Russell in [8], and the results on the asymptotic behaviour of σ_k and $\phi_k'''(0)$ when $k \to +\infty$ obtained in ([2], Lem. 2.2).

Remark 1.2. Consider $u_2 = 0$ and suppose that $\lambda \in \mathcal{G}$. Theorem 1.1 tells us that equation (1.2) is not null controllable in time $T > 0$ in $L^2(0, L)$. Being more precise, in the proof of that theorem we see that if $\lambda \in \mathcal{G}$, then there exists $n \in \mathbb{N}$, depending on $\lambda \in \mathcal{G}$, such that $\phi_n'''(0) = 0$. Furthermore, in that case, the inital states $z_0 \in L^2(0, L)$ that cannot be driven to the null state in time $T > 0$ are those that satisfy

$$\int_0^L z_0(x)\phi_n(x)\,\mathrm{d}x \neq 0.$$

Regarding the Neumann case, up to our best knowledge, there are no controllability results. In this case, the corresponding underlying spatial operator is

$$A_N : y \in \{v \in H^4(0, L) \;/\; v'' \in H_0^2(0, L)\} \subset L^2(0, L) \to -y'''' - \lambda y'' \in L^2(0, L),$$

which has a compact resolvent but is not self-adjoint because of the boundary conditions. Accordingly, we cannot follow the same strategy based on the moment theory for studying the null controllability property. With a useful characterization of this property (see Lem. 3.5), we have obtained the following negative controllability result when only one control is considered.

Theorem 1.3.

(a) *Consider the case $u_2 = 0$ and suppose that $z_0 \in L^2(0, L)$ is such that*

$$\int_0^L z_0(x)\cos(\sqrt{\lambda}x)\,\mathrm{d}x \neq 0. \tag{1.7}$$

Then, for every $u_1 \in L^2(0, T)$ the unique solution $z \in C([0, T]; L^2(0, L))$ of equation (1.3) satisfies $z(T, \cdot) \neq 0$ in $L^2(0, L)$.

(b) *Consider the case $u_1 = 0$ and suppose that $z_0 \in L^2(0, L)$ is such that*

$$\int_0^L z_0(x)\sin(\sqrt{\lambda}x)\,\mathrm{d}x \neq 0. \tag{1.8}$$

Then, for every $u_2 \in L^2(0, T)$ the unique solution $z \in C([0, T]; L^2(0, L))$ of equation (1.3) satisfies $z(T, \cdot) \neq 0$ in $L^2(0, L)$.

As in the Dirichlet case, the situation improves when two controls are considered.

Theorem 1.4. *For every $z_0 \in L^2(0, L)$, there exist $(u_1, u_2) \in L^2(0, T)^2$ such that the unique solution $z \in C([0, T]; L^2(0, L))$ of equation (1.3) satisfies $z(T, \cdot) = 0$ in $L^2(0, L)$.*

In virtue of the controllability-observability duality (see [5], Thm. 2.44 or [21], Thm. 11.2.1 for instance), we could prove Theorem 1.4 by showing the existence of a constant $C > 0$ such that

$$\|q(0, \cdot)\|^2_{L^2(0,L)} \leq C \int_0^T \left(|q(t, 0)|^2 + |q_x(t, 0)|^2\right) \mathrm{d}x, \tag{1.9}$$

for every $q = q(t, x)$ satisfying the adjoint equation

$$\begin{cases} -q_t + q_{xxxx} + \lambda q_{xx} = 0, \; (t, x) \in (0, T) \times (0, L), \\ (q_{xx} + \lambda q)(t, 0) = 0, \; (q_{xx} + \lambda q)(t, L) = 0, \; t \in (0, T), \\ (q_{xxx} + \lambda q_x)(t, 0) = 0, \; (q_{xxx} + \lambda q_x)(t, L) = 0, \; t \in (0, T), \\ q(T, x) = q_T(x), \; x \in (0, L), \end{cases} \tag{1.10}$$

with $q_T \in L^2(0,L)$. An adequate tool for obtaining observability inequality (1.9) is a Carleman estimate for (1.10) with boundary observation. However, due to the boundary conditions of (1.10), we were not able to obtain the desired Carleman estimate. Because of this difficulty, we have followed a different strategy to prove Theorem 1.4. We first prove the internal null controllability and then we use it to obtain the boundary null controllability by means of trace arguments. Consider $\omega \subset (0,L)$ as a given non-empty open interval such that $\overline{\omega} \subset (0,L)$. Denoting by $\mathbb{1}_\omega$ the characteristic function on ω, the above-mentioned internal control system is the following one.

Neumann Case – Internal Control:

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = u\mathbb{1}_\omega,\ (t,x) \in (0,T)\times(0,L), \\ z_{xx}(t,0) = 0,\ z_{xx}(t,L) = 0,\ t \in (0,T), \\ z_{xxx}(t,0) = 0,\ z_{xxx}(t,L) = 0,\ t \in (0,T), \\ z(0,x) = z_0(x),\ x \in (0,L). \end{cases} \tag{1.11}$$

We have obtained the following result.

Theorem 1.5. *For every $z_0 \in L^2(0,L)$, there exists $u \in L^2(0,T;L^2(\omega))$ such that the unique solution $z \in C([0,T];L^2(0,L))$ of equation (1.11) satisfies $z(T,\cdot) = 0$ in $L^2(0,L)$.*

Thanks to the controllability-observability duality, we can prove this theorem by showing the existence of a constant $C > 0$ such that

$$\|q(0,\cdot)\|^2_{L^2(0,L)} \leq C \int_0^T \int_\omega |q(t,x)| \mathrm{d}x\mathrm{d}t, \tag{1.12}$$

for every $q = q(t,x)$ satisfying (1.10) with $q_T \in L^2(0,L)$. We derive a suitable Carleman estimate for (1.10) with internal obervation, which allowed us to obtain (1.12) and prove Theorem 1.5. Finally, we make use of the regularizing effect of equation (1.3) when $u_1 = u_2 = 0$ (see Prop. 3.3) to prove that this internal control result implies Theorem 1.4. Note that a similar internal control result has been obtained in ([22],Thm. 1.3) for the Dirichlet case.

This paper is organized as follows. The Dirichlet case is addressed in Section 2 by presenting well-posedness results (Sect. 2.1), the characterization of the null controllability property (Sect. 2.2) and the proof of Theorems 1.1 and 1.2 (Sect. 2.3). The Neumann case is addressed in Section 3 by presenting well-posedness results (Sect. 3.1), the proof of the negative controllability result (Sect. 3.2) and the proof of Theorems 1.4 and 1.5 (Sect. 3.3). Finally, in Section 3.4 the suitable Carleman estimate used in our study is derived.

Remark 1.6. In [19] the author showed that the linear Korteweg-de Vries equation is exactly controllable, with a single boundary control, if and only if the length of the interval where the equation is posed does not belong to a critical set of lengths. Other works related to the control of the Korteweg-de Vries equation where this phenomenon is found are [4, 11]. With a suitable rescaling in the linear Kuramoto−Sivashinsky equation, we could have studied the problem of finding critical lengths L, instead of critical anti-diffusion parameters λ, for which the null controllability property fails.

2. Dirichlet case

2.1. Well-Posedness

In this section we present the well-posedness results needed for studying control system (1.2). Let us consider the equation

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = f,\ (t,x) \in (0,T)\times(0,L), \\ z(t,0) = u_1(t),\ z(t,L) = 0,\ t \in (0,T), \\ z_x(t,0) = u_2(t),\ z_x(t,L) = 0,\ t \in (0,T), \\ z(0,x) = z_0(x),\ x \in (0,L). \end{cases} \tag{2.1}$$

From [2,3,13,16] it is known that A_D, which was defined in (1.6), is an infinitesimal generator of a strongly continuous semigroup in $L^2(0,L)$. Therefore, in virtue of ([18], Cor. 2.10, Chapt. 4), it follows that equation (2.1) with $u_1 = u_2 = 0$ has a unique solution $z \in C([0,T]; H^4 \cap H_0^2(0,L)) \cap C^1([0,T]; L^2(0,L))$ provided that $f \in C^1([0,T]; L^2(0,L))$ and $z_0 \in H^4 \cap H_0^2(0,L)$. The above facts and suitable energy estimates, that can be obtained, for instance, by employing the techniques used in [1,12,13], allow us to use a density argument to conclude the following result.

Proposition 2.1. *Let $f \in L^2(0,T; L^2(0,L))$ and $u_1 = u_2 = 0$.*

(a) *If $z_0 \in L^2(0,L)$, then equation (2.1) has a unique solution $z \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^2(0,L))$. Moreover, there exists $C = C(T,L,\lambda) > 0$ such that*

$$\|z\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H_0^2(0,L))} \leq C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|z_0\|_{L^2(0,L)}\right).$$

(b) *If $z_0 \in H_0^2(0,L)$, then equation (2.1) has a unique solution $z \in C([0,T]; H_0^2(0,L)) \cap L^2(0,T; H^4(0,L))$. Moreover, there exists $C = C(T,L,\lambda) > 0$ such that*

$$\|z\|_{C([0,T];H_0^2(0,L))\cap L^2(0,T;H^4(0,L))} \leq C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|z_0\|_{H_0^2(0,L)}\right).$$

With the aid of a suitable lifting function, we can use the previous proposition to study equation (2.1) with non-homogeneous boundary conditions.

Proposition 2.2. *Let $f \in L^2(0,T; L^2(0,L))$, $(u_1,u_2) \in H^1(0,T)^2$ and $z_0 \in L^2(0,L)$. Then, equation (2.1) has a unique solution $z \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^2(0,L))$. Moreover, there exists $C = C(T,L,\lambda) > 0$ such that*

$$\|z\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))} \leq C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|(u_1,u_2)\|_{H^1(0,T)^2} + \|z_0\|_{L^2(0,L)}\right). \tag{2.2}$$

Proof. With the aid of the polynomials

$$d_1(x) := 2L^{-3}x^3 - 3L^{-2}x^2 + 1, \quad d_2(x) := L^{-2}x^3 - 2L^{-1}x^2 + x,$$

we define the lifting function

$$\psi_D(t,x) := u_1(t)d_1(x) + u_2(t)d_2(x).$$

By taking into account that $g := f - (\psi_D)_t - (\psi_D)_{xxxx} - \lambda(\psi_D)_{xx}$ and $y_0(x) := z_0(x) - \psi_D(0,x)$ are elements of $L^2(0,T; L^2(0,L))$ and $L^2(0,L)$ respectively, it follows that the equation

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} = g,\ (t,x) \in (0,T)\times(0,L), \\ y(t,0) = 0,\ y(t,L) = 0,\ t \in (0,T), \\ y_x(t,0) = 0,\ y_x(t,L) = 0,\ t \in (0,T), \\ y(0,x) = y_0(x),\ x \in (0,L), \end{cases}$$

has a unique solution $y \in C([0,T];L^2(0,L)) \cap L^2(0,T;H_0^2(0,L))$ thanks to Proposition 2.1(a). Furthermore, this solution satisfies

$$\|y\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H_0^2(0,L))} \leq C\left(\|g\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{L^2(0,L)}\right). \tag{2.3}$$

From $\psi_D(t,0) = u_1(t)$ and $(\psi_D)_x(t,0) = u_2(t)$, we get that $z := y + \psi_D \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L))$ is a solution of equation (2.1). The continuous injection $H^1(0,T) \hookrightarrow C([0,T])$ allows us to get

$$\|\psi_D\|_{C([0,T];L^2(0,L))} \leq C\|(u_1,u_2)\|_{H^1(0,T)^2},$$

which combined with $\|z\| - \|\psi_D\| \leq \|y\|$ (valid for any norm) and (2.3) give us (2.2). This inequality and the linearity of the equation yield the uniqueness of solutions. The proof of Proposition 2.2 is complete. ☐

We finish this section with a result concerning the regularizing effect of equation (2.1) when $f = 0$ and $u_1 = u_2 = 0$. This will be used in Section 3.1 when studying the regularizing effect of the uncontrolled equation associated to the Neumann case (see Prop. 3.3).

Proposition 2.3. *Let $\tau \in (0,T)$ and $z_0 \in L^2(0,L)$. Then, the unique solution $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;H_0^2(0,L))$ of equation (2.1) with $f = 0$ and $u_1 = u_2 = 0$ belongs to*

$$\mathcal{RD}(\tau,L) := C([\tau,T];H_0^2(0,L)) \cap L^2(\tau,T;H^4(0,L)) \cap H^1(\tau,T;L^2(0,L)).$$

Moreover, there exists $C = C(T,L,\lambda) > 0$ such that

$$\|z\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H_0^2(0,L))} + \|z\|_{\mathcal{RD}(\tau,L)} \leq C\left(1+\frac{1}{\tau}\right)^{1/2}\|z_0\|_{L^2(0,L)}. \tag{2.4}$$

Proof. We make all the computations needed considering $z_0 \in H^4 \cap H_0^2(0,L)$, so that equation (2.1) would have a unique solution $z \in C([0,T];H^4 \cap H_0^2(0,L)) \cap C^1([0,T];L^2(0,L))$. The case $z_0 \in L^2(0,L)$ follows from a density argument after obtaining (2.4).

Let us consider a regular function $\phi = \phi(t)$. Multiplying equation (2.1) by $z_{xxxx}\phi$ we get

$$\frac{1}{2}\int_0^L \frac{\partial\left(|z_{xx}(t,x)|^2\right)}{\partial t}\phi(t)\,\mathrm{d}x + \int_0^L |z_{xxxx}|^2\phi(t)\,\mathrm{d}x + \int_0^L \lambda z_{xx}(t,x)z_{xxxx}(t,x)\phi(t)\,\mathrm{d}x = 0. \tag{2.5}$$

Let $\tau \in (0,T)$. First, taking $\phi(t) = 1$ in (2.5) and then integrating the equation over (t,T) we get, thanks to the Cauchy inequality, that for every $t \in [\tau,T]$ it holds

$$\|z_{xx}(t,\cdot)\|^2_{L^2(0,L)} \leq \|z_{xx}(T,\cdot)\|^2_{L^2(0,L)} + 3\|z_{xxxx}\|^2_{L^2(\tau,T;L^2(0,L))} + \lambda^2\|z_{xx}\|^2_{L^2(\tau,T;L^2(0,L))}. \tag{2.6}$$

Second, taking $\phi(t) = t$ in (2.5) and then integrating the equation over $(0,T)$ we get, thanks once again to the Cauchy inequality, that

$$T\|z_{xx}(T,\cdot)\|^2_{L^2(0,L)} + \int_0^T\int_0^L |z_{xxxx}|^2 t\,\mathrm{d}x\mathrm{d}t \leq (1+T\lambda^2)\|z_{xx}\|^2_{L^2(0,T;L^2(0,L))}.$$

Here we can use that $\tau \in (0,T)$ to obtain

$$\|z_{xx}(T,\cdot)\|^2_{L^2(0,L)} + \|z_{xxxx}\|^2_{L^2(\tau,T;L^2(0,L))} \leq \frac{C}{\tau}\|z_{xx}\|^2_{L^2(0,T;L^2(0,L))}.$$

Third, from the combination of (2.2) with the previous inequality and (2.6) we arrive at

$$\|z\|^2_{C([\tau,T];H_0^2(0,L))\cap L^2(\tau,T;H^4(0,L))} \leq C\left(1+\frac{1}{\tau}\right)\|z_0\|^2_{L^2(0,L)}.$$

Note that we have used the fact that $\|v^{(n)}\|_{L^2(0,L)} + \|v\|_{L^2(0,L)}$, with $n \in \mathbb{N}$, is a norm equivalent to $\|v\|_{H^n(0,1)}$. Furthermore, the equation $z_t = -z_{xxxx} - \lambda z_{xx}$ tells us that we actually have

$$\|z\|^2_{C([\tau,T];H_0^2(0,L))\cap L^2(\tau,T;H^4(0,L))\cap H^1(\tau,T;L^2(0,L))} \leq C\left(1+\frac{1}{\tau}\right)\|z_0\|^2_{L^2(0,L)}.$$

Finally, (2.2) together with the previous inequality gives us (2.4). Furthermore, this inequality and the linearity of the equation yield the uniqueness of solutions. The proof of Proposition 2.3 is complete. □

In the forthcoming sections we will apply the results developed here to

$$\begin{cases} -q_t + q_{xxxx} + \lambda q_{xx} = G,\ (t,x) \in (0,T)\times(0,L), \\ q(t,0) = 0,\ q(t,L) = 0,\ t \in (0,T), \\ q_x(t,0) = 0,\ q_x(t,L) = 0,\ t \in (0,T), \\ q(T,x) = q_T(x),\ x \in (0,L), \end{cases} \tag{2.7}$$

which corresponds to the adjoint equation associated to equation (1.2) when $G = 0$.

2.2. Boundary control with one input

In this section we state useful characterizations of the null controllability for control system (1.2). Its proofs are very classical.

Lemma 2.4. *Consider $u_2 = 0$. Equation (1.2) is null controllable in time $T > 0$ in $L^2(0,L)$ if and only if for any $z_0 \in L^2(0,L)$ there exists $u_1 \in H^1(0,T)$ such that for every $q_T \in H_0^2(0,L)$ it holds*

$$\int_0^L z_0(x)q(0,x)\,\mathrm{d}x = \int_0^T u_1(t)q_{xxx}(t,0)\,\mathrm{d}t, \tag{2.8}$$

where $q = q(t,x)$ is the unique solution of adjoint equation (2.7) with $G = 0$.

We make use of the following fact, taken from [2], to transform the null controllability problem into a problem of moments.

(F) A_D, defined in (1.6), is a self-adjoint operator whose resolvent is compact. Its spectrum is a discrete set consisting only of real eigenvalues, denoted by $\{\sigma_k\}_{k\in\mathbb{N}}$, satisfying $\sigma_k \leq \lambda^2/4$ for all $k \in \mathbb{N}$ and $\lim_{k\to+\infty}\sigma_k = -\infty$. Its corresponding eigenfunctions, denoted by $\{\phi_k\}_{k\in\mathbb{N}}$, are elements of $H^4 \cap H_0^2(0,L)$ and form an orthonormal basis of $L^2(0,L)$.

Lemma 2.5. *Consider $u_2 = 0$. Equation (1.2) is null controllable in time $T > 0$ in $L^2(0,L)$ if and only if for any*

$$z_0(x) = \sum_{k\in\mathbb{N}} z_0^k \phi_k(x) \ \text{with} \ \sum_{k\in\mathbb{N}} |z_0^k|^2 < +\infty, \tag{2.9}$$

there exists $f \in H^1(0,T)$ such that

$$\phi_k'''(0)\int_0^T f(t)\mathrm{e}^{\sigma_k t}\,\mathrm{d}t = z_0^k \mathrm{e}^{\sigma_k T}, \quad \forall k \in \mathbb{N}. \tag{2.10}$$

The control is given by $u_1(t) := f(T-t)$.

2.3. Spectral analysis

The purpose of this section is to prove Theorem 1.1. The proof is based on the spectral analysis of the operator A_D, which was defined in (1.6).

Proof of Theorem 1.1. Every initial state $z_0 \in L^2(0,L)$ can be written as

$$z_0(x) = \sum_{k\in\mathbb{N}} z_0^k \phi_k(x), \quad \sum_{k\in\mathbb{N}} |z_0^k|^2 < +\infty.$$

In virtue of Lemma 2.5, control system (1.2) is null controllable in time $T>0$ in $L^2(0,L)$ if and only if there exists $f \in H^1(0,T)$ such that

$$\phi_k'''(0)\int_0^T f(t)\mathrm{e}^{\sigma_k t}\,\mathrm{d}t = z_0^k \mathrm{e}^{\sigma_k T}, \quad \forall k \in \mathbb{N}.$$

Hence, provided that $\phi_k'''(0) \neq 0$ for every $k \in \mathbb{N}$, we arrive at the following problem of moments. Find $f \in H^1(0,T)$ such that

$$\int_0^T f(t)\mathrm{e}^{\sigma_k t}\,\mathrm{d}t = \frac{z_0^k \mathrm{e}^{\sigma_k T}}{\phi_k'''(0)}, \quad \forall k \in \mathbb{N}.$$

This problem of moments can directly be solved by using the moment theory developed by Fattorini and Russell (see [8], Cor. 3.2), and the results on the asymptotic behaviour of σ_k and $\phi_k'''(0)$ when $k \to +\infty$ given in ([2], Lem. 2.2). Therefore, the only thing left to do is to determine when $\phi_k'''(0) \neq 0$ for every $k \in \mathbb{N}$.

In order to make the notation clearer, we omit the subscript k in the eigenvalues and eigenfunctions. Let (ϕ,σ) satisfy

$$\begin{cases} -\phi'''' - \lambda\phi'' = \sigma\phi, & x \in (0,L), \\ \phi(0) = \phi(L) = \phi'(0) = \phi'(L) = 0. \end{cases} \tag{2.11}$$

We consider the following three cases: $\sigma > 0$, $\sigma = 0$ or $\sigma < 0$. For the sake of completeness, we write again all the computations of the eigenfunctions that appear in the proof of ([2], Lem. 2.1). A similar analysis has been done in [6].

Case 1: $\sigma > 0$. Because of **(F)** we know that there exist a finite number of positive eigenvalues satisfying $\sigma < \lambda^2/4$ (it can be shown that $\sigma \neq \lambda^2/4$). Set

$$\alpha := \left(\frac{\lambda - \sqrt{\lambda^2 - 4\sigma}}{2}\right)^{1/2}, \quad \beta := \left(\frac{\lambda + \sqrt{\lambda^2 - 4\sigma}}{2}\right)^{1/2}.$$

We have that

$$\phi(x) = C_1\cos(\alpha(x - L/2)) + C_2\sin(\alpha(x - L/2)) + C_3\cos(\beta(x - L/2)) + C_4\sin(\beta(x - L/2)),$$

is a solution of equation (2.11). The real constants C_1, C_2, C_3 and C_4 make ϕ satisfy the boundary conditions in (2.11), and hence, these are the solutions of

$$\underbrace{\begin{bmatrix} \cos(\alpha L/2) & \cos(\beta L/2) \\ \alpha\sin(\alpha L/2) & \beta\sin(\beta L/2) \end{bmatrix}}_{S_1} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.12}$$

$$\underbrace{\begin{bmatrix} \sin(\alpha L/2) & \sin(\beta L/2) \\ \alpha\cos(\alpha L/2) & \beta\cos(\beta L/2) \end{bmatrix}}_{S_2} \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.13}$$

From (2.12) and (2.13) we get two finite sets of positive eigenvalues, which we denote by $\{\sigma_{1,n}\}_{n=1}^{m_1}$ and $\{\sigma_{2,n}\}_{n=1}^{m_2}$ for a certain $(m_1, m_2) \in \mathbb{N}^2$.

(a) $\{\sigma_{1,n}\}_{n=1}^{m_1}$ is obtained from the positive solutions of $\det(S_1) = 0$. Thus, they satisfy

$$\alpha\sin(\alpha L/2)\cos(\beta L/2) = \beta\sin(\beta L/2)\cos(\alpha L/2). \tag{2.14}$$

The following two possibilities are considered. The first possibility is when $\cos(\alpha L/2) \neq 0$. From (2.12), (2.13) and (2.14) we get that the eigenfunctions associated to $\sigma_{1,n}$ are

$$\begin{aligned} \phi_A(x) &= C_3\left[-\frac{\cos(\beta L/2)}{\cos(\alpha L/2)}\cos(\alpha(x-L/2)) + \cos(\beta(x-L/2))\right], \\ \phi_B(x) &= C_4\left[-\frac{\beta\cos(\beta L/2)}{\alpha\cos(\alpha L/2)}\sin(\alpha(x-L/2)) + \sin(\beta(x-L/2))\right]. \end{aligned}$$

Considering (2.14) we arrive at

$$\phi_A'''(0) = C_3\beta(\alpha^2-\beta^2)\sin(\beta L/2), \quad \phi_B'''(0) = C_4\beta(\alpha^2-\beta^2)\cos(\beta L/2).$$

Let us study when $\phi_A'''(0) \neq 0$ and $\phi_B'''(0) \neq 0$. On the one hand, if $\phi_A'''(0) = 0$, then from (2.14) we get $\sin(\alpha L/2) = 0$, allowing us to conclude, together with $\beta^2 - \alpha^2 = \sqrt{\lambda^2 - 4\sigma}$ and $\beta^2 + \alpha^2 = \lambda$, that $\lambda > 0$ should be of the form $\lambda = \left[(2j)^2 + (2k)^2\right]\pi^2/L^2$ with $(j,k) \in \mathbb{N}^2$ being such that $j < k$. On the other hand, from (2.14) we conclude that $\phi_B'''(0) \neq 0$.

The second possibility is when $\cos(\alpha L/2) = 0$. From (2.12), (2.13) and (2.14) we get that the eigenfunctions associated to $\sigma_{1,n}$ are

$$\begin{aligned} \phi_A(x) &= C_3\left[-\frac{\beta\sin(\beta L/2)}{\alpha\sin(\alpha L/2)}\cos(\alpha(x-L/2)) + \cos(\beta(x-L/2))\right], \\ \phi_B(x) &= C_4\left[-\frac{\sin(\beta L/2)}{\sin(\alpha L/2)}\sin(\alpha(x-L/2)) + \sin(\beta(x-L/2))\right], \end{aligned}$$

Considering (2.14) we arrive at

$$\phi_A'''(0) = C_3\beta(\alpha^2-\beta^2)\sin(\beta L/2), \quad \phi_B'''(0) = C_4\beta(\alpha^2-\beta^2)\cos(\beta L/2).$$

As we did before, let us study when $\phi_A'''(0) \neq 0$ and $\phi_B'''(0) \neq 0$. On the one hand, from (2.14) we conclude that $\phi_A'''(0) \neq 0$. On the other hand, from (2.14) we get $\phi_B'''(0) = 0$, allowing us to conclude, together with $\beta^2 - \alpha^2 = \sqrt{\lambda^2 - 4\sigma}$ and $\beta^2 + \alpha^2 = \lambda$, that $\lambda > 0$ should be of the form $\lambda = \left[(2j+1)^2 + (2k+1)^2\right]\pi^2/L^2$ with $(j,k) \in \mathbb{N}^2$ being such that $j < k$.

(b) $\{\sigma_{2,n}\}_{n=1}^{m_2}$ is obtained from the positive solutions of $\det(S_2) = 0$. Thus, they satisfy

$$\alpha\sin(\beta L/2)\cos(\alpha L/2) = \beta\sin(\alpha L/2)\cos(\beta L/2).$$

This time we consider the possibilities $\sin(\alpha L/2) \neq 0$ and $\sin(\alpha L/2) = 0$, where the computations and conclusions are the same as those obtained in the possibilities $\cos(\alpha L/2) = 0$ and $\cos(\alpha L/2) \neq 0$, respectively, of the previous part.

Therefore, if $\lambda \notin \left\{ \dfrac{(j^2+k^2)\pi^2}{L^2} \ / \ (j,k) \in \mathbb{N}^2 \text{ with the same parity and } j<k \right\}$, then $\phi'''(0) \neq 0$.

Case 2: $\sigma = 0$. We have that

$$\phi(x) = C_1 + C_2\,(x - L/2) + C_3 \cos\left(\sqrt{\lambda}(x-L/2)\right) + C_4 \sin\left(\sqrt{\lambda}(x-L/2)\right),$$

is a solution of equation (2.11). The real constants C_1, C_2, C_3 and C_4 make ϕ satisfy the boundary conditions in (2.11), and hence, these are the solutions of

$$\underbrace{\begin{bmatrix} 1 & \cos(\sqrt{\lambda}L/2) \\ 0 & \sqrt{\lambda}\sin(\sqrt{\lambda}L/2) \end{bmatrix}}_{S_1} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.15}$$

$$\underbrace{\begin{bmatrix} L & 2\sin(\sqrt{\lambda}L/2) \\ 1 & \sqrt{\lambda}\cos(\sqrt{\lambda}L/2) \end{bmatrix}}_{S_2} \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.16}$$

The following two possibilities are considered.

(a) Let us assume that $\det(S_1) = \sqrt{\lambda}\sin(\sqrt{\lambda}L/2) = 0$, which is when $\sqrt{\lambda}L/2 = l\pi$ with $l \in \mathbb{N}$. Since $\det(S_2) = L\sqrt{\lambda}\cos(\sqrt{\lambda}L/2) - 2\sin(\sqrt{\lambda}L/2) \neq 0$, we get that $C_2 = C_4 = 0$ is the unique solution of system (2.16), allowing us to conclude that $\phi'''(0) = -C_3\lambda\sqrt{\lambda}\sin(\sqrt{\lambda}L/2) = 0$. Note that $\lambda \neq 4l^2\pi^2/L^2$ with $l \in \mathbb{N}$ gives us $\phi'''(0) = -C_4\lambda\sqrt{\lambda}\cos(\sqrt{\lambda}L/2) \neq 0$ thank to $\det(S_1) \neq 0$.

(b) Let us assume that $\det(S_2) = L\sqrt{\lambda}\cos(\sqrt{\lambda}L/2) - 2\sin(\sqrt{\lambda}L/2) = 0$, which tells us that $\det(S_1) \neq 0$. Since the unique solution of system (2.15) is $C_1 = C_3 = 0$, we get that $\phi'''(0) = -C_4\lambda\sqrt{\lambda}\cos(\sqrt{\lambda}L/2) \neq 0$.

Therefore, if $\lambda \notin \left\{ \dfrac{4l^2\pi^2}{L^2} \ / \ l \in \mathbb{N} \right\}$, then $\phi'''(0) \neq 0$.

Case 3: $\sigma < 0$. In this case we set

$$\alpha := \left(\frac{-\lambda + \sqrt{\lambda^2 - 4\sigma}}{2} \right)^{1/2}, \quad \beta := \left(\frac{\lambda + \sqrt{\lambda - 4\sigma}}{2} \right)^{1/2}.$$

We have that

$$\phi(x) = C_1 \cosh(\alpha(x-L/2)) + C_2 \sinh(\alpha(x-L/2)) + C_3 \cos(\beta(x-L/2)) + C_4 \sin(\beta(x-L/2)),$$

is a solution of equation (2.11). The real constants C_1, C_2, C_3 and C_4 make ϕ satisfy the boundary conditions in (2.11), and hence, these are the solutions of

$$\underbrace{\begin{bmatrix} \cosh(\alpha L/2) & \cos(\beta L/2) \\ \alpha\sinh(\alpha L/2) & \beta\sin(\beta L/2) \end{bmatrix}}_{S_1} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.17}$$

$$\underbrace{\begin{bmatrix} \sinh(\alpha L/2) & \sin(\beta L/2) \\ \alpha\cosh(\alpha L/2) & \beta\cos(\beta L/2) \end{bmatrix}}_{S_2} \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.18}$$

From (2.17) and (2.18) we get two sets of negative eigenvalues, which we denote by $\{\sigma_{1,n}\}_{n\in\mathbb{N}}$ and $\{\sigma_{2,n}\}_{n\in\mathbb{N}}$.

(a) $\{\sigma_{1,n}\}_{n\in\mathbb{N}}$ is obtained from the negative solutions of $\det(S_1)=0$. Thus, they satisfy

$$-\alpha\sinh(\alpha L/2)\cos(\beta L/2)=\beta\sin(\beta L/2)\cosh(\alpha L/2). \tag{2.19}$$

Considering (2.19) we get

$$\begin{aligned}\det(S_2)&=\beta\cos(\beta L/2)\sinh(\alpha L/2)-\alpha\sin(\beta L/2)\cosh(\alpha L/2),\\ &=-(\alpha^2+\beta^2)\sin(\beta L/2)\cosh(\alpha L/2)/\alpha,\end{aligned}$$

which tells us that $C_2=C_4=0$ is the unique solution of system (2.18). Accordingly, from (2.17) we get that the eigenfunction associated to $\sigma_{1,n}$ is

$$\phi(x)=C_3\left[-\frac{\beta\sin(\beta L/2)}{\alpha\sinh(\alpha L/2)}\cosh(\alpha(x-L/2))+\cos(\beta(x-L/2))\right].$$

Once again, (2.19) allows us to conclude that

$$\phi'''(0)=-C_3\beta(\alpha^2+\beta^2)\sin(\beta L/2)\neq 0.$$

(b) $\{\sigma_{2,n}\}_{n\in\mathbb{N}}$ is obtained from the negative solutions of $\det(S_2)=0$. Thus, they satisfy

$$\alpha\sin(\beta L/2)\cosh(\alpha L/2)=\beta\cos(\beta L/2)\sinh(\alpha L/2). \tag{2.20}$$

Considering (2.20) we get

$$\begin{aligned}\det(S_2)&=-\beta\sin(\beta L/2)\cosh(\alpha L/2)-\alpha\cos(\beta L/2)\sinh(\alpha L/2),\\ &=-(\alpha^2+\beta^2)\cos(\beta L/2)\sinh(\alpha L/2)/\alpha.\end{aligned}$$

which tells us that $C_1=C_3=0$ is the unique solution of system (2.17). Accordingly, from (2.18) we get that the eigenfunction associated to $\sigma_{2,n}$ is

$$\phi(x)=C_4\left[-\frac{\sin(\beta L/2)}{\sinh(\alpha L/2)}\sinh(\alpha(x-L/2))+\sin(\beta(x-L/2))\right].$$

Once again, (2.20) allows us to conclude that

$$\phi'''(0)=-C_4\beta(\alpha^2+\beta^2)\cos(\beta L/2)\neq 0.$$

Therefore, for every $\lambda>0$ we have that $\phi'''(0)\neq 0$.

From the combination of the three above cases we conclude that $\phi_k'''(0)\neq 0$ for every $k\in\mathbb{N}$ if and only if

$$\lambda\notin\mathcal{G}:=\left\{\frac{(j^2+k^2)\pi^2}{L^2}\ /\ (j,k)\in\mathbb{N}^2 \text{ with the same parity and } j<k\right\}\cup\left\{\frac{4\pi^2l^2}{L^2}\ /\ l\in\mathbb{N}\right\}.$$

Now, it only remains to prove that equation (1.2) is not null controllable in time $T>0$ in $L^2(0,L)$ when $\lambda\in\mathcal{G}$. If $\lambda\in\mathcal{G}$, then there exists $n\in\mathbb{N}$, depending on $\lambda\in\mathcal{G}$, such that $\phi_n'''(0)=0$. Consider an initial state $z_0\in L^2(0,L)$, satisfying

$$\int_0^L z_0(x)\phi_n(x)\,\mathrm{d}x\neq 0,$$

as the initial state of equation (1.2). If we take $q_T(x) = \phi_n(x)$ as the final state of adjoint equation (2.7) with $G = 0$, then we get that $q(t,x) = \mathrm{e}^{(T-t)\sigma_n}\phi_n(x)$ is its unique solution. Hence, for every $u_1 \in H^1(0,T)$ we have that

- $\displaystyle\int_0^L z_0(x)q(0,x)\,\mathrm{d}x \neq 0.$
- $\displaystyle\int_0^T u_1(t)q_{xxx}(t,0)\,\mathrm{d}t = 0.$

Therefore, the preceding points and (2.8) of Lemma 2.4 give us the desired result. The proof of Theorem 1.1 is complete. $\square$

3. Neumann case

3.1. Well-Posedness

In this section we present the well-posedness results needed for studying control system (1.3). Let us consider the equation

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = f,\ (t,x) \in (0,T)\times(0,L), \\ z_{xx}(t,0) = u_1(t),\ z_{xx}(t,L) = 0,\ t \in (0,T), \\ z_{xxx}(t,0) = u_2(t),\ z_{xxx}(t,L) = 0,\ t \in (0,T), \\ z(0,x) = z_0(x),\ x \in (0,L), \end{cases} \tag{3.1}$$

We begin studying this equation with homogeneous boundary conditions.

Proposition 3.1. *Let $f \in C^1([0,T];L^2(0,L))$ and $z_0 \in \mathcal{N}_L := \{v \in H^4(0,L)\ /\ v'' \in H_0^2(0,L)\}$. Then, equation (3.1) with $u_1 = u_2 = 0$ has a unique solution $z \in C([0,T];\mathcal{N}_L) \cap C^1([0,T];L^2(0,L))$.*

Proof. We use the semigroup theory for the proof of this proposition. Let us consider the bilinear form $a : H^2(0,L)\times H^2(0,L) \to \mathbb{R}$ defined by

$$a(u,v) := \int_0^L u''(x)v''(x)\,\mathrm{d}x + \int_0^L \lambda u''(x)v(x)\,\mathrm{d}x.$$

Let A_N be the unbounded operator with domain

$$D(A_N) := \{u \in H^2(0,L)\ /\ v \mapsto a(u,v) \text{ is continuous over } H^2(0,L) \text{ for the } L^2(0,L)\text{-topology}\},$$

defined through $(A_N u, v)_{L^2(0,L)} = a(u,v)$. From the continuous injection $H^m(0,L) \hookrightarrow C^{m-1}([0,L])$, $m \in \mathbb{N}$, the identity

$$a(u,v) = (u'''' + \lambda u'', v)_{L^2(0,L)} + u''(x)v'(x)|_{x=0}^{x=L} - u'''(x)v(x)|_{x=0}^{x=L},$$

and the Cauchy inequality, we get that $D(A_N) = \mathcal{N}_L$ and $A_N : D(A_N) \subset L^2(0,1) \to L^2(0,L)$ is given by $A_N u = u'''' + \lambda u''$.

It turns out that A_N is the underlying spatial operator of equation (3.1). Therefore, ([18], Cor. 2.10, Chap. 4) would allow us to conclude our result if $-A_N$ is an infinitesimal generator of a strongly continuous semigroup in $L^2(0,L)$. The inequality

$$\begin{aligned} \int_0^L |v''(x)|^2\mathrm{d}x + \frac{1}{2}\int_0^L |v(x)|^2\mathrm{d}x &= \frac{1}{2}\int_0^L |v(x)|^2\mathrm{d}x - \int_0^L \lambda v''(x)v(x)\,\mathrm{d}x + a(v,v), \\ &\leq \frac{1+\lambda^2}{2}\int_0^L |v(x)|^2\mathrm{d}x + \frac{1}{2}\int_0^L |v''(x)|^2\mathrm{d}x + a(v,v), \end{aligned}$$

and the fact that $\|v''\|_{L^2(0,L)} + \|v\|_{L^2(0,L)}$ is an equivalent norm to the norm $\|v\|_{H^2(0,L)}$, lead us to the existence of a $\lambda_0 \in \mathbb{R}$ and an $\alpha > 0$ such that for every $v \in H^2(0,L)$ it holds

$$\alpha\|v\|_{H^2(0,L)} \leq \lambda_0\|v\|^2_{L^2(0,L)} + a(v,v).$$

Accordingly, ([7], Th. 3, Chap. XVII) gives us the desired property for $-A_N$. The proof of Proposition 3.1 is complete. □

With the aid of a suitable lifting function, we can use the previous proposition to study equation (3.1) with non-homogeneous boundary conditions.

Proposition 3.2. *Let $f \in L^2(0,T;L^2(0,L))$, $(u_1,u_2) \in L^2(0,T)^2$ and $z_0 \in L^2(0,L)$. Then, equation (3.1) has a unique solution $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L))$. Moreover, there exists $C = C(T,L,\lambda) > 0$ such that*

$$\|z\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))} \leq C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|(u_1,u_2)\|_{L^2(0,T)^2} + \|z_0\|_{L^2(0,L)}\right). \tag{3.2}$$

Proof. Let us assume that $f \in C^1([0,T];L^2(0,L))$, $(u_1,u_2) \in \{u \in C^2([0,T]) \ / \ u(0) = 0\}^2$ and $z_0 \in \mathcal{N}_L$. With the aid of the polynomials

$$n_1(x) := (1/10)L^{-3}x^5 - (1/4)L^{-2}x^4 + (1/2)x^2, \quad n_2(x) := (1/20)L^{-2}x^5 - (1/6)L^{-1}x^4 + (1/6)x^3,$$

we define the lifting function

$$\psi_N(t,x) := u_1(t)n_1(x) + u_2(t)n_2(x).$$

By taking into account that $g := f - (\psi_N)_t - (\psi_N)_{xxxx} - \lambda(\psi_N)_{xx}$ and $y_0(x) := z_0(x) - \psi_N(0,x) = z_0(x)$ are elements of $C^1([0,T];L^2(0,L))$ and $\mathcal{N}_L$ respectively, it follows that the equation

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} = g, \ (t,x) \in (0,T)\times(0,L), \\ y_{xx}(t,0) = 0, \ y_{xx}(t,L) = 0, \ t \in (0,T), \\ y_{xxx}(t,0) = 0, \ y_{xxx}(t,L) = 0, \ t \in (0,T), \\ y(0,x) = y_0(x), \ x \in (0,L), \end{cases}$$

has a unique solution $y \in C([0,T];\mathcal{N}_L) \cap C^1([0,T];L^2(0,L))$ in virtue of Proposition 3.1. From $(\psi_N)_{xx}(t,0) = u_1(t)$ and $(\psi_N)_{xxx}(t,0) = u_2(t)$, we get that $z := y + \psi_N \in C([0,T];H^4(0,L)) \cap C^1([0,T];L^2(0,L))$ is a solution of equation (3.1).

Multiplying equation (3.1) by z we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^L |z(t,x)|^2\mathrm{d}x\right) + \int_0^L z_{xxxx}(t,x)z(t,x)\,\mathrm{d}x + \int_0^L \lambda z_{xx}(t,x)z(t,x)\,\mathrm{d}x = \int_0^L f(t,x)z(t,x)\,\mathrm{d}x. \tag{3.3}$$

Two integrations by parts on $(0,L)$ and the Cauchy inequality lead us to

$$\begin{aligned}\int_0^L z_{xxxx}(t,x)z(t,x)\,\mathrm{d}x &= \int_0^L |z_{xx}(t,x)|^2\mathrm{d}x - z_{xx}(t,x)z_x(t,x)|_{x=0}^L + z_{xxx}(t,x)z(t,x)|_{x=0}^L\,, \\ &\geq \int_0^L |z_{xx}(t,x)|^2\mathrm{d}x - \frac{1}{2\varepsilon}\left(|u_1(t)|^2 + |u_2(t)|^2\right) - \varepsilon\|z(t,\cdot)\|^2_{W^{1,\infty}(0,L)},\end{aligned}$$

where $\varepsilon > 0$ will be chosen later. Combining the above inequality with (3.3) and then using the continuous injection $H^2(0,L) \hookrightarrow W^{1,\infty}(0,L)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^L |z(t,x)|^2\mathrm{d}x\right) + \int_0^L |z_{xx}(t,x)|^2\mathrm{d}x \le \int_0^L |f(t,x)|^2\mathrm{d}x + (1+\lambda^2)\int_0^L |z(t,x)|^2\mathrm{d}x$$
$$+ \frac{1}{2\varepsilon}\left(|u_1(t)|^2 + |u_2(t)|^2\right) + C\varepsilon\|z(t,\cdot)\|^2_{H^2(0,L)}. \tag{3.4}$$

Applying Grönwall's Lemma to (3.4) gives us

$$\|z\|^2_{C([0,T];L^2(0,L))} + \|z\|^2_{L^2(0,T;H^2(0,L))} \le C\Big(\|f\|^2_{L^2(0,T;L^2(0,L))} + \varepsilon^{-1}\|(u_1,u_2)\|^2_{L^2(0,T)^2}$$
$$+\|z_0\|^2_{L^2(0,L)} + \varepsilon\|z\|^2_{L^2(0,T;H^2(0,L))}\Big). \tag{3.5}$$

Note that here we have the fact that $\|z_{xx}\|_{L^2(0,T;L^2(0,L))} + \|z\|_{L^2(0,T;L^2(0,L))}$ is an equivalent norm to the norm $\|z\|_{L^2(0,T;H^2(0,L))}$.

Therefore, we arrive at (3.2) after the choice of $\varepsilon = 1/(2C)$ in (3.5). Since $C^1([0,T];L^2(0,L))$, $\{u \in C^2([0,T]) \;/\; u(0) = 0\}$ and $\mathcal{N}_L$ are dense in $L^2(0,T;L^2(0,L))$, $L^2(0,T)$ and $L^2(0,L)$ respectively, (3.2) allows us to use a density argument to conclude that equation (3.1) has a unique solution $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L))$ provided that $f \in L^2(0,T;L^2(0,L))$, $(u_1,u_2) \in L^2(0,T)^2$ and $z_0 \in L^2(0,L)$. Note that the uniqueness of solutions comes from the linearity of equation (3.1) together with (3.2). The proof of Proposition 3.2 is complete. □

Our next result concerns the regularizing effect of equation (3.1) when $f = 0$ and $u_1 = u_2 = 0$. This will play a key role in the proof of Theorem 1.4.

Proposition 3.3. *Let $\tau \in (0,T)$ and $z_0 \in L^2(0,L)$. Then, the unique solution $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L))$ of equation (3.1) with $f = 0$ and $u_1 = u_2 = 0$ belongs to*

$$\mathcal{RN}(\tau,L) := C([\tau,T];\mathcal{N}_L) \cap L^2(\tau,T;H^6(0,L)) \cap H^1(\tau,T;H^2(0,L)).$$

Moreover, there exists $C = C(T,L,\lambda) > 0$ such that

$$\|z\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))} + \|z\|_{\mathcal{RN}(\tau,L)} \le C\left(1+\frac{1}{\tau}\right)^{1/2}\|z_0\|_{L^2(0,L)}. \tag{3.6}$$

Proof. For a $\tau \in (0,T)$ given, consider $\tau_1 \in (0,\tau)$. By noting that we can carry out the same computations as those made in the proof of Proposition 2.3, it is possible to conclude that $z \in C([\tau_1,T];H^2(0,L)) \cap L^2(\tau_1,T;\mathcal{N}_L) \cap H^1(\tau_1,T;L^2(0,L))$ provided that $z_0 \in L^2(0,L)$.

Since $y := z_{xx}$ satisfies the equation

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} = 0, \ (t,x) \in (\tau_1,T)\times(0,L), \\ y(t,0) = 0, \ y(t,L) = 0, \ t \in (\tau_1,T), \\ y_x(t,0) = 0, \ y_x(t,L) = 0, \ t \in (\tau_1,T), \\ y(\tau_1,x) = z_{xx}(\tau_1,x), \ x \in (0,L), \end{cases}$$

and $z_{xx}(\tau_1,x)$ belongs to $L^2(0,L)$, we can use Proposition 2.3 to conclude that

$$z_{xx} \in \mathcal{RD}(\tau,L) := C([\tau,T];H_0^2(0,L)) \cap L^2(\tau,T;H^4(0,L)) \cap H^1(\tau,T;L^2(0,L)),$$

which gives us the desired result. Note that (3.6) follows from the combination of (3.2) and (2.4). The proof of Proposition 3.3 is complete. □

We finish this section by studying the well-posedness of

$$\begin{cases} -q_t + q_{xxxx} + \lambda q_{xx} = G, \, (t,x) \in (0,T) \times (0,L), \\ (q_{xx} + \lambda q)(t,0) = 0, \; (q_{xx} + \lambda q)(t,L) = 0, \, t \in (0,T), \\ (q_{xxx} + \lambda q_x)(t,0) = 0, \; (q_{xxx} + \lambda q_x)(t,L) = 0, \, t \in (0,T), \\ q(T,x) = q_T(x), \, x \in (0,L), \end{cases} \tag{3.7}$$

which corresponds to the adjoint equation associated to equation (1.3) when $G = 0$.

Proposition 3.4.

(a) *If $G \in C^1([0,T]; L^2(0,L))$ and $q_T \in \mathcal{N}_L^* := \{v \in H^4(0,L) \; / \; (v'' + \lambda v) \in H_0^2(0,L)\}$, then equation (3.7) has a unique solution $q \in C([0,T]; \mathcal{N}_L^*) \cap C^1([0,T]; L^2(0,L))$.*

(b) *If $G \in L^2(0,T; L^2(0,L))$ and $q_T \in L^2(0,L)$, then equation (3.7) has a unique solution $q \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^2(0,L))$. Moreover, there exists $C = C(T,L,\lambda) > 0$ such that*

$$\|q\|_{C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L))} \leq C \left(\|G\|_{L^2(0,T;L^2(0,L)} + \|q_T\|_{L^2(0,L)} \right).$$

Proof. Let us recall some notation introduced in Proposition 3.1. Consider the unbounded operator $A_N : \mathcal{N}_L \subset L^2(0,L) \to L^2(0,L)$ given by $A_N u = u'''' + \lambda u''$. In that proposition we have shown that $-A_N$ is an infinitesimal generator of a strongly continuous semigroup in $L^2(0,L)$.

Let $u \in \mathcal{N}_L$ and $v \in H^4(0,1)$. From

$$(A_N u, v)_{L^2(0,L)} = (u, v'''' + \lambda v'')_{L^2(0,L)} + u'(x)(v''(x) + \lambda v(x))|_{x=0}^{x=L} - u(x)(v'''(x) + \lambda v'(x))|_{x=0}^{x=L},$$

we find out that $D(A_N^*) = \mathcal{N}_L^*$ and $A_N^* v = v'''' + \lambda v''$, which allow us to conclude that A_N is not a self-adjoint operator. However, because of the above mentioned property for $-A_N$, [18, Corollary 10.6, Chapter 1] tells us that $(-A_N)^*$ is also an infinitesimal generator of a strongly continuous semigroup in $L^2(0,L)$.

Therefore, on the one hand, part (a) of this proposition follows from the application of ([18], Cor. 2.10, Chap. 4) to equation (3.7) after the change of variable $t \to T - t$. On the other hand, part (b) of this proposition follows by using the same arguments as those used in the proof of Proposition 3.2 by taking into account that $C^1([0,T]; L^2(0,L))$ and $\mathcal{N}_L^*$ are dense in $L^2(0,T; L^2(0,L))$ and $L^2(0,L)$ respectively. The proof of Proposition 3.4 is complete. ☐

3.2. Boundary control with one input

The aim of this section is to prove Theorem 1.3. Let us start with a classical characterization of the null controllability for control system (1.3) whose proof is omitted.

Lemma 3.5. *Equation (1.3) is null controllable in time $T > 0$ in $L^2(0,L)$ if and only if for any $z_0 \in L^2(0,L)$ there exists $(u_1, u_2) \in L^2(0,T)^2$ such that for every $q_T \in L^2(0,L)$ it holds*

$$\int_0^L z_0(x) q(0,x) \, \mathrm{d}x = \int_0^T u_1(t) q_x(t,0) \, \mathrm{d}t - \int_0^T u_2(t) q(t,0) \, \mathrm{d}t, \tag{3.8}$$

where $q = q(t,x)$ is the unique solution of adjoint equation (3.7) with $G = 0$.

The proof of Theorem 1.3 is a consequence of the previous lemma.

Proof of Theorem 1.3. Let $u_2 = 0$ and consider $z_0 \in L^2(0,L)$, satisfying (1.7), as the initial state of equation (1.3). If we take $q_T(x) = \cos(\sqrt{\lambda} x)$ as the final state of adjoint equation (3.7) with $G = 0$, then we get that

$q(t,x) = \cos(\sqrt{\lambda}x)$ is its unique solution. Hence, for every $u_1 \in L^2(0,T)$ we have that

- $\displaystyle\int_0^L z_0(x)q(0,x)\,\mathrm{d}x \neq 0$, thanks to (1.7).
- $\displaystyle\int_0^T u_1(t)q_x(t,0)\,\mathrm{d}t = 0$.

Therefore, the preceding points and (3.8) of Lemma 3.5 give us part (a) of this theorem. For the other part, take $u_1 = 0$ and consider $z_0 \in L^2(0,L)$, satisfying (1.8), as the initial state of equation (1.3). In this case, the final state $q_T(x) = \sin(\sqrt{\lambda}x)$ in adjoint equation (3.7) with $G = 0$ gives the unique solution $q(t,x) = \sin(\sqrt{\lambda}x)$. Hence, for every $u_2 \in L^2(0,T)$ we have that

- $\displaystyle\int_0^L z_0(x)q(0,x)\,\mathrm{d}x \neq 0$, thanks to (1.8).
- $\displaystyle\int_0^T u_2(t)q(t,0)\,\mathrm{d}t = 0$.

Accordingly, the preceding points and (3.8) of Lemma 3.5 lead us to part (b) of this theorem. The proof of Theorem 1.3 is complete. □

3.3. Boundary control with two inputs and internal control

This section is devoted to the proofs of Theorems 1.4 and 1.5.

Recall that $\omega \subset (0,L)$ is a given non-empty open interval such that $\overline{\omega} \subset (0,L)$. Throughout this section and the next one we use $Q := (0,T) \times (0,L)$ and $Q_\omega := (0,T) \times \omega$. Due to the controllability-observability duality (see [5], Thm. 2.44 or [21], Thm. 11.2.1 for instance), Theorem 1.5 can be shown by means of the following observability inequality.

Proposition 3.6. *There exists $C = C(L,\lambda,\omega) > 0$ such that*

$$\|q(0,\cdot)\|^2_{L^2(0,L)} \leq \frac{C}{T}\exp\left\{C\left(T + \frac{\max\{T,T^2\}}{T^2}\right)\right\}\iint_{Q_\omega}|q|^2\mathrm{d}x\mathrm{d}t, \tag{3.9}$$

where $q = q(t,x)$ is the unique solution of adjoint equation (3.7) with $G = 0$ and $q_T \in L^2(0,L)$.

In our case, the main tool for obtaining this observability inequality is a Carleman estimate for adjoint equation (3.7), which will be derived by following a procedure described in [9] due to Fursikov and Imanuvilov. To this end, we need to introduce some weight functions.

For a $x_0 \in \omega$ take $\omega_0 := (x_0 - \varepsilon, x_0 + \varepsilon)$ with $\varepsilon := \mathrm{dist}(x_0, \partial\omega)/2 > 0$. Then, choose $\psi \in C^4([0,L])$ satisfying:

- $\psi(x) > 0$ for every $x \in (0,L)$. (3.10)
- $\psi(0) = \psi(L) = 0$. (3.11)
- $|\psi'(x)| > 0$ for every $x \in [0,L]\backslash\omega_0$. (3.12)

Remark 3.7. Since we are working in a one-dimensional setting, it is possible to give examples for such a function $\psi \in C^4([0,L])$. Indeed, take $\psi(x) = x(L-x)\mathrm{e}^{\psi_0(x)}$ with

$$\psi_0(x) = \frac{2x_0 - L}{x_0(L-x_0)}x.$$

This $\psi \in C^\infty([0,L])$ satisfies (3.10) and (3.11). Furthermore, it can be shown that $\psi'(x) > 0$ for $x \in [0,x_0)$ and $\psi'(x) < 0$ for $x \in (x_0,L]$, allowing us to conclude that $x_0 \in \omega_0$ is the unique maximizer in $[0,L]$ for the function and that (3.12) holds.

Finally, for $\mu > 0$ define the weight functions

$$\alpha(t,x) := \frac{\mathrm{e}^{4\mu\|\psi\|_{L^\infty(0,L)}}}{t(T-t)} - \beta(t,x), \quad \beta(t,x) := \frac{\mathrm{e}^{2\mu\|\psi\|_{L^\infty(0,L)}+\mu\psi(x)}}{t(T-t)}, \quad \forall (t,x) \in \overline{Q}. \tag{3.13}$$

In virtue of the procedure that will lead us to the Carleman estimate for equation (3.7) (adjoint equation for the Neumann case), it will be needed a Carleman estimate for equation (2.7) (adjoint equation for the Dirichlet case). This is just a technicality in order to avoid some unwanted boundary terms that would appear when deriving directly the Carleman estimate for adjoint equation (3.7) (see Rem. 3.13). The Carleman estimate for adjoint equation (2.7) that we are going to use corresponds to a slightly modified version of ([10], Prop. 2.1) or ([22], Thm. 1.1).

Theorem 3.8 (Prop.2.1 in [10] or Thm. 1.1 in [22])**.** *There exist $\mu_0 \geq 1$ and $C = C(L,\lambda,\omega) > 0$ such that for every $\mu \geq \mu_0$ and $\nu \geq \max\{T, T^2\}$ we have*

$$\iint_Q \mathrm{e}^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2 + \nu^3\mu^4\beta^3|q_{xx}|^2 + \nu\mu^2\beta|q_{xxx}|^2\right)\mathrm{d}x\mathrm{d}t \\ + \iint_Q \mathrm{e}^{-2\nu\alpha}\left(\frac{|q_t|^2 + |q_{xxxx}|^2}{\nu\beta}\right)\mathrm{d}x\mathrm{d}t \leq C\iint_Q \mathrm{e}^{-2\nu\alpha}|G|^2\mathrm{d}x\mathrm{d}t \\ + C\iint_{Q_{\omega_1}} \mathrm{e}^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2\mathrm{d}x\mathrm{d}t, \tag{3.14}$$

with ω_1 being any non-empty open interval satisfying $\overline{\omega_0} \subset \omega_1 \subseteq \omega$ and $q = q(t,x)$ the unique solution of adjoint equation (2.7) with $G \in L^2(0,T;L^2(0,L))$ and $q_T \in H_0^2(0,L)$.

Now we can present our Carleman estimate whose proof will be given in Section 3.4.

Proposition 3.9. *There exist $\mu_0 \geq 1$ and $C = C(L,\lambda,\omega) > 0$ such that for every $\mu \geq \mu_0$ and $\nu \geq \max\{T, T^2\}$ we have*

$$\iint_Q \mathrm{e}^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2 + \nu^3\mu^4\beta^3|q_{xx}|^2 + \nu\mu^2\beta|q_{xxx}|^2\right)\mathrm{d}x\mathrm{d}t \\ + \iint_Q \mathrm{e}^{-2\nu\alpha}\left(\frac{|q_t|^2 + |q_{xxxx}|^2}{\nu\beta}\right)\mathrm{d}x\mathrm{d}t \leq C\iint_Q \mathrm{e}^{-2\nu\alpha}\left(\frac{1}{\mu^2}|G_{xx}|^2 + \nu^3\mu^2\beta^3|G|^2\right)\mathrm{d}x\mathrm{d}t \\ + C\iint_{Q_{\omega_1}} \mathrm{e}^{-2\nu\alpha}\nu^{11}\mu^{10}\beta^{11}|q|^2\mathrm{d}x\mathrm{d}t, \tag{3.15}$$

with ω_1 being any non-empty open interval satisfying $\overline{\omega_0} \subset \omega_1 \subseteq \omega$ and $q = q(t,x)$ the unique solution of adjoint equation (3.7) with $G \in L^2(0,T;H^2(0,L))$ and $q_T \in \mathcal{N}_L^* := \{v \in H^4(0,L) \ / \ (v'' + \lambda v) \in H_0^2(0,L)\}$.

By considering $G = 0$ in adjoint equation (3.7), a density argument together with Proposition 3.4(b) can be used to obtain from Proposition 3.9 the following result.

Corollary 3.10. *There exist $\mu_0 \geq 1$ and $C = C(L,\lambda,\omega) > 0$ such that for every $\mu \geq \mu_0$ and $\nu \geq \max\{T, T^2\}$ we have*

$$\iint_Q \mathrm{e}^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2 + \nu^3\mu^4\beta^3|q_{xx}|^2\right)\mathrm{d}x\mathrm{d}t \leq C\iint_{Q_{\omega_1}} \mathrm{e}^{-2\nu\alpha}\nu^{11}\mu^{10}\beta^{11}|q|^2\mathrm{d}x\mathrm{d}t, \tag{3.16}$$

with ω_1 being any non-empty open interval satisfying $\overline{\omega_0} \subset \omega_1 \subseteq \omega$ and $q = q(t,x)$ the unique solution of adjoint equation (3.7) with $G = 0$ and $q_T \in L^2(0,L)$.

We proceed to derive observability inequality (3.9) of Proposition 3.6.

Proof of Proposition 3.6. First, let us assume that $q_T \in \mathcal{N}_L^*$ so that adjoint equation (3.7) with $G = 0$ would have a unique solution $q \in C([0,T];\mathcal{N}_L^*) \cap C^1([0,T];L^2(0,L))$ thanks to Proposition 3.4(a). Once again, the $q_T \in L^2(0,L)$ case follows from a density argument together with Proposition 3.4(b) after obtaining observability inequality (3.9).

For $t \in [0,T]$ we define $E(t) := \|q(t,\cdot)\|^2_{L^2(0,L)}$. Multiplying adjoint equation (3.7) with $G = 0$ by q and then using the Cauchy inequality we get

$$-\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq \lambda^2 E(t).$$

This inequality together with

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda^2 t}E(t)\right)\mathrm{e}^{-\lambda^2 t} = \lambda^2 E(t) + \frac{\mathrm{d}E(t)}{\mathrm{d}t},$$

leads us to

$$\|q(0,\cdot)\|^2_{L^2(0,L)} \leq \mathrm{e}^{\lambda^2 t}\|q(t,\cdot)\|^2_{L^2(0,L)}, \quad t \in [0,T]. \tag{3.17}$$

Second, in (3.16) of Corollary 3.10 we fix $\mu = \mu_0$ and $\nu = \max\{T, T^2\}$ to obtain

$$\iint_Q \mathrm{e}^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2\mathrm{d}x\mathrm{d}t \leq C\iint_{Q_\omega}\mathrm{e}^{-2\nu\alpha}\nu^{11}\mu^{10}\beta^{11}|q|^2\mathrm{d}x\mathrm{d}t. \tag{3.18}$$

Since there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{C_1\nu^7}{T^{14}}\exp\{-\frac{C_2\nu}{T^2}\} \leq \mathrm{e}^{-2\alpha\nu}\nu^7\mu^8\beta^7, \quad \forall (t,x) \in [T/4, 3T/4]\times[0,L],$$

$$\mathrm{e}^{-2\alpha\nu}\nu^{11}\mu^{10}\beta^{11} \leq C_2, \quad \forall (t,x) \in \overline{Q},$$

from (3.18) it follows that

$$\int_{T/4}^{3T/4}\int_0^L |q|^2\mathrm{d}x\mathrm{d}t \leq \frac{CT^{14}}{\nu^7}\exp\left\{\frac{C\nu}{T^2}\right\}\iint_{Q_\omega}|q|^2\mathrm{d}x\mathrm{d}t.$$

Finally, observability inequality (3.9) is obtained from the combination of (3.17) and the previous inequality. The proof of Proposition 3.6 is complete. □

Now we are ready to show the null controllability property for control system (1.3).

Proof of Theorem 1.4. Let $z_0 \in L^2(0,L)$ and $\tau \in (0,T)$. Define

$$a_0(x) := \begin{cases} z_0(x) & , x \in (0,L), \\ 0 & , x \in (L,2L). \end{cases}$$

First, let us consider the equation

$$\begin{cases} a_t + a_{xxxx} + \lambda a_{xx} = 0, \ (t,x) \in (0,T)\times(0,2L), \\ a_{xx}(t,0) = 0, \ a_{xx}(t,2L) = 0, \ t \in (0,T), \\ a_{xxx}(t,0) = 0, \ a_{xxx}(t,2L) = 0, \ t \in (0,T), \\ a(0,x) = a_0(x), \ x \in (0,2L). \end{cases} \tag{3.19}$$

Recall that $\mathcal{N}_{2L} = \{v \in H^4(0,2L) \ / \ v'' \in H_0^2(0,2L)\}$. In virtue of Proposition 3.3 we have that equation (3.19) has a unique solution $a \in C([0,T];L^2(0,2L)) \cap L^2(0,T;H^2(0,2L))$ which belongs to

$$\mathcal{RN}(\tau,2L) = C([\tau,T];\mathcal{N}_{2L}) \cap L^2(\tau,T;H^6(0,2L)) \cap H^1(\tau,T;H^2(0,2L)).$$

Second, let $\omega \subset (L,2L)$ a non-empty open interval such that $\overline{\omega} \subset (L,2L)$. By taking into account that $a(\tau,\cdot) \in \mathcal{N}_{2L}$, an application of Theorem 1.5 tells us that there exists $u \in L^2(\tau,T;L^2(\omega))$ such that the unique solution of the equation

$$\begin{cases} b_t + b_{xxxx} + \lambda b_{xx} = u\mathbb{1}_\omega, \ (t,x) \in (\tau,T)\times(0,2L), \\ b_{xx}(t,0) = 0, \ b_{xx}(t,2L) = 0, \ t \in (\tau,T), \\ b_{xxx}(t,0) = 0, \ b_{xxx}(t,2L) = 0, \ t \in (\tau,T), \\ b(0,x) = a(\tau,x), \ x \in (0,2L), \end{cases} \tag{3.20}$$

satisfies $b(T,\cdot) = 0$ in $L^2(0,2L)$. Due to Proposition 3.2, equation (3.20) has a unique solution $b \in C([\tau,T];L^2(0,2L)) \cap L^2(\tau,T;H^2(0,2L))$. Moreover, by employing the methods used in Section 3.1, we also have that $b \in L^2(\tau,T;\mathcal{N}_{2L})$. Therefore, the continuous injection $H^4(0,2L) \hookrightarrow C^3([0,2L])$ tells us that $b_{xx}(\cdot,L)$ and $b_{xxx}(\cdot,L)$ are elements of $L^2(\tau,T)$.

Finally, let us define

$$z(t,x) := \begin{cases} a(t,x) & , \ (t,x) \in (0,\tau)\times(0,L), \\ b(t,L-x) & , \ (t,x) \in (\tau,T)\times(0,L). \end{cases}$$

From equations (3.19) and (3.20) we have that $z = z(t,x)$ satisfies the equation

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = 0, \ (t,x) \in (0,T)\times(0,L), \\ z_{xx}(t,0) = u_1(t), \ z_{xx}(t,L) = 0, \ t \in (0,T), \\ z_{xxx}(t,0) = u_2(t), \ z_{xxx}(t,L) = 0, \ t \in (0,T), \\ z(0,x) = z_0(x), \ x \in (0,L), \end{cases}$$

with u_1 and u_2 being elements of $L^2(0,T)$ defined by

$$u_1(t) := \begin{cases} 0 & , \ t \in (0,\tau), \\ b_{xx}(t,L) & , \ t \in (\tau,T). \end{cases} \quad u_2(t) := \begin{cases} 0 & , \ t \in (0,\tau), \\ -b_{xxx}(t,L) & , \ t \in (\tau,T). \end{cases}$$

Therefore, Proposition 3.2 and the fact that $z(T,\cdot) = 0$ in $L^2(0,L)$ allow us to conclude our result. The proof of Theorem 1.4 is complete. $\square$

3.4. Carleman estimate

In this section we prove Proposition 3.9. The useful properties of the weight functions defined in (3.13), which can be deduced from straightforward computations, are listed in the following lemma.

Lemma 3.11.

(a) *For every $\mu > 0$ we have*

$$\mu \le \frac{T^2}{8\|\psi\|_{L^\infty(0,L)}}\beta(t,x), \quad \forall (t,x) \in \overline{Q}.$$

(b) *Let $n \in \{1,2,3,4\}$. There exists $C > 0$, independent of $\mu > 0$, such that*

$$\left|\frac{\partial^n \alpha}{\partial x^n}(t,x)\right| \le C\sum_{k=1}^{n} \mu^k \beta(t,x), \quad \forall (t,x) \in \overline{Q}.$$

Proof of Proposition 3.9. For a parameter $\nu > 0$, which will be rather large than small, let us consider $Pw := \mathrm{e}^{-\nu\alpha}\left(-\partial_t + \partial_{xxxx} + \lambda\partial_{xx}\right)(\mathrm{e}^{\nu\alpha}w)$, with $w := \mathrm{e}^{-\nu\alpha}q$, and the decomposition $Pw = P_1w + P_2w + P_3w$ given by

$$\begin{aligned}
P_1w :=& -w_t + 4\nu^3\alpha_x^3 w_x + 4\nu\alpha_x w_{xxx},\\
P_2w :=& \, \nu^4\alpha_x^4 w + 6\nu^2\alpha_x^2 w_{xx} + w_{xxxx},\\
P_3w :=& -\nu\alpha_t w + 6\nu^3\alpha_{xx}\alpha_x^2 w + 3\nu^2\alpha_{xx}^2 w + 4\nu^2\alpha_{xxx}\alpha_x w + \nu\alpha_{xxxx}w\\
&+ 12\nu^2\alpha_{xx}\alpha_x w_x + 4\nu\alpha_{xxx}w_x + 6\nu\alpha_{xx}w_{xx} + \lambda\left(\nu^2\alpha_x^2 w + \nu\alpha_{xx}w + 2\nu\alpha_x w_x + w_{xx}\right).
\end{aligned}$$

We remark that this decomposition for Pw is slightly different to the decompositions considered in ([1], Thm. 3.1 and [12], Prop. 3). Regardless of the above, the structure of the decomposition gives us

$$\|P_1w\|_{L^2(Q)}^2 + 2\left(P_1w, P_2w\right)_{L^2(Q)} + \|P_2w\|_{L^2(Q)}^2 = \|Pw - P_3w\|_{L^2(Q)}^2. \tag{3.21}$$

We shall deduce the desired Carleman estimate from this equality and Theorem 3.8. This will be done in seven steps in order to do a clearer proof. Steps 1, 2 and 3 are classical computations when proving a Carleman estimate for $w = \mathrm{e}^{-\nu\alpha}q$. Step 4 takes care of boundary terms. In Step 5, we apply Theorem 3.8 to $p := (q_{xx} + \lambda q)$, which satisfies Dirichlet boundary condition as in equation (2.7). Step 6 deal with rest terms, as those in P_3w. Finally, Step 7 is the conclusion where the Carleman estimate for q is obtained.

Step 1. We proceed to handle the terms $\|P_1w\|_{L^2(Q)}^2$ and $\|P_2w\|_{L^2(Q)}^2$. To this end, we introduce the quantities

$$\|w\|_A := \iint_Q \left(\nu^7\mu^8\beta^7|w|^2 + \nu^5\mu^6\beta^5|w_x|^2 + \nu^3\mu^4\beta^3|w_{xx}|^2 + \nu\mu^2\beta|w_{xxx}|^2\right)\mathrm{d}x\mathrm{d}t,$$

$$\|w\|_B := \iint_Q \left(\frac{|w_t|^2 + |w_{xxxx}|^2}{\nu\beta}\right)\mathrm{d}x\mathrm{d}t.$$

From the weight functions defined in (3.13) we have that $\alpha_x(t,x) = -\mu\psi'(x)\beta(t,x)$ for every $(t,x) \in \overline{Q}$. The inequalities

$$\iint_Q \frac{|w_t|^2}{\nu\beta}\mathrm{d}x\mathrm{d}t \le \iint_Q \frac{|P_1w|^2}{\nu\beta}\mathrm{d}x\mathrm{d}t + C\iint_Q \left(\nu^5\mu^6\beta^5|w_x|^2 + \nu\mu^2\beta|w_{xxx}|^2\right)\mathrm{d}x\mathrm{d}t,$$

$$\iint_Q \frac{|w_{xxxx}|^2}{\nu\beta}\mathrm{d}x\mathrm{d}t \le \iint_Q \frac{|P_2w|^2}{\nu\beta}\mathrm{d}x\mathrm{d}t + C\iint_Q \left(\nu^7\mu^8\beta^7|w|^2 + \nu^3\mu^4\beta^3|w_{xx}|^2\right)\mathrm{d}x\mathrm{d}t,$$

together with the fact that $\beta(t,x) \ge 4/T^2$ for every $(t,x) \in \overline{Q}$ allow us to conclude that

$$\|w\|_B \le C\left(\|P_1w\|_{L^2(Q)}^2 + \|P_2w\|_{L^2(Q)}^2 + \|w\|_A\right), \quad \forall \nu \ge T^2. \tag{3.22}$$

Step 2. We proceed to compute $(P_1w, P_2w)_{L^2(Q)}$. For $i,j = 1,2,3$ we denote by $I_{i,j}$ the L^2-product in Q between the ith term of P_1w with the jth term of P_2w. Note that with this notation we have

$$(P_1w, P_2w)_{L^2(Q)} = \sum_{i,j=1}^{3} I_{i,j}.$$

Before going any further, we remark that in virtue of Proposition 3.4(b) we have that $q \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^2(0,L))$. Furthermore, since $p := q_{xx} + \lambda q$ satisfies the equation

$$\begin{cases} -p_t + p_{xxxx} + \lambda p_{xx} = G_{xx} + \lambda G, \ (t,x) \in (0,T)\times(0,L),\\ p(t,0) = 0, \ p(t,L) = 0, \ t \in (0,T),\\ p_x(t,0) = 0, \ p_x(t,L) = 0, \ t \in (0,T),\\ p(T,x) = p_T(x), \ x \in (0,L), \end{cases} \tag{3.23}$$

with $p_T(x) := q_T''(x) + \lambda q_T(x) \in H_0^2(0,L)$, due to $q_T \in \mathcal{N}_L^* := \{v \in H^4(0,L) \;/\; (v'' + \lambda v) \in H_0^2(0,L)\}$, and $G_{xx} + \lambda G \in L^2(0,T;L^2(0,L))$, Proposition 2.1(b) tells us that $p \in C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))$. Hence, in particular, when performing the computations needed it is considered that $w \in C([0,T];H^2(0,L)) \cap L^2(0,T;H^4(0,L)) \cap H^1(0,T;L^2(0,L))$ satisfies $w(0,x) = w(T,x) = 0$ for every $x \in [0,L]$. The latter is due to $w = \mathrm{e}^{-\nu\alpha} q$ and the choice of the weight functions defined in (3.13).

Integrations by parts are performed and each resulting term is labeled to indicate where it is going to be considered later. Each resulting expression for $I_{i,j}$ is listed below.

- $$I_{1,1} = \underbrace{\frac{\nu^4}{2}\iint_Q \left(\alpha_x^4\right)_t |w|^2 \mathrm{d}x\mathrm{d}t}_{R(w)}.$$

- $$I_{1,2} = \underbrace{-\,6\nu^2 \iint_Q \alpha_x^2 w_t w_{xx}\,\mathrm{d}x\mathrm{d}t}_{R(w)}.$$

- $$I_{1,3} = \underbrace{-\iint_Q w_t w_{xxxx}\,\mathrm{d}x\mathrm{d}t}_{R(w)}.$$

- $$I_{2,1} = \underbrace{-\,2\nu^7 \iint_Q \left(\alpha_x^7\right)_x |w|^2 \mathrm{d}x\mathrm{d}t}_{M_0(w)} + \underbrace{2\nu^7 \int_0^T \alpha_x^7 |w|^2\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)}.$$

- $$I_{2,2} = \underbrace{-\,12\nu^5 \iint_Q \left(\alpha_x^5\right)_x |w_x|^2 \mathrm{d}x\mathrm{d}t}_{M_1(w)} + \underbrace{12\nu^5 \int_0^T \alpha_x^5 |w_x|^2\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)}.$$

- $$\begin{aligned} I_{2,3} = &\underbrace{-\,2\nu^3 \iint_Q \left(\alpha_x^3\right)_{xxx} |w_x|^2 \mathrm{d}x\mathrm{d}t}_{R(w)} + \underbrace{6\nu^3 \iint_Q \left(\alpha_x^3\right)_x |w_{xx}|^2 \mathrm{d}x\mathrm{d}t}_{M_2(w)} + \underbrace{2\nu^3 \int_0^T \left(\alpha_x^3\right)_{xx} |w_x|^2\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)} \\ &\underbrace{-\,2\nu^3 \int_0^T \alpha_x^3 |w_{xx}|^2\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)} \underbrace{-\,4\nu^3 \int_0^T \left(\alpha_x^3\right)_x w_x w_{xx}\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)} + \underbrace{4\nu^3 \int_0^T \alpha_x^3 w_x w_{xxx}\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)}. \end{aligned}$$

- $$\begin{aligned} I_{3,1} = &\underbrace{-\,2\nu^5 \iint_Q \left(\alpha_x^5\right)_{xxx} |w|^2 \mathrm{d}x\mathrm{d}t}_{R(w)} + \underbrace{6\nu^5 \iint_Q \left(\alpha_x^5\right)_x |w_x|^2 \mathrm{d}x\mathrm{d}t}_{M_1(w)} + \underbrace{2\nu^5 \int_0^T \left(\alpha_x^5\right)_{xx} |w|^2\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)} \\ &\underbrace{-\,2\nu^5 \int_0^T \alpha_x^5 |w_x|^2\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)} \underbrace{-\,4\nu^5 \int_0^T \left(\alpha_x^5\right)_x w w_x\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)} + \underbrace{4\nu^5 \int_0^T \alpha_x^5 w w_{xx}\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)}. \end{aligned}$$

- $I_{3,2} = \underbrace{-12\nu^3 \iint_Q \left(\alpha_x^3\right)_x |w_{xx}|^2 \mathrm{d}x\mathrm{d}t}_{M_2(w)} + \underbrace{12\nu^3 \int_0^T \alpha_x^3 |w_{xx}|^2\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)}.$

- $I_{3,3} = \underbrace{-2\nu \iint_Q \alpha_{xx} |w_{xxx}|^2 \mathrm{d}x\mathrm{d}t}_{M_3(w)} + \underbrace{2\nu \int_0^T \alpha_x |w_{xxx}|^2\Big|_{x=0}^{x=L} \mathrm{d}t}_{B(w,\cdot)}.$

Accordingly, by adding all the above terms we get

$$(P_1 w, P_2 w)_{L^2(Q_T)} = \sum_{k=0}^{3} M_k(w) + B(w,L) - B(w,0) + R(w), \tag{3.24}$$

where we have defined the main terms

$$M_0(w) := -14\nu^7 \iint_{Q_T} \alpha_x^6 \alpha_{xx} |w|^2 \mathrm{d}x\mathrm{d}t, \tag{3.25}$$

$$M_1(w) := -30\nu^5 \iint_{Q_T} \alpha_x^4 \alpha_{xx} |w_x|^2 \mathrm{d}x\mathrm{d}t, \tag{3.26}$$

$$M_2(w) := -18\nu^3 \iint_{Q_T} \alpha_x^2 \alpha_{xx} |w_{xx}|^2 \mathrm{d}x\mathrm{d}t, \tag{3.27}$$

$$M_3(w) := -2\nu \iint_{Q_T} \alpha_{xx} |w_{xxx}|^2 \mathrm{d}x\mathrm{d}t, \tag{3.28}$$

the boundary terms for $x \in \{0, L\}$

$$\begin{aligned} B(w,x) := {}& 2\nu^7 \int_0^T \alpha_x^7 |w|^2 \mathrm{d}t + 12\nu^5 \int_0^T \alpha_x^5 |w_x|^2 \mathrm{d}t + 2\nu^3 \int_0^T \left(\alpha_x^3\right)_{xx} |w_x|^2 \mathrm{d}t \\ & - 2\nu^3 \int_0^T \alpha_x^3 |w_{xx}|^2 \mathrm{d}t - 4\nu^3 \int_0^T \left(\alpha_x^3\right)_x w_x w_{xx} \mathrm{d}t + 4\nu^3 \int_0^T \alpha_x^3 w_x w_{xxx} \mathrm{d}t \\ & + 2\nu^5 \int_0^T \left(\alpha_x^5\right)_{xx} |w|^2 \mathrm{d}t - 2\nu^5 \int_0^T \alpha_x^5 |w_x|^2 \mathrm{d}t - 4\nu^5 \int_0^T \left(\alpha_x^5\right)_x w w_x \mathrm{d}t \\ & + 4\nu^5 \int_0^T \alpha_x^5 w w_{xx} \mathrm{d}t + 12\nu^3 \int_0^T \alpha_x^3 |w_{xx}|^2 \mathrm{d}t + 2\nu \int_0^T \alpha_x |w_{xxx}|^2 \mathrm{d}t, \end{aligned} \tag{3.29}$$

and finally, the rest terms

$$R(w) := \frac{\nu^4}{2}\iint_Q \left(\alpha_x^4\right)_t |w|^2\mathrm{d}x\mathrm{d}t - 6\nu^2\iint_Q \alpha_x^2 w_t w_{xx}\,\mathrm{d}x\mathrm{d}t - \iint_Q w_t w_{xxxx}\,\mathrm{d}x\mathrm{d}t \\ - 2\nu^3\iint_Q \left(\alpha_x^3\right)_{xxx}|w_x|^2\mathrm{d}x\mathrm{d}t - 2\nu^5\iint_Q \left(\alpha_x^5\right)_{xxx}|w|^2\mathrm{d}x\mathrm{d}t. \tag{3.30}$$

Step 3. We proceed to handle the main terms defined in (3.25)–(3.28). From the weight functions defined in (3.13) it follows that

$$\alpha_x(t,x) = -\mu\psi'(x)\beta(t,x), \quad \alpha_{xx}(t,x) = -\mu\psi''(x)\beta(t,x) - \mu^2\psi'(x)^2\beta(t,x), \quad \forall (t,x)\in\overline{Q}. \tag{3.31}$$

By plugging them into the main terms defined in (3.25)–(3.28) and then considering that $|\psi'(x)| > 0$ for every $x \in [0,L]\setminus\omega_0$, we see that there exist $C>0$ such that

$$\sum_{k=0}^{3} M_k(w) \geq C\|w\|_A - \frac{1}{\mu}\|w\|_A \\ - \iint_{Q_{\omega_0}} \left(\nu^7\mu^8\beta^7|w|^2 + \nu^5\mu^6\beta^5|w_x|^2 + \nu^3\mu^4\beta^3|w_{xx}|^2 + \nu\mu^2\beta|w_{xxx}|^2\right)\mathrm{d}x\mathrm{d}t. \tag{3.32}$$

Let us handle the last three terms of the right-hand side of this inequality. To this end, let us consider a non-negative function $\chi \in C_0^\infty(\omega_1)$ such that $\chi(x) = 1$ for every $x \in \omega_0$. Recall that $\overline{\omega_0} \subset \omega_1 \subseteq \omega$.

First, some integration by parts gives us

$$\iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi|w_x|^2\mathrm{d}x\mathrm{d}t = \underbrace{\frac{1}{2}\iint_{Q_{\omega_1}} \nu^5\mu^6\left(\beta^5\chi\right)_{xx}|w|^2\mathrm{d}x\mathrm{d}t}_{A_1} - \underbrace{\iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi w_{xx}w\,\mathrm{d}x\mathrm{d}t}_{A_2}. \tag{3.33}$$

On the one hand, the property $\alpha_x(t,x) = -\beta_x(t,x)$ for every $(t,x)\in\overline{Q}$ together with (3.31) and Lemma 3.11(b) allow us to obtain

$$A_1 = \frac{1}{2}\iint_{Q_{\omega_1}} \nu^5\mu^6\left(20\beta^3\beta_x^2\chi + 5\beta^4\beta_{xx}\chi + 10\beta^4\beta_x\chi' + \beta^5\chi''\right)|w|^2\mathrm{d}x\mathrm{d}t, \\ \leq \frac{C}{\nu^2}\left(1 + \frac{1}{\mu} + \frac{1}{\mu^2}\right)\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^5|w|^2\mathrm{d}x\mathrm{d}t.$$

If in this inequality we take into account Lemma 3.11(a), then we arrive at

$$A_1 \leq \frac{C}{\mu^2}\iint_Q \nu^7\mu^8\beta^7|w|^2\mathrm{d}x\mathrm{d}t, \quad \forall \mu\geq 1, \quad \forall \nu \geq T^2. \tag{3.34}$$

On the other hand, the Cauchy inequality leads us to

$$|A_2| \leq \frac{C}{\rho_1}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2\mathrm{d}x\mathrm{d}t + C\rho_1\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2\mathrm{d}x\mathrm{d}t, \quad \forall\rho_1>0. \tag{3.35}$$

Accordingly, from the combination of (3.33), (3.34) and (3.35) we get

$$\iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi|w_x|^2\mathrm{d}x\mathrm{d}t \leq \frac{C}{\mu^2}\iint_Q \nu^7\mu^8\beta^7|w|^2\mathrm{d}x\mathrm{d}t + \frac{C}{\rho_1}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2\mathrm{d}x\mathrm{d}t$$
$$+ C\rho_1\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2\mathrm{d}x\mathrm{d}t, \quad \forall\rho_1 > 0, \quad \forall\mu \geq 1, \quad \forall\nu \geq T^2. \tag{3.36}$$

Second, note that the same arguments presented above can be used to obtain the following inequalities.

$$\iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2\mathrm{d}x\mathrm{d}t \leq \frac{C}{\mu^2}\iint_Q \nu^5\mu^6\beta^5|w_x|^2\mathrm{d}x\mathrm{d}t + \frac{C}{\rho_2}\iint_Q \nu\mu^2\beta|w_{xxx}|^2\mathrm{d}x\mathrm{d}t$$
$$+ C\rho_2\iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi|w_x|^2\mathrm{d}x\mathrm{d}t, \quad \forall\rho_2 > 0, \quad \forall\mu \geq 1, \quad \forall\nu \geq T^2, \tag{3.37}$$

$$\iint_{Q_{\omega_1}} \nu\mu^2\beta\chi|w_{xxx}|^2\mathrm{d}x\mathrm{d}t \leq \frac{C}{\mu^2}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2\mathrm{d}x\mathrm{d}t + \frac{C}{\rho_3}\iint_Q \frac{|w_{xxxx}|^2}{\nu\beta}\mathrm{d}x\mathrm{d}t$$
$$+ C\rho_3\iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2\mathrm{d}x\mathrm{d}t, \quad \forall\rho_3 > 0, \quad \forall\mu \geq 1, \quad \forall\nu \geq T^2. \tag{3.38}$$

Third, plugging (3.36) into (3.37) we obtain

$$\iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2\mathrm{d}x\mathrm{d}t \leq \frac{C}{\mu^2}(1+\rho_2)\iint_Q \left(\nu^7\mu^8\beta^7|w|^2 + \nu^5\mu^6\beta^5|w_x|^2\right)\mathrm{d}x\mathrm{d}t$$
$$+ C\left(\frac{1}{\rho_2} + \frac{\rho_2}{\rho_1}\right)\iint_Q \left(\nu^3\mu^4\beta^3|w_{xx}|^2 + \nu\mu^2\beta|w_{xxx}|^2\right)\mathrm{d}x\mathrm{d}t + C\rho_1\rho_2\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2\mathrm{d}x\mathrm{d}t,$$
$$\forall(\rho_1,\rho_2) \in \mathbb{R}_+^2, \quad \forall\mu \geq 1, \quad \forall\nu \geq T^2. \tag{3.39}$$

Then, plugging (3.39) into (3.38) we get

$$\iint_{Q_{\omega_1}} \nu\mu^2\beta\chi|w_{xxx}|^2\mathrm{d}x\mathrm{d}t \leq \frac{C}{\mu^2}(1+\rho_3+\rho_2\rho_3)\iint_Q \left(\nu^7\mu^8\beta^7|w|^2 + \nu^5\mu^6\beta^5|w_x|^2 + \nu^3\mu^4\beta^3|w_{xx}|^2\right)\mathrm{d}x\mathrm{d}t$$
$$+ C\left(\frac{\rho_3}{\rho_2} + \frac{\rho_2\rho_3}{\rho_1}\right)\iint_Q \left(\nu^3\mu^4\beta^3|w_{xx}|^2 + \nu\mu^2\beta|w_{xxx}|^2\right)\mathrm{d}x\mathrm{d}t + \frac{C}{\rho_3}\iint_Q \frac{|w_{xxxx}|^2}{\nu\beta}\mathrm{d}x\mathrm{d}t$$
$$+ C\rho_1\rho_2\rho_3\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2\mathrm{d}x\mathrm{d}t, \quad \forall(\rho_1,\rho_2,\rho_3) \in \mathbb{R}_+^3, \quad \forall\mu \geq 1, \quad \forall\nu \geq T^2. \tag{3.40}$$

Here we note that if we choose $\rho_2 = \rho > 0$ and $\rho_1 = \rho^2$ in (3.39), then it follows that

$$\iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2\mathrm{d}x\mathrm{d}t \leq \frac{C}{\mu^2}(1+\rho)\|w\|_A + \frac{C}{\rho}\|w\|_A$$
$$+ C\rho^3\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2\mathrm{d}x\mathrm{d}t, \forall\rho > 0, \quad \forall\mu \geq 1, \quad \forall\nu \geq T^2.$$

In a similar way, the choice of $\rho_3 = \rho > 0$, $\rho_2 = \rho^2$ and $\rho_1 = \rho^4$ in (3.40) lead us to

$$\iint_{Q_{\omega_1}} \nu\mu^2\beta\chi|w_{xxx}|^2 \mathrm{d}x\mathrm{d}t \leq \frac{C}{\mu^2}(1+\rho+\rho^3)\|w\|_A + \frac{C}{\rho}\left(\|w\|_A + \|w\|_B\right) + C\rho^7 \iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2 \mathrm{d}x\mathrm{d}t, \quad \forall \rho > 0, \quad \forall \mu \geq 1, \quad \forall \nu \geq T^2.$$

Therefore, from the combination of (3.36) with the two previous inequalities we obtain

$$\iint_{Q_{\omega_1}} \left(\nu^5\mu^6\beta^5\chi|w_x|^2 + \nu^3\mu^4\beta^3\chi|w_{xx}|^2 + \nu\mu^2\beta\chi|w_{xxx}|^2\right) \mathrm{d}x\mathrm{d}t \leq \frac{C}{\mu^2}\left(1+\rho+\rho^3\right)\|w\|_A + \frac{C}{\rho}\left(\|w\|_A + \|w\|_B\right) + C(\rho+\rho^3+\rho^7) \iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2 \mathrm{d}x\mathrm{d}t, \quad \forall \rho > 0, \quad \forall \mu \geq 1, \quad \forall \nu \geq T^2. \tag{3.41}$$

Finally, since

$$\iint_{Q_{\omega_0}} \nu^{7-2n}\mu^{8-2n}\beta^{7-2n}\left|\frac{\partial^n w}{\partial x^n}\right|^2 \mathrm{d}x\mathrm{d}t \leq \iint_{Q_{\omega_1}} \nu^{7-2n}\mu^{8-2n}\beta^{7-2n}\chi\left|\frac{\partial^n w}{\partial x^n}\right|^2 \mathrm{d}x\mathrm{d}t, \quad n = 1, 2, 3,$$

we can use (3.32) together with (3.41) to see that by setting $\rho = \rho_0$ with $\rho_0 \geq 1$ large enough give us

$$\sum_{k=0}^{3} M_k(w) \geq C\|w\|_A - \frac{1}{\rho_0}\|w\|_B - \rho_0^7 \iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2 \mathrm{d}x\mathrm{d}t, \quad \forall \mu \geq \rho_0^3, \quad \forall \nu \geq T^2. \tag{3.42}$$

Step 4. We proceed to handle the boundary terms defined in (3.29). The Cauchy inequality and Lemma 3.11 allow us to obtain the following inequalities that are valid for $x \in \{0, L\}$.

- $$\left|2\nu^3 \int_0^T \left(\alpha_x^3\right)_{xx} |w_x|^2 \mathrm{d}t\right| = \left|2\nu^3 \int_0^T \left(6\alpha_x\alpha_{xx}^2 + 3\alpha_x^2\alpha_{xxx}\right)|w_x|^2 \mathrm{d}t\right| \leq \frac{C}{\nu^2}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)^2 \frac{1}{\mu^2}\int_0^T \nu^5\mu^5\beta^5|w_x|^2 \mathrm{d}t, \quad \forall \mu \geq 1.$$

- $$\left|4\nu^3 \int_0^T \left(\alpha_x^3\right)_x w_x w_{xx}\,\mathrm{d}t\right| = \left|12\nu^3 \int_0^T \alpha_x^2\alpha_{xx} w_x w_{xx}\,\mathrm{d}t\right| \leq C\nu^4 \int_0^T \alpha_x^2\alpha_{xx}^2|w_x|^2 \mathrm{d}t + C\nu^2 \int_0^T \alpha_x^2|w_{xx}|^2 \mathrm{d}t \leq \frac{C}{\nu}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)\frac{1}{\mu}\int_0^T \left(\nu^5\mu^5\beta^5|w_x|^2 + \nu^3\mu^3\beta^3|w_{xx}|^2\right)\mathrm{d}t, \quad \forall \mu \geq 1.$$

- $$\left|4\nu^3 \int_0^T \alpha_x^3 w_x w_{xxx}\,\mathrm{d}t\right| \leq 4\nu^5 \int_0^T |\alpha_x^5||w_x|^2 \mathrm{d}t + \nu \int_0^T |\alpha_x||w_{xxx}|^2 \mathrm{d}t.$$

- $$\left|2\nu^5 \int_0^T \left(\alpha_x^5\right)_{xx} |w|^2 \mathrm{d}t\right| = \left|2\nu^5 \int_0^T \left(20\alpha_x^3\alpha_{xx}^2 + 5\alpha_x^4\alpha_{xxx}\right)|w|^2 \mathrm{d}t\right| \leq \frac{C}{\nu^2}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)^2 \frac{1}{\mu^2}\int_0^T \nu^7\mu^7\beta^7|w|^2 \mathrm{d}t, \quad \forall \mu \geq 1.$$

- $$\left|4\nu^5 \int_0^T \left(\alpha_x^5\right)_x w w_x \, \mathrm{d}t\right| = \left|20\nu^5 \int_0^T \alpha_x^4 \alpha_{xx} w w_x \, \mathrm{d}t\right|$$
$$\leq C\nu^6 \int_0^T \alpha_x^4 \alpha_{xx}^2 |w|^2 \mathrm{d}t + C\nu^4 \int_0^T \alpha_x^4 |w_x|^2 \mathrm{d}t$$
$$\leq \frac{C}{\nu} \left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right) \frac{1}{\mu} \int_0^T \left(\nu^7\mu^7\beta^7|w|^2 + \nu^5\mu^5\beta^5|w_x|^2\right) \mathrm{d}t, \quad \forall \mu \geq 1.$$

- $$\left|4\nu^5 \int_0^T \alpha_x^5 w w_{xx} \, \mathrm{d}t\right| \leq \nu^7 \int_0^T |\alpha_x^7||w|^2 \mathrm{d}t + 4\nu^3 \int_0^T |\alpha_x^3||w_{xx}|^2 \mathrm{d}t.$$

A useful feature of the function $\psi \in C^4([0,L])$ satisfying (3.10)–(3.12) is that $\psi'(0) > 0$ and $\psi'(L) < 0$. From this feature and the combination of the six above inequalities with the boundary terms defined in (3.29), it follows that the choice of $\mu_0 \geq 1$ large enough gives us

$$B(w,0) \leq 0, \quad B(w,L) \geq 0, \quad \forall \mu \geq \mu_0, \quad \forall \nu \geq T^2. \tag{3.43}$$

Step 5. Before handling the rest terms defined in (3.30), we are going to apply Theorem 3.8 to the equation satisfied by $p := q_{xx} + \lambda q$, which is equation (3.23). Let $\omega_{1/2}$ be a non-empty open interval such that $\overline{\omega_0} \subset \omega_{1/2}$ and $\overline{\omega_{1/2}} \subset \omega_1$. The above theorem in particular yields

$$\iint_Q \mathrm{e}^{-2\nu\alpha} \left(\nu^7\mu^8\beta^7|p|^2 + \nu^5\mu^6\beta^5|p_x|^2 + \nu^3\mu^4\beta^3|p_{xx}|^2\right) \mathrm{d}x\mathrm{d}t$$
$$\leq C \iint_Q \mathrm{e}^{-2\nu\alpha}|G_{xx} + \lambda G|^2 \mathrm{d}x\mathrm{d}t + C \iint_{Q_{\omega_{1/2}}} \mathrm{e}^{-2\nu\alpha}\nu^7\mu^8\beta^7|p|^2 \mathrm{d}x\mathrm{d}t, \quad \forall \mu \geq \mu_0, \quad \forall \nu \geq \max\{T, T^2\},$$

from which it follows that

$$\iint_Q \mathrm{e}^{-2\nu\alpha} \left(\nu^7\mu^8\beta^7|q_{xx}|^2 + \nu^5\mu^6\beta^5|q_{xxx}|^2 + \nu^3\mu^4\beta^3|q_{xxxx}|^2\right) \mathrm{d}x\mathrm{d}t \leq$$
$$C \iint_Q \mathrm{e}^{-2\nu\alpha} \left(|G_{xx}|^2 + |G|^2\right) \mathrm{d}x\mathrm{d}t + C \iint_Q \mathrm{e}^{-2\nu\alpha} \left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2 + \nu^3\mu^4\beta^3|q_{xx}|^2\right) \mathrm{d}x\mathrm{d}t$$
$$+ C \iint_{Q_{\omega_{1/2}}} \mathrm{e}^{-2\nu\alpha}\nu^7\mu^8\beta^7 \left(|q_{xx}|^2 + |q|^2\right) \mathrm{d}x\mathrm{d}t, \quad \forall \mu \geq \mu_0, \quad \forall \nu \geq \max\{T, T^2\}. \tag{3.44}$$

Now, let us consider a non-negative function $\chi \in C_0^\infty(\omega_1)$ such that $\chi(x) = 1$ for every $x \in \omega_{1/2}$. Then, by using the same arguments as those used in Step 3 it is possible to obtain

$$\iint_{Q_{\omega_{1/2}}} \mathrm{e}^{-2\nu\alpha}\nu^7\mu^8\beta^7|q_{xx}|^2\mathrm{d}x\mathrm{d}t \le \frac{C}{\rho}\iint_Q \mathrm{e}^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q_{xx}|^2 + \nu^5\mu^6\beta^5|q_{xxx}|^2\right)\mathrm{d}x\mathrm{d}t$$
$$+ C(1+\rho)(1+\rho^2)\iint_{Q_{\omega_1}} \mathrm{e}^{-2\nu\alpha}\nu^{11}\mu^{12}\beta^{11}|q|^2\mathrm{d}x\mathrm{d}t, \quad \forall\rho > 0, \quad \forall\mu \ge 1, \quad \forall\nu \ge T^2.$$

Accordingly, by plugging the previous inequality into (3.44) we see that $\rho > 0$ can be choosen in such a way that for $\mu_0 \ge 1$ large enough we get

$$\iint_Q \mathrm{e}^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q_{xx}|^2 + \nu^5\mu^6\beta^5|q_{xxx}|^2 + \nu^3\mu^4\beta^3|q_{xxxx}|^2\right)\mathrm{d}x\mathrm{d}t \le$$
$$C\iint_Q \mathrm{e}^{-2\nu\alpha}\left(|G_{xx}|^2 + |G|^2\right)\mathrm{d}x\mathrm{d}t + C\iint_Q \mathrm{e}^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2\right)\mathrm{d}x\mathrm{d}t$$
$$+ C\iint_{Q_{\omega_1}} \mathrm{e}^{-2\nu\alpha}\nu^{11}\mu^{12}\beta^{11}|q|^2\mathrm{d}x\mathrm{d}t, \quad \forall\mu \ge \mu_0, \quad \forall\nu \ge \max\{T, T^2\}.$$

Furthermore, in view of $-q_t + q_{xxxx} + \lambda q_{xx} = G$ and Lemma 3.11(a) we actually have

$$\iint_Q \mathrm{e}^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q_{xx}|^2 + \nu^5\mu^6\beta^5|q_{xxx}|^2 + \nu^3\mu^4\beta^3|q_{xxxx}|^2 + \nu^3\mu^4\beta^3|q_t|^2\right)\mathrm{d}x\mathrm{d}t \le$$
$$C\iint_Q \mathrm{e}^{-2\nu\alpha}\left(|G_{xx}|^2 + \nu^3\mu^4\beta^3|G|^2\right)\mathrm{d}x\mathrm{d}t + C\iint_Q \mathrm{e}^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2\right)\mathrm{d}x\mathrm{d}t$$
$$+ C\iint_{Q_{\omega_1}} \mathrm{e}^{-2\nu\alpha}\nu^{11}\mu^{12}\beta^{11}|q|^2\mathrm{d}x\mathrm{d}t, \quad \forall\mu \ge \mu_0, \quad \forall\nu \ge \max\{T, T^2\}. \tag{3.45}$$

Step 6. We proceed to handle the rest terms defined in (3.30) and those in $P_3 w$. All of these can be handled in the same way as we did in Step 4. Nevertheless, we pay special attention to the first three terms of the right-hand side of (3.30) and to the term $\nu\alpha_t w$ in $P_3 w$.

The Cauchy inequality, (3.31) and Lemma 3.11(a) allow us to obtain the following inequalities.

- $$\frac{\nu^4}{2}\iint_Q \left(\alpha_x^4\right)_t|w|^2\mathrm{d}x\mathrm{d}t = 2\nu^4\iint_Q \alpha_x^3\alpha_{xt}|w|^2\mathrm{d}x\mathrm{d}t$$
$$\le \frac{C}{\nu^3}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)^2 T\frac{1}{\mu^7}\iint_Q \nu^7\mu^8\beta^7|w|^2\mathrm{d}x\mathrm{d}t. \tag{3.46}$$

- $$6\nu^2\iint_Q \alpha_x^2 w_t w_{xx}\,\mathrm{d}x\mathrm{d}t \le C\iint_Q \nu^2\mu^2\beta^2|w_t|^2\mathrm{d}x\mathrm{d}t + \frac{C}{\nu}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)\frac{1}{\mu^3}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2\mathrm{d}x\mathrm{d}t.$$

- $$\iint_Q w_t w_{xxxx}\,\mathrm{d}x\mathrm{d}t \le C\iint_Q \nu^2\mu^2\beta^2|w_t|^2\mathrm{d}x\mathrm{d}t + C\iint_Q \frac{|w_{xxxx}|^2}{\nu^2\mu^2\beta^2}\mathrm{d}x\mathrm{d}t.$$

- $$\nu^2\iint_Q \alpha_t^2|w|^2\mathrm{d}x\mathrm{d}t \le \frac{C}{\nu^5}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)^3 T^2\frac{1}{\mu^{10}}\iint_Q \nu^7\mu^8\beta^7|w|^2\mathrm{d}x\mathrm{d}t. \tag{3.47}$$

Note that here we have used the facts that $|\alpha_{xt}(t,x)| \leq CT\beta^2(t,x)$ and $|\alpha_t(t,x)| \leq T\beta^2(t,x)$ hold for every $(t,x) \in \overline{Q}$. Accordingly, from the combination of the above inequalities with the rest term defined in (3.30) and $P_3 w$ we get

$$|R(w)| \leq \frac{C}{\mu}\|w\|_A + C\iint_Q \nu^2\mu^2\beta^2|w_t|^2 \mathrm{d}x\mathrm{d}t + C\iint_Q \frac{|w_{xxxx}|^2}{\nu^2\mu^2\beta^2}\mathrm{d}x\mathrm{d}t, \quad \forall\mu \geq 1, \quad \forall\nu \geq \max\{T, T^2\}. \tag{3.48}$$

$$\|P_3 w\|^2_{L^2(Q)} \leq \frac{C}{\mu}\|w\|_A, \quad \forall\mu \geq 1, \quad \forall\nu \geq \max\{T, T^2\}. \tag{3.49}$$

We remark that $\nu \geq \max\{T, T^2\}$ was asked because of (3.46) and (3.47).

Step 7. We proceed to obtain the desired Carleman estimate. First, from the combination of (3.21), (3.24), (3.42) and (3.43), it follows that for $\rho_0 \geq 1$ large enough we have

$$\begin{aligned} C\left(\|P_1 w\|_{L^2(Q)} + \|P_2 w\|_{L^2(Q)} + \|w\|_A\right) &\leq \|Pw - P_3 w\|^2_{L^2(Q)} + \frac{1}{\rho_0}\|w\|_B \\ &\quad + |R(w)| + \rho_0^7 \iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2 \mathrm{d}x\mathrm{d}t, \quad \forall\mu \geq \rho_0^3, \quad \forall\nu \geq T^2. \end{aligned}$$

Hence, from this inequality and (3.22) we see that by taking $\rho_0 \geq 1$ large enough gives us the existence of a $\mu_0 \geq 1$ also large enough such that

$$C\left(\|w\|_A + \|w\|_B\right) \leq \|Pw - P_3 w\|^2_{L^2(Q)} + |R(w)| + \iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2 \mathrm{d}x\mathrm{d}t, \quad \forall\mu \geq \mu_0, \quad \forall\nu \geq T^2.$$

Second, by plugging (3.48) together with (3.49) into this inequality and then taking $\mu_0 \geq 1$ large enough lead us to

$$\begin{aligned} C\left(\|w\|_A + \|w\|_B\right) &\leq \|Pw\|^2_{L^2(Q)} + \iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2 \mathrm{d}x\mathrm{d}t \\ &\quad + \iint_Q \nu^2\mu^2\beta^2|w_t|^2 \mathrm{d}x\mathrm{d}t + \iint_Q \frac{|w_{xxxx}|^2}{\nu^2\mu^2\beta^2}\mathrm{d}x\mathrm{d}t, \quad \forall\mu \geq \mu_0, \quad \forall\nu \geq \max\{T, T^2\}. \end{aligned} \tag{3.50}$$

Third, by taking into account that the inequalities

$$\iint_Q \mathrm{e}^{-2\nu\alpha}\frac{1}{\nu\beta}\left(|(\mathrm{e}^{\nu\alpha}w)_t|^2 + |(\mathrm{e}^{\nu\alpha}w)_{xxxx}|^2\right)\mathrm{d}x\mathrm{d}t \leq C\left(\|w\|_A + \|w\|_B\right),$$

$$\sum_{n=0}^{3}\iint_Q \mathrm{e}^{-2\nu\alpha}\nu^{7-2n}\mu^{8-2n}\beta^{7-2n}\left|\frac{\partial^n(\mathrm{e}^{\nu\alpha}w)}{\partial x^n}\right|^2 \mathrm{d}x\mathrm{d}t \leq C\|w\|_A,$$

$$|w_t|^2 \leq C\mathrm{e}^{-2\nu\alpha}\left(|q_t|^2 + T^2\nu^2\beta^4|q|^2\right), \quad \forall(t,x) \in \overline{Q},$$

$$|w_{xxxx}|^2 \leq C\mathrm{e}^{-2\nu\alpha}\left(\nu^8\mu^8\beta^8|q|^2 + \nu^6\mu^6\beta^6|q_x|^2 + \nu^4\mu^4\beta^4|q_{xx}|^2 + \nu^2\mu^2\beta^2|q_{xxx}|^2\right), \quad \forall(t,x) \in \overline{Q},$$

hold for every $\mu \geq \mu_0$, with $\mu_0 \geq 1$ large enough, and $\nu \geq \max\{T, T^2\}$, we see from (3.50), $w = \mathrm{e}^{-\nu\alpha} q$ and $Pw := \mathrm{e}^{-\nu\alpha} \left(-\partial_t + \partial_{xxxx} + \lambda \partial_{xx}\right) \left(\mathrm{e}^{\nu\alpha} w\right)$ that it follows

$$\begin{aligned}
&\iint_Q \mathrm{e}^{-2\nu\alpha} \left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2 + \nu^3\mu^4\beta^3|q_{xx}|^2 + \nu\mu^2\beta|q_{xxx}|^2\right) \mathrm{d}x\mathrm{d}t \\
&\qquad + \iint_Q \mathrm{e}^{-2\nu\alpha} \left(\frac{|q_t|^2 + |q_{xxxx}|^2}{\nu\beta}\right) \mathrm{d}x\mathrm{d}t \leq C \iint_Q \mathrm{e}^{-2\nu\alpha}|G|^2 \mathrm{d}x\mathrm{d}t + C \iint_{Q_{\omega_1}} \mathrm{e}^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2 \mathrm{d}x\mathrm{d}t, \\
&\qquad\qquad + \frac{C}{\mu^2} \iint_Q \mathrm{e}^{-2\nu\alpha} \left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2 + \nu^3\mu^4\beta^3|q_{xx}|^2 + \nu\mu^2\beta|q_{xxx}|^2\right) \mathrm{d}x\mathrm{d}t \\
&\qquad\qquad\qquad + \frac{C}{\mu^2} \iint_Q \mathrm{e}^{-2\nu\alpha}\nu^3\mu^4\beta^3|q_t|^2 \mathrm{d}x\mathrm{d}t, \quad \forall \mu \geq \mu_0, \quad \forall \nu \geq \max\{T, T^2\}.
\end{aligned}$$

Finally, from the combination of this inequality with (3.45) we obtain (3.15) by choosing $\mu_0 \geq 1$ large enough. The proof of Proposition 3.9 is complete.

Remark 3.12. We were not able to prove the null controllability of the Kuramoto−Sivashinsky equation with Neumann boundary condition and either internal or boundary controls. The reason is that our Carleman estimate (47) is not good enough. In the the right-hand side of (47), we see the term $\iint_Q \mathrm{e}^{-2\nu\alpha} \frac{1}{\mu^2} |G_{xx}|^2 \mathrm{d}x\mathrm{d}t$, which requires the right-hand side G of the adjoint equation to be in $L^2(0, T; H^2(0, L))$. Thus, if duality arguments are applied, then the solution of the controlled equation would be in $L^2(0, T; H^{-2}(0, L))$ and then in $C([0, T], H^{-4}(0, L))$. At this level of regularity, it is hard to deal with the nonlinearity of the equation.

Remark 3.13. As mentioned in the Introduction, we do not obtain directly a Carleman estimate in the Neumann case, because of the boundary conditions. For instance, if we perform integration by parts in $I_{1,2}$ and $I_{1,3}$ (see Step 2 in the proof of Prop. 3.9), then we would obtain

- $I_{1,2} = 6\nu^2 \iint_Q \left(\alpha_x^2\right)_x w_t w_x \, \mathrm{d}x\mathrm{d}t - 3\nu^2 \iint_Q \left(\alpha_x^2\right)_t |w_x|^2 \mathrm{d}x\mathrm{d}t - \underbrace{6\nu^2 \int_0^T \alpha_x^2 w_t w_x \big|_{x=0}^{x=L} \mathrm{d}t}_{B_1}.$

- $I_{1,3} = \underbrace{\int_0^T w_{tx} w_{xx} \big|_{x=0}^{x=L} \mathrm{d}t}_{B_2} - \underbrace{\int_0^T w_t w_{xxx} \big|_{x=0}^{x=L} \mathrm{d}t}_{B_3}.$

The boundary conditions of adjoint equation (3.7) makes difficult to handle the boundary terms B_1, B_2 and B_3. The difficulty arises when trying to handle the terms $w_t(t, x)$ and $w_{tx}(t, x)$ at $x \in \{0, L\}$. Note that this difficulty does not appear when deriving Carleman estimates for adjoint equation (2.7) because B_1, B_2 and B_3 vanish.

References

[1] L. Baudouin, E. Cerpa, E. Crépeau and A. Mercado, Lipschitz stability in an inverse problem for the Kuramoto−Sivashinsky equation. *Appl. Anal.* **92** (2013) 2084–2102.

[2] E. Cerpa, Null controllability and stabilization of the linear Kuramoto−Sivashinsky equation. *Commun. Pure Appl. Anal.* **9** (2010) 91–102.

[3] E. Cerpa and A. Mercado, Local exact controllability to the trajectories of the 1-D Kuramoto−Sivashinsky equation. *J. Differ. Equ.* **250** (2011) 2024–2044.

[4] E. Cerpa, I. Rivas and B.-Y. Zhang, Boundary controllability of the Korteweg-de Vries equation on a bounded domain. *SIAM J. Control Optim.* **51** (2013) 2976–3010.

[5] J.-M. Coron, Control and Nonlinearity. Vol. 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI (2007).
[6] E. Crépeau, Exact controllability of the Boussinesq equation on a bounded domain. *Differ. Integral Equ.* **16** (2003) 303–326.
[7] R. Dautray and J.-L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 5. Springer Verlag, Berlin, Heidelberg (2000).
[8] H.O. Fattorini and D.L. Russell, Exact controllability theorems for linear parabolic equation in one space dimension. *Arch. Rat. Mech. Anal.* **43** (1971) 272–292.
[9] A.V. Fursikov and O.Yu. Imanuvilov, Controllability of Evolution Equations. Vol. 34 of *Lecture Notes Series*. Seoul National University, Korea (1996).
[10] P. Gao, Insensitizing controls for the Cahn-Hilliard type equation. *Electron. J. Qual. Theory Differ. Equ.* **35** (2014) 1–22.
[11] O. Glass and S. Guerrero, Controllability of the Korteweg-de Vries equation from the right Dirichlet boundary condition. *Systems Control Lett.* **59** (2010) 390–395.
[12] P. Guzmán Meléndez, Lipschitz stability in an inverse problem for the main coefficient of a Kuramoto−Sivashinsky type equation. *J. Math. Anal. Appl.* **408** (2013) 275–290.
[13] C. Hu and R. Temam, Robust control of the Kuramoto−Sivashinsky equation. *Dyn. Contin. Discrete Impuls. Syst. Ser. B* **8** (2001) 315–338.
[14] Y. Kuramoto and T. Tsuzuki, On the formation of dissipative structures in reaction-diffusion systems: Reductive perturbation approach. *Prog. Theor. Phys.* **54** (1975) 687–699.
[15] Y. Kuramoto and T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium. *Prog. Theor. Phys.* **55** (1976) 356–369.
[16] W.-J. Liu and M. Krstić, Stability enhancement by boundary control in the Kuramoto−Sivashinsky equation. *Nonlin. Anal. Ser. A* **43** (2001) 485–507.
[17] D.M. Michelson, G.I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames II: Numerical experiments. *Acta Astronaut.* **4** (1977) 1207–1221.
[18] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations. Vol. 44 of *Appl. Math. Sci.* Springer Verlag, New York (1983).
[19] L. Rosier, Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. *ESAIM: COCV* **2** (1997) 33–55.
[20] G.I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames I: Derivation of basic equations. *Acta Astronaut.* **4** (1977) 1177–1206.
[21] M. Tucsnak and G. Weiss, Observation and Control for Operator Semigroups. Birkhäuser Verlag, Basel, Boston, Berlin (2009).
[22] Z. Zhou, Observability estimate and null controllability for one-dimensional fourth order parabolic equation. *Taiwanese J. Math.* **16** (2012) 1991–2017.