On the controllability of the improved Boussinesq equation*

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Abstract

The improved Boussinesq equation is studied in this paper. Control properties for this equation posed on a bounded interval are first considered. When the control acts through the Dirichlet boundary condition the linearized system is proved to be approximately but not spectrally controllable. In a second part, the equation is posed on the one-dimensional torus and distributed moving controls are considered. Under some condition on the velocity to which the control moves, exact controllability results for both linear and nonlinear improved Boussinesq equations are obtained applying the moment method and a fixed point argument.

Key words. Boussinesq type equation, exact controllability, approximate controllability, spectral analysis, moving control, moment method, fixed point theorem

AMS subject classifications. 35Q35, 93B05, 93C10

1 Introduction

In [1], Boussinesq derived the so called "bad" Boussinesq equation, written

$$y_{tt} - y_{xx} - y_{xxxx} = (y^2)_{xx}. \tag{1}$$

This equation describes the flow of shallow water waves with small amplitudes in a flat bottom canal. It is called "bad" due to its poor existence and uniqueness properties. For instance, unlikely the "good" Boussinesq equation, which reads as

$$y_{tt} - y_{xx} + y_{xxxx} = (y^2)_{xx}, \tag{2}$$

there is no local well-posedness results for equation (1). However, in [12], Makhankov proved that the "bad" Boussinesq equation (1) can be approached by the following one, called improved Boussinesq equation,

$$y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx}. \tag{3}$$

The well-posedness problem for the improved Boussinesq equation (3) with Dirichlet boundary conditions has been studied by Zhijian in [19].

Concerning the control of these equations, due to the lack of a well-posedness framework, there is no control results dealing with the "bad" Boussinesq equation. On the

*This work has been partially supported by Fondecyt 1140741, MathAmsud COSIP, and CONICYT-Basal Project FB0008 AC3E.

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other hand, the boundary controllability for the "good" Boussinesq equation (2) posed on a bounded domain was addressed in [6]. In that paper, a local controllability result for the nonlinear equation is obtained with the help of the Hilbert Uniqueness Method for the controllability of the linearized equation and a fixed point theorem to obtain the local controllability of the nonlinear one.

In this paper, we are concerned with the controllability of the improved Boussinesq equation (3) posed either on a bounded or periodic domain. In the case of a bounded domain $[0,1]$ with boundary control, we prove that the linearized equation is not spectrally controllable, and consequently, not null controllable. Despite these negative results, we prove an approximate controllability result. Those results of non controllability are due to the existence of a finite accumulation point in the spectrum. Previous results of bad control properties due to the spectrum have already been noticed by Russell [18] for the beam equation with internal damping, by Leugering and Schmidt [10] for the plate equation, by Micu [14] for the linearized Benjamin-Bona-Mahony equation and by Rosier and Rouchon [15] for the structurally damped wave equation.

In order to improve the control properties of our equation, we study the improved Boussinesq equation (3) posed on a periodic domain and we consider a moving distributed control. This kind of moving actuators has been considered previously in the literature since the work [11] by Lions. In that paper, the wave equation with moving point control was considered (see also [8]). For the same equation, we find the more recent papers [2, 3, 13, 5]. Concerning parabolic equations, we can cite the papers [7, 4] dealing with semi linear and linear heat equations. In [16], Rosier and Zhang proved that the BBM equation posed in the torus with a moving distributed control is locally exactly controllable for a control time large enough.

In the second part of this paper, we are able to prove the local exact controllability of the improved Boussinesq equation under a condition on the velocity the control moves. The controllability is proved with the moment method for the linearized equation, and then with a fixed point argument in order to deal with the nonlinearity.

The rest of this paper is organized as follows. In Section 2 we consider the linearized improved Boussinesq equation posed on a bounded domain with a boundary control. The approximate controllability and the lack of exact controllability are proved. The periodic case with a moving control is studied in Section 3, where exact control results are obtained for both linear and nonlinear improved Boussinesq equations.

2 Boundary control on a bounded domain

In this section we look at the boundary controllability of the linearized improved Boussinesq equation posed on the finite interval $[0,1]$. Namely, given a time $T>0$, an initial condition (y^0,y^1) and a target (y^0_T,y^1_T) on an appropriate space, we wonder if we can find a control function $h=h(t)$ such that the solution of the following linear problem

$$(4)\qquad \begin{cases} y_{tt}-y_{xx}-y_{xxtt}=0, & (x,t)\in(0,1)\times(0,T),\\ y(0,t)=0,\ y(1,t)=h(t), & t\in(0,T),\\ y(x,0)=y^0(x),\ y_t(x,0)=y^1(x), & x\in(0,1),\end{cases}$$

satisfies $y(T)=y^0_T$ and $y_t(T)=y^1_T$.

2.1 Well posedness

We first look at the well posedness of the homogeneous improved Boussinesq problem on a bounded domain

$$(5)\qquad \begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in (0,1). \end{cases}$$

The well posedness of this problem has already been studied by Zhijian in [19]. He proved that the equation (5) with a nonlinear term $(y^2)_{xx}$ is well posed, locally in time, for $(y^0, y^1) \in (H^2 \cap H_0^1(0,1))^2$ and the solution belongs to $C^2([0, T_0[, H^2 \cap H_0^1(0,1))$. We will establish this kind of results by using spectral methods for the linear equation (5).

We can rewrite the homogeneous system (5) as follows,

$$(6)\qquad y_{tt} + Ay = 0,$$

where, for $D(A) = H^2 \cap H_0^1(0,1)$, we define the operator $A : D(A) \subset L^2(0,1) \longrightarrow L^2(0,1)$ by means of

$$A : w \longmapsto -(I - \partial_{xx})^{-1} \partial_{xx} w.$$

Proposition 1 *There exists a basis of $L^2(0,1)$ formed by eigenfunctions $\{f_k\}_{k\in\mathbb{N}^*}$ of the operator A. Moreover, this family is given by $f_k(x) = \sqrt{2}\sin(k\pi x)$ and the corresponding eigenvalues are $\lambda_k = \frac{k^2\pi^2}{k^2\pi^2+1}$, for any $k \in \mathbb{N}^*$.*

Proof. Let $\lambda \in \mathbb{C}$ and $y \in H^2 \cap H_0^1(0,1)$ such that $Ay = \lambda y$, then

$$(7)\qquad \begin{cases} y_{xx} = -\lambda(y - y_{xx}), \\ y(0) = y(1) = 0. \end{cases}$$

From this, we see that

$$(8)\qquad \begin{cases} y_{xx} = \frac{\lambda}{\lambda-1} y, \\ y(0) = y(1) = 0, \end{cases}$$

and then, the eigenvalues are $\lambda_k = \frac{k^2\pi^2}{k^2\pi^2+1}$ and the corresponding eigenfunctions are $f_k(x) = \sqrt{2}\sin(k\pi x)$, for $k \in \mathbb{N}^*$. ■

Remark 2 *We can easily remark that the eigenvalues $\lambda_k \in \mathbb{R}^+$ are simple, and in addition $\lim_{k\to+\infty} \lambda_k = 1$. Thus, the spectrum of A admits a finite point of accumulation.*

By using the spectral decomposition of A and the asymptotic behavior of λ_k, we can write the solutions of system (5) in the space $\mathcal{H}^s(0,1)$ defined in the following classical way for any $s \geq 0$:

$$\mathcal{H}^s(0,1) = \left\{ \sum_{k\geq 1} a_k f_k(x) \Big/ \sum_{k\geq 1} k^{2s} |a_k|^2 < \infty \right\}.$$

We can remark that:

- for $s \leq 1/2$, $\mathcal{H}^s(0,1) = H^s(0,1)$,

- for $1/2 < s \leq 3/2$, $\mathcal{H}^s(0,1) = H_0^s(0,1)$,
- and for $3/2 < s \leq 2$, $\mathcal{H}^s(0,1) = H^s(0,1) \cap H_0^1(0,1)$,

where $H^s(0,1)$ and $H_0^s(0,1)$ are the usual Sobolev spaces. Thus, we explicitely obtain the following.

Proposition 3 *Let $s \geq 0$. For every $(y^0, y^1) \in \mathcal{H}^s(0,1)^2$ the solution y of equation (5) belongs to $C^1([0,+\infty[, \mathcal{H}^s(0,1))$ and can be written as*

$$y(x,t) = \sum_{k\geq 1} \left(a_k \cos(\sqrt{\lambda_k} t) + \frac{b_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \right) f_k(x), \tag{9}$$

where $y^0 = \sum_{k\geq 1} a_k f_k$ and $y^1 = \sum_{k\geq 1} b_k f_k$.

Remark 4 *From the cosine and sine functions, we see that there is no gain of regularity for the linear improved Boussinesq equation. Moreover, from the asymptotic behavior of eigenvalues λ_k, we see the fact that the position y and the velocity y_t have the same regularity.*

Remark 5 *As we will see later, we need to have solutions y of the homogeneous problem such that the trace $y_x(1,t)$ exists as a function. From the previous proposition, we see that if $s > 3/2$, then we obtain a solution such that the desired trace is a function. Another way to see that is through the following simple computation,*

$$\begin{aligned} |y_x(1,t)| &\leq 2\pi\sqrt{2} \sum_{n\geq 1} \left(n|a_n| + n|b_n| \right) \\ &\leq 2\pi\sqrt{2} \Big(\sum_{n\geq 1} n^{2-2s} \Big)^{1/2} \Big(\sum_{n\geq 1} n^{2s} |a_n|^2 \Big)^{1/2} + 2\pi\sqrt{2} \Big(\sum_{n\geq 1} n^{2-2s} \Big)^{1/2} \Big(\sum_{n\geq 1} n^{2s} |b_n|^2 \Big)^{1/2}, \end{aligned}$$

which gives the desired result when we are in regularity $\mathcal{H}^s(0,1)$ with $s > 3/2$.

We work now with the initial boundary value problem (4).

Proposition 6 *Let $y^0, y^1 \in L^2(0,1)$ and $h \in H^2(0,T)$. Then, the equation (4) has a unique solution in the space $y \in C^1([0,T], L^2(0,1))$.*

Proof. If y is the solution of (4), then $\varphi(x,t) := y(x,t) - xh(t)$ is the solution of

$$\begin{cases} \varphi_{tt} - \varphi_{xx} - \varphi_{xxtt} = -x\ddot{h}, & (x,t) \in (0,1) \times (0,T), \\ \varphi(0,t) = \varphi(1,t) = 0, & t \in (0,T), \\ \varphi(x,0) = y^0 - xh(0),\ \varphi_t(x,0) = y^1 - x\dot{h}(0), & x \in (0,1), \end{cases} \tag{10}$$

where $\dot{h}$ and $\ddot{h}$ denote the first and second derivative in time of h, respectively.

By introducing $v = \varphi$ and $w = \varphi_t$, we can write this equation as a first order system as follows,

$$\begin{cases} \dfrac{d}{dt} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & I \\ (I - \partial_{xx})^{-1} \partial_{xx} & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ -\ddot{h}(I - \partial_{xx})^{-1} x \end{pmatrix}, \\ \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} y^0 - xh(0) \\ y^1 - x\dot{h}(0) \end{pmatrix}. \end{cases} \tag{11}$$

The forcing term belongs to $L^2(0,T;\mathcal{H}^s(0,1))$, for any $0 \le s \le 1/2$ (notice that the function $f(x) = x$ belongs to $H^s(0,1)$ for any $s < 1/2$). On the other hand, the matrix operator defines a group of isometries in $L^2(0,1)^2$ (see Proposition 3). Thus, by using a classical result, we obtain a well-posedness result for (11) giving solutions φ in $C^1([0,T], L^2(0,1))$. Going back to y we obtain the wanted result (see [9, page 13] for a similar argument). ■

2.2 Lack of exact controllability

We want to study the exact controllability of

$$\left\{\begin{array}{ll} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = 0,\ y(1,t) = h(t), & t \in (0,T), \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in (0,1), \end{array}\right. \tag{12}$$

where $h \in H^2(0,T)$ is the boundary control and $y^0, y^1 \in L^2(0,1)$ are the initial data. Let z be the solution of the adjoint problem of (12), which is given by

$$\left\{\begin{array}{ll} z_{tt} - z_{xx} - z_{xxtt} = 0, & (x,t) \in (0,1) \times (0,T), \\ z(0,t) = z(1,t) = 0, & t \in (0,T), \\ z(x,T) = z_T^0(x),\ z_t(x,T) = z_T^1(x), & x \in (0,1), \end{array}\right. \tag{13}$$

for $z_T^0, z_T^1 \in H^2 \cap H_0^1(0,1) = \mathcal{H}^2(0,1)$. We decompose the initial data of z in Fourier series,

$$z_T^0 = \sum_{n\ge 1} \tilde{a}_n f_n(x),\ z_T^1 = -\sum_{n\ge 1} \tilde{b}_n f_n(x),$$

with $\sum_{n\ge 1}(n^4|\tilde{a}_n|^2 + n^4|\tilde{b}_n|^2) < \infty$, in order to write the solution of (13) as

$$z(x,t) = \sum_{n\ge 1} \Big(\tilde{a}_n \cos(\sqrt{\lambda_n}(T-t)) + \frac{\tilde{b}_n}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n}(T-t))\Big) f_n(x).$$

Equivalently, in its complex form, this solution is given by

$$z(x,t) = \sum_{n\ge 1} \Big(\tilde{c}_n e^{i\sqrt{\lambda_n}(T-t)} + \tilde{d}_n e^{-i\sqrt{\lambda_n}(T-t)}\Big) f_n(x), \tag{14}$$

where

$$\tilde{c}_n = \frac{1}{2}\Big(\tilde{a}_n - \frac{i\tilde{b}_n}{\sqrt{\lambda_n}}\Big) \text{ and } \tilde{d}_n = \frac{1}{2}\Big(\tilde{a}_n + \frac{i\tilde{b}_n}{\sqrt{\lambda_n}}\Big).$$

We are now in position to prove the following non controllability result.

Theorem 7 *The control system* (12) *is not spectrally controllable in* $L^2(0,1)$.

Proof. We prove that no nontrivial finite combination of eigenfunctions $\{f_n\}_{n\geq 1}$ can be driven to zero in finite time. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of real numbers such that there exists $N \in \mathbb{N}$ with $a_n = b_n = 0$ for all $n > N$.

Suppose that the system (12) is spectrally controllable. Then, there exists a boundary control $h \in H^2(0,T)$ such that the solution of (12) with initial data

$$y^0 = \sum_{n\geq 1} a_n f_n, \quad y^1 = \sum_{n\geq 1} b_n f_n,$$

satisfies $y(T) = y_t(T) = 0$.

We multiply (12) by z and integrate in space and time over $(0,1)\times(0,T)$. Thus, we obtain

$$\int_0^1 \Big[y^1\Big(z(x,0) - z_{xx}(x,0)\Big) - y^0\Big(z_t(x,0) - z_{xxt}(x,0)\Big)\Big]dx = \int_0^T \Big(h(t) + \ddot{h}(t)\Big) z_x(1,t)dt$$

for any solution z of (13). Using this equation with appropriate trajectories, first with $z(x,t) = e^{i\sqrt{\lambda_n}(T-t)} f_n(x)$ and then with $z(x,t) = e^{-i\sqrt{\lambda_n}(T-t)} f_n(x)$, we have that the control h is the solution of the moment problem composed, for any $n \geq 1$, of the following equations

$$\left\{\begin{array}{l} (1+n^2\pi^2)\Big(i\sqrt{\lambda_n}a_n + b_n\Big) = \sqrt{2}(n\pi)(-1)^n \displaystyle\int_0^T \Big(h(t) + \ddot{h}(t)\Big)e^{-i\sqrt{\lambda_n}t}dt, \\ (1+n^2\pi^2)\Big(-i\sqrt{\lambda_n}a_n + b_n\Big) = \sqrt{2}(n\pi)(-1)^n \displaystyle\int_0^T \Big(h(t) + \ddot{h}(t)\Big)e^{i\sqrt{\lambda_n}t}dt. \end{array}\right. \tag{15}$$

We proceed as in [14] and [15]. Let us define the complex function

$$F(z) := \int_0^T (h(t) + \ddot{h}(t))e^{izt}dt.$$

Due to the Paley-Wiener theorem, F is an entire function and it satisfies $F(\pm\sqrt{\lambda_n}) = 0$, for all $n > N$. Because the asymptotic behavior of the eigenvalues ($\sqrt{\lambda_n} \to 1$ as $n \to \infty$), we see that F vanishes on a set with a finite accumulation point. Therefore, we conclude that $F \equiv 0$. From (15), we easily obtain that $a_n = b_n = 0$ for each $n \geq 1$. That means that the trivial state is the only one which can be steered to zero. ■

Remark 8 *System* (12) *being not spectrally controllable, we consequently know that* (12) *is neither exact nor null controllable. This is clear from the proof. Moreover, all the computations in this proof involve a finite number of modes. In this way, this applies for any regularity framework* $\mathcal{H}^s(0,1)$ *where the control system is well-posed.*

2.3 Approximate controllability

In spite of the lack of exact controllability from the boundary, we now prove that system (12) is approximately controllable. This property is equivalent to a unique continuation property for the adjoint system.

Theorem 9 *System* (12) *is approximately controllable in* $L^2(0,1)$ *for any time* $T > 0$.

Proof. Thanks to the linearity of system (12), we only have to prove the approximate controllability from the initial state ($y^0 = 0, y^1 = 0$). Let us define the map

$$\Lambda : h \in H^2(0,T) \longmapsto (y(T), y_t(T)) \in L^2(0,1)^2.$$

We have to prove that the range of this linear operator Λ is dense in $L^2(0,1)^2$. Let $(w_T^0, w_T^1) \in L^2(0,1)^2$ such that

$$-\int_0^1 y_t(T) w_T^0 \, dx + \int_0^1 y(T) w_T^1 \, dx = 0. \tag{16}$$

Let us define $z_T^0, z_T^1 \in H^2 \cap H_0^1(0,1)$ such that

$$z_T^0 - \partial_{xx} z_T^0 = w_T^0, \qquad \text{and} \qquad z_T^1 - \partial_{xx} z_T^1 = w_T^1,$$

and consider z as the solution of (13) with initial condition (at $t = T$) given by z_T^0, z_T^1.

We multiply (12) by z and integrate in space and time over $(0,1) \times (0,T)$. Thus, we obtain

$$\int_0^T \Big(h(t) + \ddot{h}(t) \Big) z_x(1,t) dt = 0,$$

where we have used (16) and the fact that ($y^0 = y^1 = 0$). We prove now that we must have $z = 0$. This would imply that $z_T^0 = z_T^1 = 0$ and consequently $w_T^0 = w_T^1 = 0$, which ends the proof. Indeed, let us choose $h(t) = e^{i\frac{2\pi n}{T}t}$, for $n \in \mathbb{Z}$. Then,

$$0 = \int_0^T (h(t) + \ddot{h}(t)) z_x(1,t) dt = \left(1 - \left(\frac{2\pi n}{T} \right)^2 \right) \int_0^T e^{i\frac{2\pi n}{T}t} z_x(1,t) dt,$$

and consequently, the integral term in the right hand side must be zero for any $n \neq \pm\frac{T}{2\pi}$. Thus, we see that:

- If $\frac{T}{2\pi} \notin \mathbb{Z}$, then $z_x(1,\cdot)$ is orthogonal to any function $e^{i\frac{2\pi n}{T}t}$. In conclusion, we get $z_x(1,\cdot) = 0$.
- If $\frac{T}{2\pi} \in \mathbb{Z}$, then $z_x(1,\cdot)$ is orthogonal to any function $e^{i\frac{2\pi n}{T}t}$ except at most when $n = \pm\frac{T}{2\pi}$. In conclusion, we get $z_x(1,\cdot) \in \mathrm{Span}\{e^{it}, e^{-it}\}$.

In both cases, we can write that there exist α and $\beta \in \mathbb{C}$ such that we have the expression $z_x(1,t) = \alpha e^{it} + \beta e^{-it}$. In this way, we get

$$z_x(1,t) = \sum_{n \geq 1} \left[\tilde{c}_n e^{i\sqrt{\lambda_n}(T-t)} + \tilde{d}_n e^{-i\sqrt{\lambda_n}(T-t)} \right] \sqrt{2}(n\pi)(-1)^n = \alpha e^{it} + \beta e^{-it},$$

where we have used (14) with the corresponding coefficients. From this, we see that

$$\sum_{n \geq 1} \left[\tilde{c}_n e^{i\sqrt{\lambda_n}(T-t)} + \tilde{d}_n e^{-i\sqrt{\lambda_n}(T-t)} \right] \sqrt{2}(n\pi)(-1)^n - \alpha e^{it} - \beta e^{-it} = 0, \tag{17}$$

for all $t \in (0,T)$. As this function is analytic, it vanishes for any $t \in \mathbb{R}$.

By using (17), and noting that $\sqrt{\lambda_m} \neq 1$ for any $m \geq 1$, we obtain the following

$$0 = \lim_{S\to+\infty} \frac{1}{2S}\int_{-S}^{S} \Big(\sum_{n\geq 1} \left[\tilde{c}_n e^{i\sqrt{\lambda_n}(T-t)} + \tilde{d}_n e^{-i\sqrt{\lambda_n}(T-t)}\right] \sqrt{2}(n\pi)(-1)^n$$

$$- \alpha e^{it} - \beta e^{-it}\Big) e^{i\sqrt{\lambda_m}t} dt = (-1)^m \sqrt{2}(m\pi)\tilde{c}_m e^{i\sqrt{\lambda_m}T},$$

and

$$0 = \lim_{S\to+\infty} \frac{1}{2S}\int_{-S}^{S} \Big(\sum_{n\geq 1} \left[\tilde{c}_n e^{i\sqrt{\lambda_n}(T-t)} + \tilde{d}_n e^{-i\sqrt{\lambda_n}(T-t)}\right] \sqrt{2}(n\pi)(-1)^n$$

$$- \alpha e^{it} - \beta e^{-it}\Big) e^{-i\sqrt{\lambda_m}t} dt = (-1)^m \sqrt{2}(m\pi)\tilde{d}_m e^{-i\sqrt{\lambda_m}T}.$$

In consequence, for any $m \geq 1$, we obtain that $\tilde{c}_m = \tilde{d}_m = 0$. This fact implies that $z = 0$, which ends the proof of Theorem 9. ■

Remark 10 *Concerning the nonlinear improved Boussinesq equation controlled from the boundary, one can prove the local approximately controllability around the origin by using the wellposedness of the equation, the approximately controllability for the linearized equation stated in Theorem 9 and a perturbative argument. On the other hand, we can wonder if the exact controllability holds for the nonlinear equation. Our conjecture is that it does not. The lack of exact controllability comes from the bad behavior of the eigenvalues and this affects in a severe way the controllability property.*

3 Moving distributed control on a periodic domain

The previous results of non exact controllability lead us to study another type of control whose support moves on the torus $\mathbb{T} = \mathbb{R} \setminus (2\pi\mathbb{Z})$. This moving control is supposed to be supported in a set that moves at a constant velocity c. In this section, we look at the distributed control problem of the improved Boussinesq equation posed on the torus $\mathbb{T}$. Namely, given a time $T > 0$, an initial condition (y^0, y^1) and a target (y^0_T, y^1_T) on appropriate spaces, we wonder wether we can find a control function $h = h(x,t)$ such that the solution of the following problem

$$\text{(18)} \qquad \begin{cases} y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx} + b(x+ct)h(x,t), & x \in \mathbb{T},\ t > 0, \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in \mathbb{T}, \end{cases}$$

satisfies $y(T) = y^0_T$ and $y_t(T) = y^1_T$, where $b = b(x)$ is a given nonzero smooth function. In order to deal with the control problem, we first linearize it around the origin to obtain

$$\text{(19)} \qquad \begin{cases} y_{tt} - y_{xx} - y_{xxtt} = b(x+ct)h(x,t), & x \in \mathbb{T},\ t > 0, \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in \mathbb{T}. \end{cases}$$

We study this linear control system and then we come back to the nonlinear one by means of a fixed-point argument.

3.1 Well-posedness

We first study the well-posedness of equation (19) without control,

$$
\left\{\begin{array}{ll} y_{tt} - y_{xxtt} - y_{xx} = 0, & x \in \mathbb{T},\ t > 0, \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in \mathbb{T}. \end{array}\right. \tag{20}
$$

For $s \geq 0$, $H^s(\mathbb{T})$ is the usual Sobolev space on the torus $\mathbb{T}$, namely

$$
H^s(\mathbb{T}) = \Big\{ u : \mathbb{T} \to \mathbb{R} \Big/ \|u\|_{H^s(\mathbb{T})} := \|(1-\partial_x^2)^{\frac{s}{2}} u\|_{L^2(\mathbb{T})} < \infty \Big\}.
$$

As in the previous section, we easily obtain the following propositions.

Proposition 11 *Let $s \geq 0$. For every $(y^0, y^1) \in H^s(\mathbb{T})^2$ the solution y of equation (20) belongs to $C^1([0,+\infty[, H^s(\mathbb{T}))$ and can be decomposed as*

$$
y(x,t) = (\alpha_0 + \beta_0 t) + \sum_{k\in\mathbb{Z}^*} \left(\alpha_k \cos\left(\sqrt{\frac{k^2}{k^2+1}} t \right) + \beta_k \sqrt{\frac{k^2+1}{k^2}} \sin\left(\sqrt{\frac{k^2}{k^2+1}} t \right) \right) e^{ikx}.
$$

where $y^0 = \sum_{k\in\mathbb{Z}} \alpha_k e^{ikx}$ and $y^1 = \sum_{k\in\mathbb{Z}} \beta_k e^{ikx}$. We easily obtain that there exists $C_0 > 0$ such that

$$
\|y\|_{C^1([0,+\infty[,H^s(\mathbb{T}))} \leq C_0 \left(\|y^0\|_{H^s(\mathbb{T})} + \|y^1\|_{H^s(\mathbb{T})} \right).
$$

Proposition 12 *Let $(y^0, y^1) \in H^s(\mathbb{T})^2$ with $s \geq 0$ and $F \in L^1(0,T; H^{s-2}(\mathbb{T}))$. Then the solution of the linear system*

$$
y_{tt} - y_{xxtt} - y_{xx} = F,
$$

with $y(.,0) = y^0$ and $y_t(.,0) = y^1$, satisfies $y \in C^1([0,+\infty[, H^s(\mathbb{T}))$ and the solution depends continuously on data, i.e. there exists $C_2 > 0$ such that

$$
\|y\|_{C^1([0,+\infty[,H^s(\mathbb{T}))} \leq C_2 \left(\|y^0\|_{H^s(\mathbb{T})} + \|y^1\|_{H^s(\mathbb{T})} + \|F\|_{L^1(0,T;H^{s-2}(\mathbb{T}))} \right).
$$

3.2 Gap condition

Let us prove that if we choose a sufficiently large c, the terms $kc \pm \sqrt{\frac{k^2}{k^2+1}}$ are all different and have an asymptotical gap. This will be useful to study the exact controllability wanted. We define two families of sequences, for $k \in \mathbb{Z}$,

$$
\lambda_k^+ = \left(ck + \frac{|k|}{\sqrt{1+k^2}} \right) \text{ and } \lambda_k^- = \left(ck - \frac{|k|}{\sqrt{1+k^2}} \right).
$$

We can easily prove the following lemma concerning the asymptotical gap in these eigenvalues distribution.

Lemma 13 *Let us denote by δ the asymptotical gap between the eigenvalues. If $\frac{2}{c} \notin \mathbb{Z}$, then*

$$
\delta \geq \Delta = \left| c\Big(dist(\frac{2}{c}, \mathbb{Z}) \Big) \right|.
$$

Proof. Let $k, k' \in \mathbb{N}$. We easily get that

$$\lambda^+_{k'} - \lambda^+_k = \lambda^-_{-k} - \lambda^-_{-k'} = c(k'-k) + \frac{1}{2k'^2} - \frac{1}{2k^2} + o\Big(\frac{1}{k^2}\Big) + o\Big(\frac{1}{k'^2}\Big)$$
$$\lambda^-_{k'} - \lambda^+_k = c\Big(k' - k - \frac{2}{c}\Big) - \frac{1}{2k'^2} - \frac{1}{2k^2} + o\Big(\frac{1}{k^2}\Big) + o\Big(\frac{1}{k'^2}\Big)$$

Thus the asymptotical gap is larger than $\Big|c\Big(\mathrm{dist}(\frac{2}{c}, \mathbb{Z})\Big)\Big|$. ■

Remark 14 *For instance, if $c \geq 4$, then we can take as asymptotical gap $\Delta = 2$.*

3.3 Exact controllability of the linear system

We look at the following internal control problem,

$$(21) \qquad \begin{cases} y_{tt} - y_{xxtt} + y_{xx} = b(x+ct)h(x,t), & x \in \mathbb{T},\ t > 0, \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in \mathbb{T}. \end{cases}$$

We want to prove the following result, where $\Delta > 0$ is defined in Lemma 13 as the gap for some eigenvalues.

Theorem 15 *Let $s \geq 2$ and c be such that $|c| > 2$. Let $b = b(x) \in C^\infty(\mathbb{T})$ be such that $\{x \in \mathbb{T},\ b(x) \neq 0\} \neq \emptyset$. Then, for all $T > \frac{2\pi}{\Delta}$ and all (y^0, y^1), $(y^0_T, y^1_T) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$, there exists a control $h \in L^2(0,T; H^{s-2}(\mathbb{T}))$ such that the linear problem (21) admits a unique solution $y \in C^1([0,T], H^s(\mathbb{T}))$ such that $y(x,T) = y^0_T(x)$ and $y_t(x,T) = y^1_T(x)$. Furthermore, there exist $C_1 > 0$ such that*

$$\|h\|_{L^2(0,T,H^{s-2}(\mathbb{T}))} \leq C_1 \left(\|(y^0, y^1)\|_{H^s(\mathbb{T})^2} + \|(y^0_T, y^1_T)\|_{H^s(\mathbb{T})^2} \right)$$

Remark 16 *The condition $|c| > 2$ is useful in two ways. To be sure the asymptotic gap Δ is positive ($\frac{2}{c} \notin \mathbb{Z}$) and to avoid the existence of different k, m such that*

$$ck + \sqrt{\frac{k^2}{k^2+1}} = cm - \sqrt{\frac{m^2}{m^2+1}}.$$

The latter is needed in order to solve the moment problem with no additional compatibility conditions on the initial and final data.

Proof. The adjoint problem is written as follows,

$$(22) \qquad \varphi_{tt} - \varphi_{xxtt} - \varphi_{xx} = 0, \quad x \in \mathbb{T},\ t > 0.$$

We easily remark that if y is solution of the direct problem (21) with $h = 0$, then $\varphi(x,t) = y(2\pi - x, T - t)$ is a solution of the adjoint problem (22).

Let us multiply equation (21) by $\bar{\varphi}$ where φ is a solution of (22) and integrate by parts on $[0,T] \times \mathbb{T}$. Then, we obtain

$$(23) \qquad \int_{\mathbb{T}} \left[y_t(\bar{\varphi} - \bar{\varphi}_{xx}) - y(\bar{\varphi}_t - \bar{\varphi}_{xxt}) \right] \Big|_{t=0}^{T} dx = \int_0^T \int_{\mathbb{T}} h(x,t) b(x+ct) \bar{\varphi}(x,t) dx dt.$$

We take, for $k \in \mathbb{Z}$, $\bar{\varphi}^+(x,t) = e^{i\sqrt{\frac{k^2}{k^2+1}}(T-t)} e^{ikx}$ and $\bar{\varphi}^-(x,t) = e^{-i\sqrt{\frac{k^2}{k^2+1}}(T-t)} e^{ikx}$. Thus, (23) becomes

(24)
$$(1+k^2)\left(\langle y_t(T)-e^{\pm i\sqrt{\frac{k^2}{k^2+1}}T}y_t(0),e^{ikx}\rangle \mp i\sqrt{\frac{k^2}{k^2+1}}\langle y(T)-e^{\pm i\sqrt{\frac{k^2}{k^2+1}}T}y(0),e^{ikx}\rangle\right)$$
$$=\int_0^T\int_{\mathbb{T}}h(x,t)b(x+ct)e^{\pm i\sqrt{\frac{k^2}{k^2+1}}(T-t)}e^{ikx}dxdt.$$

where $\langle f,e^{ikx}\rangle$ stands for the coordinate f_k in the Fourier decomposition $f=\sum_{k\in\mathbb{Z}}f_ke^{ikx}$.

Taking $\varphi=t$ in (23), we obtain

$$T\int_{\mathbb{T}}y_t(T)\,dx-\int_{\mathbb{T}}y(T)\,dx+\int_{\mathbb{T}}y(0)\,dx=\int_0^T\int_{\mathbb{T}}h(x,t)b(x+ct)tdxdt. \tag{25}$$

By a simple change of variables we can rewrite the right hand side of (24) and (25) as follows,

$$\int_0^T\int_{\mathbb{T}}h(x,t)b(x+ct)e^{\pm i\sqrt{\frac{k^2}{k^2+1}}(T-t)}e^{ikx}dxdt$$
$$=\int_0^T\int_{\mathbb{T}}h(x-ct,t)b(x)e^{\pm i\sqrt{\frac{k^2}{k^2+1}}(T-t)}e^{ik(x-ct)}\,dxdt,$$

$$\int_0^T\int_{\mathbb{T}}h(x,t)b(x+ct)t\,dxdt=\int_0^T\int_{\mathbb{T}}h(x-ct,t)b(x)t\,dxdt.$$

Let us define $\tilde{h}(x,t)=h(x-ct,t)$. Then the moment problem becomes in finding a control $\tilde{h}$ such that for all $k\in\mathbb{Z}$,

(26)
$$(1+k^2)\left(\langle e^{\pm i\sqrt{\frac{k^2}{k^2+1}}T}y_t(T)-y_t(0),e^{ikx}\rangle \pm i\sqrt{\frac{k^2}{k^2+1}}\langle e^{\pm i\sqrt{\frac{k^2}{k^2+1}}T}y(T)-y(0),e^{ikx}\rangle\right)$$
$$=\int_0^T\int_{\mathbb{T}}\tilde{h}(x,t)b(x)e^{-i(kc\mp\sqrt{\frac{k^2}{k^2+1}})t}e^{ikx}dxdt,$$

and

$$T\int_{\mathbb{T}}y_t(T)\,dx-\int_{\mathbb{T}}y(T)\,dx+\int_{\mathbb{T}}y(0)\,dx=\int_0^T\int_{\mathbb{T}}\tilde{h}(x,t)b(x)t\,dxdt. \tag{27}$$

Because of the asymptotical gap Δ and the fact that the time exponential functions are all different, we can apply standard results on complex exponential functions (see [17, section 2]) to prove, for any $T>\frac{2\pi}{\Delta}$, the existence of a function $h\in L^2(0,T,H^{s-2}(\mathbb{T}))$ solving (26)-(27). Indeed, we choose $\{q_0,\tilde{q}_0\}\cup\{q_k^{\pm}\}_{k\in\mathbb{Z}^*}\subset L^2(0,T)$, the biorthogonal family to

$$\Big\{1,t\Big\}\cup\Big\{e^{-i(kc\mp\sqrt{\frac{k^2}{k^2+1}})t}\Big\}_{k\in\mathbb{Z}^*}.$$

We follow [16], and look for a control in the form

$$\tilde{h}(x,t) = b(x)\Big\{ f_0 q_0(t) + \tilde{f}_0 \tilde{q}_0(t) + \sum_{j\in\mathbb{Z}^*} \Big(f_j^+ q_j^+(t) + f_j^- q_j^-(t)\Big) e^{-ijx} \Big\}$$

where $f_0, \tilde{f}_0, f_j^{\pm}$ are scalars chosen to satisfy the moment problem (26)-(27). Thus, we obtain, for $k \in \mathbb{Z}^*$,

$$(28)\quad f_k^+ = \frac{(1+k^2)}{\int_{\mathbb{T}} b^2(x)dx}\Big(\langle e^{i\sqrt{\frac{k^2}{k^2+1}}T} y_t(T) - y_t(0), e^{ikx}\rangle + i\sqrt{\frac{k^2}{k^2+1}} \langle e^{i\sqrt{\frac{k^2}{k^2+1}}T} y(T) - y(0), e^{ikx}\rangle \Big),$$

$$(29)\quad f_k^- = \frac{(1+k^2)}{\int_{\mathbb{T}} b^2(x)dx}\Big(\langle e^{-i\sqrt{\frac{k^2}{k^2+1}}T} y_t(T) - y_t(0), e^{ikx}\rangle - i\sqrt{\frac{k^2}{k^2+1}} \langle e^{-i\sqrt{\frac{k^2}{k^2+1}}T} y(T) - y(0), e^{ikx}\rangle \Big),$$

and

$$(30)\qquad f_0 = \frac{1}{\int_{\mathbb{T}} b^2(x)dx}\Big(\int_{\mathbb{T}} y_t(T)\,dx - \int_{\mathbb{T}} y_t(0)\,dx \Big),$$

$$(31)\qquad \tilde{f}_0 = \frac{1}{\int_{\mathbb{T}} b^2(x)dx}\Big(T\int_{\mathbb{T}} y_t(T)\,dx - \int_{\mathbb{T}} y(T)\,dx + \int_{\mathbb{T}} y(0)\,dx \Big),$$

where $\int_{\mathbb{T}} b^2(x)dx \neq 0$ by hypothesis. Furthermore, from equations (28)-(29)-(30)-(31) we obtain the continuity of the control with respect to data. More explicitly, we obtain the existence of a constant $C > 0$ such that

(32)

$$\begin{aligned}
\|\tilde{h}\|^2_{L^2(0,T;H^{s-2}(\mathbb{T}))} &= \int_0^T \|b(x)\Big\{ f_0 q_0(t) + \tilde{f}_0 \tilde{q}_0(t) + \sum_{j\in\mathbb{Z}^*} \Big(f_j^+ q_j^+(t) + f_j^- q_j^-(t)\Big) e^{-ijx} \Big\}\|^2_{H^{s-2}(\mathbb{T})} dt \\
&\leq C \int_0^T \Big\{ |f_0 q_0(t)|^2 + |\tilde{f}_0 \tilde{q}_0(t)|^2 + \sum_{j\in\mathbb{Z}^*} (1+j^2)^{s-2} |f_j^+ q_j^+(t) + f_j^- q_j^-(t)|^2 \Big\} dt \\
&\leq 2C \Big\{ |f_0|^2 + |\tilde{f}_0|^2 + \sum_{j\in\mathbb{Z}^*} (1+j^2)^{s-2} (|f_j^+|^2 + |f_j^-|^2) \Big\} \\
&\leq 2C (\|y^0\|^2_{H^s(\mathbb{T})} + \|y^1\|^2_{H^s(\mathbb{T})} + \|y^0_T\|^2_{H^s(\mathbb{T})} + \|y^1_T\|^2_{H^s(\mathbb{T})})
\end{aligned}$$

Then, $h(x,t) = \tilde{h}(x+ct, t)$ is the desired control function that drives the system from (y^0, y^1) to (y_T^0, y_T^1), which concludes the proof. ■

3.4 Local exact controllability of the nonlinear system

We follow the proof of the local exact controllability of the Boussinesq equation given in [6]. We decompose the solution of the nonlinear problem,

$$
\left\{ \begin{array}{ll} y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx} + b(x+ct)h(x,t), & x \in \mathbb{T},\, t > 0, \\ y(x,0) = y^0(x),\, y_t(x,0) = y^1(x), & x \in \mathbb{T}, \end{array} \right. \tag{33}
$$

in the following way, $y = \alpha + \beta + \gamma$, where α is the solution of the linear problem with initial data and with no control,

$$
\left\{ \begin{array}{ll} \alpha_{tt} - \alpha_{xx} - \alpha_{xxtt} = 0, & x \in \mathbb{T},\, t > 0, \\ \alpha(x,0) = y^0(x),\, \alpha_t(x,0) = y^1(x), & x \in \mathbb{T}, \end{array} \right. \tag{34}
$$

β is the solution of the linear problem with control but with null initial data,

$$
\left\{ \begin{array}{ll} \beta_{tt} - \beta_{xx} - \beta_{xxtt} = b(x+ct)h(x,t), & x \in \mathbb{T},\, t > 0, \\ \beta(x,0) = 0,\, \beta_t(x,0) = 0, & x \in \mathbb{T}, \end{array} \right. \tag{35}
$$

and γ is the solution of the linear problem with a second member term F,

$$
\left\{ \begin{array}{ll} \gamma_{tt} - \gamma_{xx} - \gamma_{xxtt} = F, & x \in \mathbb{T},\, t > 0, \\ \gamma(x,0) = 0,\, \gamma_t(x,0) = 0, & x \in \mathbb{T}. \end{array} \right. \tag{36}
$$

(Later, the source term F will be taken as the nonlinearity $(y^2)_{xx}$.)

We study the nonlinear problem in the following regularity framework:

$$
(y^0, y^1) \in H^2(\mathbb{T})^2,\ h \in L^2(0,T,L^2(\mathbb{T})),\ F \in L^1(0,T;L^2(\mathbb{T})) \text{ and } b \in C^\infty(\mathbb{T}),
$$

where b is such that $\{x \in \mathbb{T},\, b(x) \neq 0\} \neq \emptyset$.

Let us consider the following maps, which are well-defined, linear and continuous by Propositions 11 and 12.

- The initial data-to-solution map:
$$
\psi_0 : (y^0, y^1) \in H^2(\mathbb{T})^2 \longmapsto \alpha \in C^1([0,T], H^2(\mathbb{T})),
$$
where α is the solution of (34).
- The control-to-solution map:
$$
\psi_1 : h \in L^2(0,T,L^2(\mathbb{T})) \longmapsto \beta \in C^1([0,T], H^2(\mathbb{T})),
$$
where β is the solution of (35).
- The source term-to-solution map:
$$
\psi_2 : F \in L^1(0,T;L^2(\mathbb{T})) \longmapsto \gamma \in C^1([0,T], H^2(\mathbb{T})),
$$
where γ is the solution of (36).

In order to deal with the nonlinearity, we need the following proposition.

Proposition 17 ([6]) *The map*

$$
\phi \in L^2(0,T,H^2(\mathbb{T})) \longmapsto (\phi^2)_{xx} \in L^1(0,T,L^2(\mathbb{T})),
$$

is well-defined and continuous. We have the existence of $K > 0$ such that

$$
\|(\phi^2)_{xx} - (\psi^2)_{xx}\|_{L^1(0,T,L^2(\mathbb{T}))} \leq K \left(\|\phi\|_{L^2(0,T,H^2(\mathbb{T}))} + \|\psi\|_{L^2(0,T,H^2(\mathbb{T}))} \right) \|\phi - \psi\|_{L^2(0,T,H^2(\mathbb{T}))}.
$$

Proof. This proof is exactly the same as in [6, Proposition 6], where the result is obtained in the spatial domain $[0, L]$. ■

Thanks to equation (32), we can define a continuous map

$$\Gamma : (y_T^0, y_T^1) \in H^2(\mathbb{T})^2 \longmapsto h \in L^2(0, T, L^2(\mathbb{T})),$$

such that the solution $\beta \in C^1([0,T], H^2(\mathbb{T}))$ of (35) satisfies $(\beta(T), \beta_t(T)) = (y_T^0, y_T^1)$. Let us define the following map

$$\Pi : y \in C^1([0,T], H^2(\mathbb{T})) \mapsto \Pi(y) \in C^1([0,T], H^2(\mathbb{T}))$$

where

$$\Pi(y) = \psi_0(y^0, y^1) + \psi_2((y^2)_{xx})$$
$$+ \psi_1 \circ \Gamma\Big((y_T^0, y_T^1) - \big(\psi_0(y^0, y^1)(T), \psi_{0t}(y^0, y^1)(T)\big) - \big(\psi_2((y^2)_{xx})(T), \psi_{2t}((y^2)_{xx})(T)\big)\Big).$$

We see that a fixed-point of Π is a trajectory $y = y(x,t)$ of (33) going from (y^0, y^1) at time $t = 0$ to (y_T^0, y_T^1) at time $t = T$.

Let us apply the Banach fixed-point theorem. We consider small data with $\epsilon > 0$ to be fixed later:

$$\|(y^0, y^1)\|_{H^2(\mathbb{T})^2} \leq \epsilon, \quad \|(y_T^0, y_T^1)\|_{H^2(\mathbb{T})^2} \leq \epsilon.$$

We have to find $R > 0$ and $D \in (0,1)$ such that

1. $\Pi(B_R) \subset B_R$ where B_R is the closed ball

   $$B_R = \Big\{ v \in C^1([0,T], H^2(\mathbb{T})) \Big/ \|v\| \leq R \Big\}.$$

   (In this section, $\|\cdot\|$ stands for the norm in $C^1([0,T], H^2(\mathbb{T}))$.)

2. $\|\Pi(v) - \Pi(w)\| \leq D\|v - w\|$.

Taking in mind the constants C_0, C_2, K, C_1 in Propositions 11, 12, 17 and Theorem 15, respectively, we obtain for $v \in B_R$

$$\|\Pi(v)\| \leq C_0\epsilon + C_2C_1(\epsilon + C_0\epsilon + C_2KR^2) + C_2KR^2.$$

Thus, we obtain the first condition to satisfy:

$$C_0\epsilon + C_2C_1(\epsilon + C_0\epsilon + C_2KR^2) + C_2KR^2 \leq R. \tag{37}$$

On the other hand, for $v, w \in B_R$,

$$\Pi(v) - \Pi(w) = \psi_1 \circ \Gamma\Big(\psi_2((w^2)_{xx} - (v^2)_{xx})(T), \psi_{2t}((w^2)_{xx} - (v^2)_{xx})(T)\Big) + \psi_2((w^2)_{xx} - (v^2)_{xx}),$$

and then

$$\|\Pi(v) - \Pi(w)\| \leq 2C_2C_1C_2KR\|v - w\| + 2C_2KR\|v - w\| \leq 2\tilde{C}R\|v - w\|.$$

In this way, we obtain the second condition to satisfy:

$$\tilde{C}R < 1/2. \tag{38}$$

By chosing $\epsilon, R > 0$ small enough in order to satisfy (37) and (38), we can apply the Banach fixed-point theorem to prove the following result.

Theorem 18 *Let c be such that $|c| > 2$. Let $b = b(x) \in C^\infty(\mathbb{T})$ be such that $\{x \in \mathbb{T}, b(x) \neq 0\} \neq \emptyset$. Then, for all $T > \frac{2\pi}{\Delta}$, there exists $\epsilon > 0$ such that for any (y^0, y^1), $(y_T^0, y_T^1) \in H^2(\mathbb{T})^2$ satisfying*

$$\|(y^0, y^1)\|_{H^2(\mathbb{T})^2} \leq \epsilon, \quad \|(y_T^0, y_T^1)\|_{H^2(\mathbb{T})^2} \leq \epsilon,$$

there exists a control $h \in L^2(0, T, L^2(\mathbb{T}))$ such that the problem (33) admits a unique solution $y \in C^1([0,T], H^2(\mathbb{T}))$ such that

$$y(x,T) = y_T^0(x), \quad \text{and} \quad y_t(x,T) = y_T^1(x).$$

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