Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

BOUNDARY CONTROLLABILITY OF THE KORTEWEG-DE VRIES EQUATION ON A TREE-SHAPED NETWORK

Eduardo Cerpa

Departamento de Matemática, Universidad Técnica Federico Santa María,
Avda. España 1680
Valparaíso, Chile

Emmanuelle Crépeau*

Univ. Grenoble Alpes, CNRS, Grenoble INP, LJK
38000 Grenoble, France

Julie Valein

Institut Elie Cartan de Lorraine
Université de Lorraine & Inria (Project-Team SPHINX)
BP 70 239, F-54506 Vandœuvre-les-Nancy Cedex, France

(Communicated by the associate editor name)

Abstract. Controllability of coupled systems is a complex issue depending on the coupling conditions and the equations themselves. Roughly speaking, the main challenge is controlling a system with less inputs than equations. In this paper this is successfully done for a system of Korteweg-de Vries equations posed on an oriented tree shaped network. The couplings and the controls appear only on boundary conditions.

1. **Introduction and main result.** Partial differential equations (PDE) appear in many contexts to model different phenomena. Most of time, these equations are coupled and their study gets much more difficult than when a single equation appear. The control of such systems is not the exception and thus it is very important to understand how we can get controllable systems by using the properties of the single equations and the couplings.

The most known PDE are parabolic and hyperbolic equations. Thus, it is very natural to find many works concerning the controllability of coupled systems involving them. If we restrict our attention to boundary controllability, which is the main issue of this paper, we can mention among a huge literature, [1, 5, 16] for parabolic equations and [3, 13, 19] for hyperbolic equations.

In this context, a network is a particular kind of coupled systems in which different PDE are posed on different domains (edges of the network) with coupled boundary conditions (acting on the nodes of the network). Depending on the topology of the structure edges-nodes we define star-shaped, tree-shaped or just general

2010 *Mathematics Subject Classification.* Primary: 35Q35, 93B05; Secondary: 93C10.

Key words and phrases. KdV equation, control theory, coupled systems, network, mutlipliers method.

This work has been partially supported by FONDECYT 1180528, Math-Amsud ICoPS 17-MATH-04, ANR project ISDEEC (ANR-16-CE40-0013) and Basal Project FB0008 AC3E.

* Corresponding author: Emmanuelle Crépeau.

networks. For this particular kind of coupled systems we already find some boundary controllability results. In fact, we can mention [7] for parabolic systems and [4, 14, 15, 18, 21, 25] for hyperbolic systems.

In this work, we are interested in the controllability of oriented networks for the Korteweg-de Vries (KdV) equation. In the literature there is already a good understanding of the control of the single KdV equation. When we deal with the KdV equation with homogeneous Dirichlet conditions and right Neumann condition on a bounded domain, the length L of the interval where the equation is set plays a role in the ability of controlling the solution of the equation ([9, 12, 23]). Indeed, it is well-known that if $L = 2\pi$, there exists a stationary solution ($y(x,t) = 1-\cos x$) of the linearized system around 0 which has constant energy. More generally, defining the set of critical lengths

$$\mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}},\ k,l \in \mathbb{N}^* \right\},$$

one can recall that the linearized equation around 0 is exactly controllable with only one right Neumann control if and only if $L \notin \mathcal{N}$ (see [23]) and the local exact controllability result holds for the nonlinear KdV equation (using a fixed point argument) if $L \notin \mathcal{N}$. Further results show that the nonlinear KdV equation is in fact locally exactly controllable for all critical lengths contrary to the linear KdV equation (see [8, 10, 12]). See also [9] and [24] for a complete bibliographical review.

As we have a rather complete understanding of the boundary controllability of this equation, we deal here with the KdV equation posed on a network. Recently, we find two papers dealing with the controllability of the KdV equation on a network. In both, the topology considered is a star-shaped network, having in this way one central node and several external nodes. These two papers giving a positive answer to the controllability of the nonlinear KdV equation on network are [2] with $N+1$ boundary controls for N edges (the main topic of that work is the stabilization) and [11] with N boundary controls for N edges.

Generally speaking, the main differences between papers [2] and [11] and the present work are: the sense of the propagation of the water wave on the first edge; the transmission conditions at the central node; and the fact that we improve the previous results having one control less here.

More precisely, in this paper we consider a tree-shaped network $\mathcal{R}$ of $(N+1)$ edges e_i (where $N \in \mathbb{N}^*$), of lengths $l_i > 0$, $i \in \{1,..,N+1\}$, connected at one vertex that we assume to be 0 for all the edges. We assume that the first edge e_1 is parametrized on the interval $I_1 := (-l_1, 0)$ and the N other edges e_i are parametrized on the interval $I_i := (0, l_i)$ (see Figure 1).

On each edge we pose a nonlinear Korteweg-de Vries (KdV) equation. On the first edge ($i = 1$) we put no control and on the other edges ($i = 2, \cdots, N+1$) we

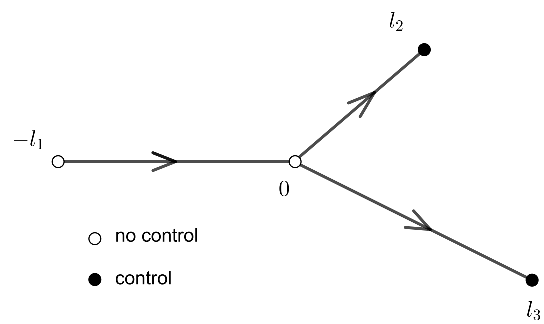

FIGURE 1. A tree-shaped network with 3 edges ($N = 2$)

consider Neumann boundary controls. Thus, we can write the system

$$\begin{cases} (y_{i,t} + y_{i,x} + y_{i,xxx} + y_i y_{i,x})(x,t) = 0, \ i \in \{1, \cdots, N+1\}, \ x \in I_i, \ t > 0, \\ y_1(-l_1, t) = 0, & t > 0, \\ \left.\begin{array}{l} y_i(l_i,t) = 0, \\ y_{i,x}(l_i,t) = h_i(t), \\ y_1(0,t) = \alpha_i y_i(0,t), \end{array}\right\} & \forall i \in \{2, \cdots, N+1\}, \ t > 0, \\ y_{1,x}(0,t) = \displaystyle\sum_{i=2}^{N+1} \beta_i y_{i,x}(0,t), & t > 0, \\ y_{1,xx}(0,t) = \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i} y_{i,xx}(0,t), & t > 0, \\ y_i(x,0) = y_{i0}(x), & \forall i \in \{1, \cdots, N+1\}, \ x \in I_i, \end{cases} \tag{1}$$

where $y_i(x,t)$ is the amplitude of the water wave on the edge e_i at position $x \in I_i$ at time t, $h_i = h_i(t)$ is the control on the edge e_i ($i \in \{2, \cdots, N+1\}$) belonging to $L^2(0,T)$ and α_i and β_i ($i \in \{2, \cdots, N+1\}$) are positive constants. The initial data y_{i0} are supposed to be L^2 functions of the space variable.

It is worth to mention that the transmission conditions at the central node 0 are inspired by the recent papers [20] and [6]. It is not the only possible choice, and the main motivation is that they guarantee uniqueness of the regular solutions of the KdV equation linearized around 0 (see [6, 26]). A characterization of boundary conditions that imply a well-posedness dynamics for the linear Airy-type evolution equation ($u_t = \alpha u_{xxx} + \beta u_x$, where $\alpha \in \mathbb{R}^*$, $\beta \in \mathbb{R}$) on star graphs of half-lines are given in [20].

Let us introduce some notations. First, for any function $f : \mathcal{R} \to \mathbb{R}$ we set

$$f_i = f\,|_{e_i} \ \text{ the restriction of } f \text{ to the edge } e_i.$$

In the sequel, we shall use the following notations :

$$L^2(\mathcal{R}) = \left\{f : \mathcal{R} \to \mathbb{R},\ f_i \in L^2(I_i),\ \forall i \in \{1, ..., N+1\}\right\},$$
$$H_0^1(\mathcal{R}) = \left\{f : \mathcal{R} \to \mathbb{R},\ f_i \in H^1(I_i),\ \forall i \in \{1, ..., N+1\},\right.$$
$$\left. f_1(-l_1) = f_i(l_i) = 0,\ f_1(0) = \alpha_i f_i(0), \forall i \in \{2, ..., N+1\}\right\}.$$

For shortness, for $f \in L^1(\mathcal{R}) = \left\{f : \mathcal{R} \to \mathbb{R},\ f_i \in L^1(I_i),\ \forall i \in \{1, ..., N+1\}\right\}$ we often write,

$$\int_{\mathcal{R}} f dx = \int_{-l_1}^{0} f_1(x)dx + \sum_{i=2}^{N+1} \int_0^{l_i} f_i(x)dx.$$

Then the inner products and the norms of the Hilbert spaces $L^2(\mathcal{R})$ and $H_0^1(\mathcal{R})$ are defined by

$$\|f\|^2_{L^2(\mathcal{R})} = \int_{\mathcal{R}} |f|^2 dx \text{ and } \langle f, g\rangle_{L^2(\mathcal{R})} = \int_{\mathcal{R}} f(x)g(x)dx,$$
$$\|f\|^2_{H_0^1(\mathcal{R})} = \int_{\mathcal{R}} |f_x|^2 dx \text{ and } \langle f, g\rangle_{H_0^1(\mathcal{R})} = \int_{\mathcal{R}} f_x(x)g_x(x)dx.$$

The main goal of this paper is to study the controllability of the nonlinear KdV equation on the tree shaped network of $N+1$ edges with N controls. The controllability problem can be stated as following. For any $T > 0$, $l_i > 0$, $y_0 \in L^2(\mathcal{R})$ and $y_T \in L^2(\mathcal{R})$, is it possible to find N Neumann boundary controls $h_i \in L^2(0, T)$ such that the solution y to (1) on the tree shaped network of $N+1$ edges satisfies $y(\cdot, 0) = y_0$ and $y(\cdot, T) = y_T$?

The main result of this paper gives a positive answer if the time of control is large enough and the lengths of the edges are small enough.

Theorem 1.1. *Let $l_i > 0$ satisfying*

$$L := \max_{i=1,\cdots,N+1} l_i < \sqrt{3}\pi \left(\frac{\min(1, \frac{\alpha_i}{N\beta_i})}{\max(1, \frac{\alpha_i}{N\beta_i})}\right)^{1/2} \frac{1}{\sqrt{2\pi^2\left(1 - \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2}\right) + 1}} \tag{2}$$

and assume that

$$\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} \leq 1 \qquad \textit{and} \qquad \sum_{i=2}^{N+1} \beta_i^2 = 1. \tag{3}$$

There exists a positive constant $T_{\min}$ such that the system (1) is locally exactly controllable in any time $T > T_{\min}$. More precisely, there exists $r > 0$ sufficiently small such that for any states $y_0 \in L^2(\mathcal{R})$ and $y_T \in L^2(\mathcal{R})$ with

$$\|y_0\|_{L^2(\mathcal{R})} < r \quad \textit{and} \quad \|y_T\|_{L^2(\mathcal{R})} < r,$$

there exist N Neumann boundary controls $h_i \in L^2(0, T)$ such that the solution y to (1) on the tree shaped network of $N+1$ edges satisfies $y(\cdot, 0) = y_0$ and $y(\cdot, T) = y_T$ for $T > T_{\min}$.

Remark 1. A same type of result can be obtained for a general tree with $N+1$ external vertices, we get the controllability result with only N Neumann controls. For a sake of clarity on the notations, we choose to write our result for a simplified tree with only one internal vertex.

Remark 2. An open problem is whether it is possible to reduce the number of controls at the external vertices and still having a control result. What is clear is that if we make zero one of our controls h_i, then in that branch i the exact controllability does not hold anymore. It is known that the single KdV equation controlled from the left (here, through the couplings) is null controllable only [17]. Regarding other similar systems, we can mention the well known result of controllability of the wave equation on a network where we can reduce the number of control if the ratio of the lengths is not rational (see for instance [14]). In our case the conclusion is not easy to obtain.

In order to prove Theorem 1.1 we prove first the exact controllability result of the KdV equation linearized around 0. Our proof is based on an observability inequality for the linear backward adjoint system obtained by a multiplier approach. We recall that the KdV equation linearized around 0 writes

$$\begin{cases} y_{i,t}(x,t) + y_{i,x}(x,t) + y_{i,xxx}(x,t) = 0, \ \forall i \in \{1, \cdots, N+1\}, \ x \in I_i, \ t > 0, \\ y_1(-l_1, t) = 0, & t > 0, \\ \left.\begin{array}{l} y_i(l_i,t) = 0, \\ y_{i,x}(l_i,t) = h_i(t), \\ y_1(0,t) = \alpha_i y_i(0,t), \end{array}\right\} & \forall i \in \{2, \cdots, N+1\}, \ t > 0, \\ y_{1,x}(0,t) = \displaystyle\sum_{i=2}^{N+1} \beta_i y_{i,x}(0,t), & t > 0, \\ y_{1,xx}(0,t) = \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i} y_{i,xx}(0,t), & t > 0, \\ y(x,0) = y_0(x), & x \in \mathcal{R}. \end{cases} \tag{4}$$

We then get the local exact controllability result of the nonlinear KdV equation applying a fixed point argument. The drawback of this method is that we do not obtain sharp conditions on the lengths l_i and on the time of control $T_{\min}$. However, we get an explicit constant of observability.

The paper is organized as follows. Section 2 is devoted to the necessary preliminary step dealing with the well-posedness and regularity of the solutions of the linear and nonlinear KdV equation. Section 3 will develop the proof of the local controllability result stated in Theorem 1.1 with a first step concerning the linearized KdV equation and a second step dealing with the original nonlinear system.

2. **Well-posedness and regularity results.**

In this section, we follow [23] (see also [2, 9, 11]). We first study the homogeneous linear system (without control), then the linear KdV equation with regular initial data and controls, and by density and the multiplier method, with less regularity on the data. Secondly, we consider the case of the linear system with a source term in order to pass to the nonlinear KdV equation by a fixed point argument.

2.1. **Study of the linear equation.**

We begin by proving the well-posedness of the linear KdV equation (4) with $h_i = 0$ for any $i \in \{2, \cdots, N+1\}$. We consider the operator A defined by

$$A : y = (y_1, \cdots, y_{N+1}) \in \mathcal{D}(A) \subset L^2(\mathcal{R}) \mapsto -y_x - y_{xxx} \in L^2(\mathcal{R}),$$

with domain

$$\mathcal{D}(A) = \left\{ y \in \left(\prod_{i=1}^{N+1} H^3(I_i) \right) \cap V \,\middle|\, y_{1,xx}(0) = \sum_{i=2}^{N+1} \frac{1}{\alpha_i}\, y_{i,xx}(0) \right\},$$

where

$$V = \Big\{ y \in \prod_{i=1}^{N+1} H^2(I_i) \,|\, y_1(-l_1) = y_i(l_i) = y_{i,x}(l_i) = 0 \quad (i \in \{2, \cdots, N+1\}) \,,$$
$$y_1(0) = \alpha_i y_i(0) \quad (i \in \{2, \cdots, N+1\}),\; y_{1,x}(0) = \sum_{i=2}^{N+1} \beta_i y_{i,x}(0) \Big\}.$$

Then we can rewrite the homogeneous linear KdV equation (4) with $h_i = 0$ for any $i \in \{2, \cdots, N+1\}$ as

$$\begin{cases} y_t(t) = Ay(t), \quad t > 0, \\ y(0) = y_0 \in L^2(\mathcal{R}). \end{cases} \tag{5}$$

It is not difficult to show that the adjoint of A, denoted by A^*, is defined by

$$A^* : z = (z_1, \cdots, z_{N+1}) \in \mathcal{D}(A^*) \subset L^2(\mathcal{R}) \mapsto z_x + z_{xxx} \in L^2(\mathcal{R}),$$

with domain

$$\mathcal{D}(A^*) =$$
$$\left\{ z \in \left(\prod_{i=2}^{N+1} H^3(I_i) \right) \cap \widetilde{V} \,\middle|\, z_{1,xx}(0) = \sum_{i=2}^{N+1} \frac{1}{\alpha_i}\, z_{i,xx}(0) + \left(\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} - 1 \right) z_1(0) \right\},$$

where

$$\widetilde{V} = \Big\{ z \in \prod_{i=2}^{N+1} H^2(I_i) \,|\, z_1(-l_1) = z_i(l_i) = z_{1,x}(-l_1) = 0 \,,$$
$$z_1(0) = \alpha_i z_i(0),\; z_{1,x}(0) = \frac{1}{\beta_i}\, z_{i,x}(0), i \in \{2, \cdots, N+1\} \Big\}.$$

Proposition 1. *Assuming that*

$$\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} \leq 1 \qquad \text{and} \qquad \sum_{i=2}^{N+1} \beta_i^2 \leq 1, \tag{6}$$

the operators A and A^* are dissipative.

Proof. We first prove that the operator A is dissipative. Let $y = (y_1, \cdots, y_{N+1}) \in \mathcal{D}(A)$. Then we have with Cauchy-Schwarz inequality

$$\begin{aligned}
&\langle Ay, y \rangle_{L^2(\mathcal{R})} = -\int_{\mathcal{R}} (y_x + y_{xxx}) y \, dx \\
&= \frac{1}{2} \left(\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} - 1 \right) y_1^2(0) - y_1(0) \left(y_{1,xx}(0) - \sum_{i=2}^{N+1} \frac{1}{\alpha_i}\, y_{i,xx}(0) \right) - \frac{1}{2} y_{1,x}^2(-l_1) \\
&\quad + \frac{1}{2} \left(\sum_{i=2}^{N+1} \beta_i y_{i,x}(0) \right)^2 - \frac{1}{2} \sum_{i=2}^{N+1} y_{i,x}^2(0) \\
&\leq \frac{1}{2} \left(\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} - 1 \right) y_1^2(0) - \frac{1}{2} y_{1,x}^2(-l_1) + \frac{1}{2} \left(\sum_{i=2}^{N+1} \beta_i^2 - 1 \right) \sum_{i=2}^{N+1} y_{i,x}^2(0).
\end{aligned}$$

If we take α_i and β_i such that (6) holds, then $\langle Ay, y\rangle_{L^2(\mathcal{R})} \leq 0$, which means that the operator A is dissipative.

We now prove that the adjoint operator A^* is also dissipative.
Let $z = (z_1, \cdots, z_{N+1}) \in \mathcal{D}(A^*)$. Then we have

$$\begin{aligned}
\langle A^* z, z\rangle_{L^2(\mathcal{R})} &= \int_{\mathcal{R}} (z_x + z_{xxx}) z \, dx \\
&= \frac{1}{2}\left(1 - \sum_{i=2}^{N+1} \frac{1}{\alpha_i^2}\right) z_1^2(0) + z_1(0)\left(z_{1,xx}(0) - \sum_{i=2}^{N+1} \frac{1}{\alpha_i} z_{i,xx}(0)\right) - \frac{1}{2}\sum_{i=2}^{N+1} z_{i,x}^2(l_i) \\
&\quad + \frac{1}{2}\left(\sum_{i=2}^{N+1} \beta_i^2 - 1\right) z_{1,x}^2(0) \\
&= \frac{1}{2}\left(\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} - 1\right) z_1^2(0) - \frac{1}{2}\sum_{i=2}^{N+1} z_{i,x}^2(l_i) + \frac{1}{2}\left(\sum_{i=2}^{N+1} \beta_i^2 - 1\right) z_{1,x}^2(0).
\end{aligned}$$

If we take α_i and β_i such that (6) holds, then $\langle A^* z, z\rangle_{L^2(\mathcal{R})} \leq 0$, which means that the operator A^* is dissipative. $\square$

Consequently, A generates a strongly continuous semigroup of contractions S on $L^2(\mathcal{R})$ (see [22]). We denote by $\{S(t), t \geq 0\}$ the semigroup of contractions associated with A. For any $y_0 \in L^2(\mathcal{R})$ there exists a unique mild solution $y = S(.)y_0 \in C([0,T], L^2(\mathcal{R}))$ of (5). Moreover, if $y_0 \in \mathcal{D}(A)$, then the solution of (5) is classical and satisfies $y \in C([0,T], \mathcal{D}(A)) \cap C^1([0,T], L^2(\mathcal{R}))$.

We now prove the well-posedness result for the linear equation (4) with regular initial data and controls. More precisely, we assume that the N boundary controls h_i belong to $C_0^2([0,T])$ for any $i \in \{2, \cdots, N+1\}$ where $C_0^2([0,T]) = \{h \in C^2([0,T]), h(0) = 0\}$.

Proposition 2. *Assume that* (6) *holds. Let $y_0 \in \mathcal{D}(A)$ and $h_i \in C_0^2([0,T])$ for any $i \in \{2, \cdots, N+1\}$. Then there exists a unique solution $y \in C([0,T], \mathcal{D}(A)) \cap C^1([0,T], L^2(\mathcal{R}))$ of* (4).

Proof. Let $y_0 \in \mathcal{D}(A)$ and $h_i \in C_0^2([0,T])$ for any $i \in \{2, \cdots, N+1\}$. We first take $N+1$ functions $\phi_i \in C^2([0,T], C^\infty(I_i))$ $(i \in \{1, \cdots, N+1\})$ satisfying

$$\begin{cases}
\phi_1(-l_1, t) = 0, & t > 0, \\
\left.\begin{array}{l} \phi_i(l_i, t) = 0, \\ \phi_{i,x}(l_i, t) = h_i(t), \\ \phi_1(0,t) = \alpha_i \phi_i(0,t), \end{array}\right\} \forall i \in \{2, \cdots, N+1\}, & t > 0, \\
\phi_{1,x}(0,t) = \displaystyle\sum_{i=2}^{N+1} \beta_i \phi_{i,x}(0,t), & t > 0, \\
\phi_{1,xx}(0,t) = \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i} \phi_{i,xx}(0,t), & t > 0.
\end{cases}$$

This choice is possible by taking, for instance, for all $i \in \{2, \cdots, N+1\}$, the functions $\phi_i(x,t) = \frac{-x(l_i - x)}{l_i} h_i(t)$.

Moreover, we can define the function $\phi_1(x,t) = a(t)x^3 + b(t)x^2 + c(t)x$ with

$$c(t) = -\sum_{i=2}^{N+1} \beta_i h_i(t), \quad b(t) = \sum_{i=2}^{N+1} \frac{1}{\alpha_i l_i} h_i(t), \quad \text{and} \quad a(t) = \frac{b(t) l_1 - c(t)}{l_1^2}.$$

We now define $z = y - \phi$, which satisfies

$$\begin{cases} z_t(x,t) = (-z_x - z_{xxx} + g)(x,t), & x \in \mathcal{R},\ t > 0, \\ z_1(-l_1,t) = 0, & t > 0, \\ \left.\begin{array}{l} z_i(l_i,t) = 0, \\ z_{i,x}(l_i,t) = 0, \\ z_1(0,t) = \alpha_i z_i(0,t), \end{array}\right\} \forall i \in \{2,\cdots,N+1\}, & t > 0, \\ z_{1,x}(0,t) = \displaystyle\sum_{i=2}^{N+1} \beta_i z_{i,x}(0,t), & t > 0, \\ z_{1,xx}(0,t) = \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i} z_{i,xx}(0,t), & t > 0, \\ z(x,0) = y_0(x), & x \in \mathcal{R}, \end{cases} \tag{7}$$

where $g(x,t) = -\phi_t(x,t) - \phi_x(x,t) - \phi_{xxx}(x,t) \in C^1([0,T], L^2(\mathcal{R}))$. We deduce from classical results on semigroup theory (see [22]) and from the fact that A generates a strongly continuous semigroup of contractions on $L^2(\mathcal{R})$ that there exists a unique classical solution $z \in C([0,T], \mathcal{D}(A)) \cap C^1([0,T], L^2(\mathcal{R}))$ of (7). Consequently, there exists a unique solution $y \in C([0,T], \mathcal{D}(A)) \cap C^1([0,T], L^2(\mathcal{R}))$ of (4). $\square$

We now study the same system but with less regularity on the data, using a density argument and the multiplier method.

Proposition 3. *Assume that (6) holds. Let $y_0 \in L^2(\mathcal{R})$ and $h_i \in L^2(0,T)$ for any $i \in \{2,\cdots,N+1\}$. Then, there exists a unique solution $y \in C([0,T], L^2(\mathcal{R})) \cap L^2(0,T,H_0^1(\mathcal{R}))$ of (4). Moreover $y_{1,x}(-l_1,\cdot) \in L^2(0,T)$ and there exists $C > 0$ such that the following estimates hold:*

$$\|y\|^2_{C([0,T],L^2(\mathcal{R}))} + \|y\|^2_{L^2(0,T,H_0^1(\mathcal{R}))} \le C \left(\|y_0\|^2_{L^2(\mathcal{R})} + \sum_{i=2}^{N+1} \|h_i\|^2_{L^2(0,T)} \right), \tag{8}$$

$$\|y_{1,x}(-l_1,\cdot)\|^2_{L^2(0,T)} \le \|y_0\|^2_{L^2(\mathcal{R})} + \sum_{i=2}^{N+1} \|h_i\|^2_{L^2(0,T)}. \tag{9}$$

Proof. We first assume $y_0 \in \mathcal{D}(A)$ and $h_i \in C_0^2([0,T])$ for $i \in \{2,\cdots,N+1\}$. By Proposition 2, there exists a unique solution $y \in C([0,T], \mathcal{D}(A)) \cap C^1([0,T], L^2(\mathcal{R}))$ of (4). Let $s \in [0,T]$ and $q \in C^\infty(\mathcal{R} \times [0,s])$. By multiplying $y_t + y_x + y_{xxx} = 0$ by

qy and integrating by parts on $\mathcal{R} \times [0,s]$, we get after some computations,

$$\begin{aligned}
&\int_{\mathcal{R}} q(x,s)y^2(x,s)dx - \int_{\mathcal{R}} q(x,0)y^2(x,0)dx - \int_0^s \int_{\mathcal{R}} (q_t + q_x + q_{xxx})y^2 dxdt \\
&+ 3\int_0^s \int_{\mathcal{R}} q_x y_x^2 dxdt = \sum_{i=2}^{N+1} \int_0^s q_i(l_i,t)h_i^2(t)dt - \int_0^s q_1(-l_1,t)y_{1,x}^2(-l_1,t)dt \\
&- \int_0^s q_1(0,t)y_1^2(0,t)dt + \sum_{i=2}^{N+1} \int_0^s q_i(0,t)y_i^2(0,t)dt \\
&- \int_0^s q_{1,xx}(0,t)y_1^2(0,t)dt + \sum_{i=2}^{N+1} \int_0^s q_{i,xx}(0,t)y_i^2(0,t)dt \\
&+ 2\int_0^s q_{1,x}(0,t)y_1(0,t)y_{1,x}(0,t)dt - 2\sum_{i=2}^{N+1} \int_0^s q_{i,x}(0,t)y_i(0,t)y_{i,x}(0,t)dt \\
&+ \int_0^s q_1(0,t)y_{1,x}^2(0,t)dt - \sum_{i=2}^{N+1} \int_0^s q_i(0,t)y_{i,x}^2(0,t)dt \\
&- 2\int_0^s q_1(0,t)y_1(0,t)y_{1,xx}(0,t)dt + 2\sum_{i=2}^{N+1} \int_0^s q_i(0,t)y_i(0,t)y_{i,xx}(0,t)dt. \qquad (10)
\end{aligned}$$

• Taking now $q(x,t) = 1$ in (10), we obtain

$$\begin{aligned}
&\int_{\mathcal{R}} y^2(x,s)dx - \int_{\mathcal{R}} y_0^2(x)dx = \sum_{i=2}^{N+1} \int_0^s h_i^2(t)dt - \int_0^s y_{1,x}^2(-l_1,t)dt \\
&- \int_0^s y_1^2(0,t)dt + \sum_{i=2}^{N+1} \int_0^s y_i^2(0,t)dt + \int_0^s y_{1,x}^2(0,t)dt \\
&- \sum_{i=2}^{N+1} \int_0^s y_{i,x}^2(0,t)dt - 2\int_0^s y_1(0,t)y_{1,xx}(0,t)dt + 2\sum_{i=2}^{N+1} \int_0^s y_i(0,t)y_{i,xx}(0,t)dt.
\end{aligned}$$

Using the boundary condition of (4) at the internal node 0, we have

$$\begin{aligned}
&\int_{\mathcal{R}} y^2(x,s)dx + \int_0^s y_{1,x}^2(-l_1,t)dt = \int_{\mathcal{R}} y_0^2(x)dx + \sum_{i=2}^{N+1} \int_0^s h_i^2(t)dt \\
&+ \left(\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} - 1\right) \int_0^s y_1^2(0,t)dt + \int_0^s \left(\sum_{i=2}^{N+1} \beta_i y_{i,x}(0,t)\right)^2 dt - \sum_{i=2}^{N+1} \int_0^s y_{i,x}^2(0,t)dt,
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{\mathcal{R}} y^2(x,s)dx + \int_0^s y_{1,x}^2(-l_1,t)dt \le \int_{\mathcal{R}} y_0^2(x)dx + \sum_{i=2}^{N+1} \int_0^s h_i^2(t)dt \\
&+ \left(\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} - 1\right) \int_0^s y_1^2(0,t)dt + \left(\sum_{i=2}^{N+1} \beta_i^2 - 1\right) \sum_{i=2}^{N+1} \int_0^s y_{i,x}^2(0,t)dt.
\end{aligned}$$

Using (6), we obtain

$$\max_{s\in[0,T]}\int_{\mathcal{R}} y^2(x,s)dx \leq \int_{\mathcal{R}} y_0^2(x)dx + \sum_{i=2}^{N+1}\int_0^T h_i^2(t)dt \tag{11}$$

and

$$\int_0^T y_{1,x}^2(-l_1,t)dt \leq \int_{\mathcal{R}} y_0^2(x)dx + \sum_{i=2}^{N+1}\int_0^T h_i^2(t)dt. \tag{12}$$

Note that (11) and (12) mean that $y \in C([0,T],L^2(\mathcal{R}))$ and $y_x(-l_1,\cdot) \in L^2(0,T)$ (it is a hidden regularity property) provided that $y_0 \in L^2(\mathcal{R})$ and $h_i \in L^2(0,T)$ for any $i \in \{2,\cdots,N+1\}$. Moreover, (11) implies that

$$\int_0^T\int_{\mathcal{R}} y^2(x,t)dxdt \leq T\left(\int_{\mathcal{R}} y_0^2(x)dx + \sum_{i=2}^{N+1}\int_0^T h_i^2(t)dt\right). \tag{13}$$

• Picking $s=T$, $q_1(x,t)=x$ and $q_i(x,t)=\alpha_i\beta_i x$, for $i=2,\dots N+1$ in (10), we obtain

$$\begin{aligned}\int_{\mathcal{R}} q(x,T)y^2(x,T)dx - \int_{\mathcal{R}} q(x,0)y_0^2(x)dx - \int_0^T\int_{\mathcal{R}} q_x y^2 dxdt + 3\int_0^T\int_{\mathcal{R}} q_x y_x^2 dxdt \\ = \sum_{i=2}^{N+1}\alpha_i\beta_i l_i\int_0^T h_i^2(t)dt + l_1\int_0^T y_{1,x}^2(-l_1,t)dt + 2\int_0^T y_1(0,t)y_{1,x}(0,t)dt \\ - 2\sum_{i=2}^{N+1}\int_0^T \alpha_i\beta_i y_i(0,t)y_{i,x}(0,t)dt.\end{aligned}$$

Using again the boundary condition of (4) at the central node 0, we have

$$\begin{aligned}3\int_0^T\int_{\mathcal{R}} q_x y_x^2 dxdt = \int_{\mathcal{R}} q(x,0)y_0^2(x)dx - \int_{\mathcal{R}} q(x,T)y^2(x,T)dx + \int_0^T\int_{\mathcal{R}} q_x y^2 dxdt \\ + \sum_{i=2}^{N+1}\int_0^T \alpha_i\beta_i l_i h_i^2(t)dt + \int_0^T l_1 y_{1,x}^2(-l_1,t)dt.\end{aligned} \tag{14}$$

We then deduce from (12), (13) and (14) that

$$\int_0^T\int_{\mathcal{R}} y_x^2 dxdt \leq \frac{(T+3L)\max\limits_{i=2,\dots,N+1}(1,\alpha_i\beta_i)}{3\min\limits_{i=2,\dots,N+1}(1,\alpha_i\beta_i)}\left(\int_{\mathcal{R}} y_0^2(x)dx + \sum_{i=2}^{N+1}\int_0^T h_i^2(t)dt\right), \tag{15}$$

which means that $y \in L^2(0,T,H_0^1(\mathcal{R}))$ provided that $y_0 \in L^2(\mathcal{R})$ and $h_i \in L^2(0,T)$ for any $i \in \{2,\cdots,N+1\}$.

Consequently, by density of $\mathcal{D}(A)$ in $L^2(\mathcal{R})$ and of $C_0^2([0,T])$ in $L^2(0,T)$ and using (11) and (15), we can extend the notion of solution for less regular data $y_0 \in L^2(\mathcal{R})$ and $h_i \in L^2(0,T)$ for any $i \in \{2,\cdots,N+1\}$ and we obtain a solution in the space $C([0,T],L^2(\mathcal{R})) \cap L^2(0,T,H_0^1(\mathcal{R}))$ and the estimates (8) and (9). □

2.2. **KdV linear equation with a source term.** In order to prove the well-posedness result for the nonlinear KdV equation (1), we use a well-posedness and regularity result for the linear KdV equation with a source term:

$$\begin{cases} (y_{i,t}+y_{i,x}+y_{i,xxx})(x,t)=f_i(x,t), \ \forall i\in\{1,\cdots,N+1\}, \ x\in I_i, \ t>0, \\ y_1(-l_1,t)=0, \hfill t>0, \\ \left.\begin{array}{l} y_i(l_i,t)=0, \\ y_{i,x}(l_i,t)=h_i(t), \\ y_1(0,t)=\alpha_i y_i(0,t), \end{array}\right\} \hfill \forall i\in\{2,\cdots,N+1\}, \ t>0, \\ y_{1,x}(0,t)=\displaystyle\sum_{i=2}^{N+1}\beta_i y_{i,x}(0,t), \hfill t>0, \\ y_{1,xx}(0,t)=\displaystyle\sum_{i=2}^{N+1}\frac{1}{\alpha_i}\, y_{i,xx}(0,t), \hfill t>0, \\ y(x,0)=y_0(x), \hfill x\in\mathcal{R}, \end{cases} \tag{16}$$

where $f=(f_1,f_2,\cdots,f_{N+1})\in L^1(0,T,L^2(\mathcal{R}))$, $y_0\in L^2(\mathcal{R})$ and $h_i\in L^2(0,T)$ for any $i\in\{2,\cdots,N+1\}$.

Proposition 4. *Assume that (6) holds. Let $y_0\in L^2(\mathcal{R})$, $f=(f_1,\cdots,f_{N+1})\in L^1(0,T,L^2(\mathcal{R}))$ and $h_i\in L^2(0,T)$ for any $i\in\{2,\cdots,N+1\}$. Then, there exists a unique solution $y\in C([0,T],L^2(\mathcal{R}))\cap L^2(0,T,H_0^1(\mathcal{R}))$ of (16). Moreover, there exists $C>0$ such that the following estimate holds:*

$$\|y\|^2_{C([0,T],L^2(\mathcal{R}))}+\|y\|^2_{L^2(0,T,H_0^1(\mathcal{R}))} \leq C\left(\|y_0\|^2_{L^2(\mathcal{R})}+\|f\|^2_{L^1(0,T,L^2(\mathcal{R}))}+\sum_{i=2}^{N+1}\|h_i\|^2_{L^2(0,T)}\right). \tag{17}$$

Proof. Using Proposition 3, it suffices to consider the case $y_0=0$ and $h_i=0$ for any $i\in\{2,\cdots,N+1\}$. Since A generates a strongly continuous semigroup of contractions on $L^2(\mathcal{R})$, if $f\in L^1(0,T,L^2(\mathcal{R}))$, there exists a unique mild solution $y\in C([0,T],L^2(\mathcal{R}))$ (see [22]) given by the Duhamel's formula

$$y(t)=S(t)y_0+\int_0^t S(t-s)f(s)ds, \qquad t>0,$$

and there exists $C>0$ such that

$$\|y\|^2_{C([0,T],L^2(\mathcal{R}))}\leq C\,\|f\|^2_{L^1(0,T,L^2(\mathcal{R}))}.$$

It remains to prove that $y\in L^2(0,T,H_0^1(\mathcal{R}))$ and that

$$\|y\|^2_{L^2(0,T,H_0^1(\mathcal{R}))}\leq C\,\|f\|^2_{L^1(0,T,L^2(\mathcal{R}))}.$$

To prove this we follow exactly the steps of the proof of Proposition 3 paying attention to the fact that the right hand side terms are not homogeneous anymore but involve the source f. □

2.3. **Well-posedness result of the nonlinear equation.** We endow the space $\mathcal{B}=C([0,T],L^2(\mathcal{R}))\cap L^2(0,T,H_0^1(\mathcal{R}))$ with the norm

$$\|y\|_{\mathcal{B}}=\max_{t\in[0,T]}\|y(.,t)\|_{L^2(\mathcal{R})}+\left(\int_0^T\|y(.,t)\|^2_{H_0^1(\mathcal{R})}\,dt\right)^{1/2}.$$

To prove the well-posedness result of the nonlinear system (1), we follow [12] (see also [9]). The first step is to show that the nonlinear term yy_x can be considered as a source term of the linear equation (16).

Proposition 5. *Let $T > 0$, $l > 0$ and $y \in L^2(0,T,H^1(0,l)) := L^2(H^1)$. Then, $yy_x \in L^1(0,T,L^2(0,l))$ and the map*

$$y \in L^2(H^1) \mapsto yy_x \in L^1(0,T,L^2(0,l))$$

is continuous. In particular, there exists $K > 0$ such that, for any $y, \tilde{y} \in L^2(H^1)$, we have

$$\int_0^T \|yy_x - \tilde{y}\tilde{y}_x\|_{L^2(0,l)} \leq K \left(\|y\|_{L^2(H^1)} + \|\tilde{y}\|_{L^2(H^1)} \right) \|y - \tilde{y}\|_{L^2(H^1)} .$$

Proof. The proof can be found in [23] or [9]. □

Let $y_0 \in L^2(\mathcal{R})$ and $h_i \in L^2(0,T)$ (for any $i \in \{2, \cdots, N+1\}$) such that

$$\|y_0\|_{L^2(\mathcal{R})} + \sum_{i=2}^{N+1} \|h_i\|_{L^2(0,T)} \leq r$$

where $r > 0$ is chosen small enough later. Given $y \in \mathcal{B}$, we consider the map $\Phi : \mathcal{B} \to \mathcal{B}$ defined by $\Phi(y) = \tilde{y}$ where $\tilde{y}$ is the solution of

$$\begin{cases} \tilde{y}_t(x,t) + \tilde{y}_x(x,t) + \tilde{y}_{xxx}(x,t) = -y(x,t)y_x(x,t), & x \in \mathcal{R},\ t > 0, \\ \tilde{y}_1(-l_1,t) = 0, & t > 0, \\ \left. \begin{array}{l} \tilde{y}_i(l_i,t) = 0, \\ \tilde{y}_{i,x}(l_i,t) = h_i(t), \\ \tilde{y}_1(0,t) = \alpha_i \tilde{y}_i(0,t), \end{array} \right\} & \forall i \in \{2, \cdots, N+1\},\ t > 0, \\ \tilde{y}_{1,x}(0,t) = \displaystyle\sum_{i=2}^{N+1} \beta_i \tilde{y}_{i,x}(0,t), & t > 0, \\ \tilde{y}_{1,xx}(0,t) = \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i} \tilde{y}_{i,xx}(0,t), & t > 0, \\ \tilde{y}(x,0) = y_0(x), & x \in \mathcal{R}. \end{cases}$$

Clearly $y \in \mathcal{B}$ is a solution of (1) if and only if y is a fixed point of the map Φ. From (17) and Proposition 5, we get

$$\begin{aligned} \|\Phi(y)\|_{\mathcal{B}} = \|\tilde{y}\|_{\mathcal{B}} &\leq C \left(\|y_0\|_{L^2(\mathcal{R})} + \sum_{i=2}^{N+1} \|h_i\|_{L^2(0,T)} + \int_0^T \|yy_x(t)\|_{L^2(\mathcal{R})} dt \right) \\ &\leq C \left(\|y_0\|_{L^2(\mathcal{R})} + \sum_{i=2}^{N+1} \|h_i\|_{L^2(0,T)} + \|y\|^2_{\mathcal{B}} \right). \end{aligned}$$

Moreover, for the same reasons, we have

$$\begin{aligned} \|\Phi(y_1) - \Phi(y_2)\|_{\mathcal{B}} &\leq C \int_0^T \|-y_1 y_{1,x} + y_2 y_{2,x}\|_{L^2(\mathcal{R})} dt \\ &\leq C \left(\|y_1\|_{\mathcal{B}} + \|y_2\|_{\mathcal{B}} \right) \|y_1 - y_2\|_{\mathcal{B}} . \end{aligned}$$

We consider Φ restricted to the closed ball $\overline{B(0,R)} = \{y \in \mathcal{B}, \|y\|_{\mathcal{B}} \leq R\}$ with $R > 0$ to be chosen later. Then $\|\Phi(y)\|_{\mathcal{B}} \leq C\left(r + R^2\right)$ and $\|\Phi(y_1) - \Phi(y_2)\|_{\mathcal{B}} \leq 2CR \|y_1 - y_2\|_{\mathcal{B}}$ so that if we take R and r satisfying

$$R < \frac{1}{2C} \qquad \text{and} \qquad r < \frac{R}{2C},$$

then $\|\Phi(y)\|_{\mathcal{B}} < R$ and $\|\Phi(y_1) - \Phi(y_2)\|_{\mathcal{B}} \leq 2CR\|y_1 - y_2\|_{\mathcal{B}}$, with $2CR < 1$. Then $\Phi(\overline{B(0,R)}) \subset \overline{B(0,R)}$ and $\|\Phi(y_1) - \Phi(y_2)\|_{\mathcal{B}} \leq \tilde{C}\|y_1 - y_2\|_{\mathcal{B}}$, with $\tilde{C} < 1$. Consequently, we can apply the Banach fixed point theorem and the map Φ has a unique fixed point. We have then shown the following proposition.

Proposition 6. *Let $T > 0$, $l_i > 0$ and assume that (6) holds. Then, there exist $r > 0$ and $C > 0$ such that for every $y_0 \in L^2(\mathcal{R})$ and $h_i \in L^2(0,T)$ (for any $i \in \{2, \cdots, N+1\}$) verifying*

$$\|y_0\|_{L^2(\mathcal{R})} + \sum_{i=2}^{N+1} \|h_i\|_{L^2(0,T)} \leq r,$$

there exists a unique $y \in \mathcal{B}$ solution of system (1) which satisfies

$$\|y\|_{\mathcal{B}} \leq C \left(\|y_0\|_{L^2(\mathcal{R})} + \sum_{i=2}^{N+1} \|h_i\|_{L^2(0,T)} \right).$$

3. **Controllability results.** We first prove the exact controllability result of the linear system (4) by using a duality argument and the multiplier method in order to prove the observability inequality. Then, we obtain the local exact controllability result of the nonlinear system (1) by a fixed point theorem.

3.1. **Linear system.** Due to the linearity of the system (4), we can consider the case of a null initial data, i.e. by taking $y_0 = 0$ on $\mathcal{R}$. It can be easily seen that the exact controllability of (4) is equivalent to the surjectivity of the operator

$$\Lambda : (h_2, \cdots, h_{N+1}) \in L^2(0,T)^N \mapsto (y_1(\cdot,T), y_2(\cdot,T), \cdots, y_{N+1}(\cdot,T)) \in L^2(\mathcal{R}),$$

where $y = (y_1, y_2, \cdots, y_{N+1})$ is the solution of (4) when controls $(h_2, \cdots, h_{N+1})$ are chosen.

It is known that the surjectivity of this operator is equivalent to an observability inequality for the adjoint operator of Λ, which is given by

$$\Lambda^* : \varphi_T = (\varphi_1^T, \varphi_2^T, \cdots, \varphi_{N+1}^T) \in L^2(\mathcal{R}) \mapsto (\varphi_{2,x}(l_2,\cdot), \cdots, \varphi_{N+1,x}(l_{N+1},\cdot)) \in L^2(0,T)^N,$$

where $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_{N+1})$ is the solution of the backward adjoint system

$$\begin{cases} \varphi_t(x,t) + \varphi_x(x,t) + \varphi_{xxx}(x,t) = 0, & x \in \mathcal{R},\ t > 0, \\ \varphi_1(-l_1,t) = 0, & t > 0, \\ \varphi_{1,x}(-l_1,t) = 0, & t > 0, \\ \left.\begin{array}{l} \varphi_i(l_i,t) = 0, \\ \varphi_1(0,t) = \alpha_i \varphi_i(0,t), \\ \varphi_{1,x}(0,t) = \dfrac{1}{\beta_i}\, \varphi_{i,x}(0,t), \end{array}\right\} & \forall i \in \{2, \cdots, N+1\},\ t > 0, \\ \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i}\, \varphi_{i,xx}(0,t) = \varphi_{1,xx}(0,t) + \left(1 - \sum_{i=2}^{N+1} \frac{1}{\alpha_i^2}\right) \varphi_1(0,t), & t > 0, \\ \varphi(x,T) = \varphi_T(x), & x \in \mathcal{R}. \end{cases} \tag{18}$$

The first step is to prove an observability inequality for the backward adjoint system (18), stated below and obtained by a multiplier method.

Theorem 3.1. *Let $l_i > 0$ for any $i \in \{1, \cdots, N+1\}$ satisfying (2) and assume that (3) holds. There exists a positive constant $T_{\min}$ such that if $T > T_{\min}$, then we have the following observability inequality*

$$\|\varphi_T\|^2_{\mathcal{R}} \leq C \sum_{i=2}^{N+1} \|\varphi_{i,x}(l_i, t)\|^2_{L^2(0,T)}, \qquad \forall \varphi_T \in L^2(\mathcal{R}), \tag{19}$$

where $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_{N+1})$ is the solution of (18) with final condition $\varphi_T = (\varphi_1^T, \varphi_2^T, \cdots, \varphi_{N+1}^T) \in L^2(\mathcal{R})$ and C is a positive constant.

Proof. Let $s \in [0, T]$ and $q \in C^\infty(\mathcal{R} \times [s, T])$. By multiplying $\varphi_t + \varphi_x + \varphi_{xxx} = 0$ by $q\varphi$ and integrating by parts on $\mathcal{R} \times [s, T]$, we get after some computations

$$\begin{aligned}
\int_{\mathcal{R}} q(x,T)\varphi^2(x,T)dx - \int_{\mathcal{R}} q(x,s)\varphi^2(x,s)dx &= \int_s^T \int_{\mathcal{R}} (q_t + q_x + q_{xxx})\varphi^2 dxdt \\
- 3\int_s^T \int_{\mathcal{R}} q_x \varphi_x^2 dxdt - \int_s^T q_1(0,t)\varphi_1^2(0,t)dt &+ \sum_{i=2}^{N+1} \int_s^T q_i(0,t)\varphi_i^2(0,t)dt \\
- 2\int_s^T q_1(0,t)\varphi_1(0,t)\varphi_{1,xx}(0,t)dt &+ 2\sum_{i=2}^{N+1} \int_s^T q_i(0,t)\varphi_i(0,t)\varphi_{i,xx}(0,t)dt \\
+ 2\int_s^T q_{1,x}(0,t)\varphi_1(0,t)\varphi_{1,x}(0,t)dt &- 2\sum_{i=2}^{N+1} \int_s^T q_{i,x}(0,t)\varphi_i(0,t)\varphi_{i,x}(0,t)dt \\
- \int_s^T q_{1,xx}(0,t)\varphi_1^2(0,t)dt &+ \sum_{i=2}^{N+1} \int_s^T q_{i,xx}(0,t)\varphi_i^2(0,t)dt \\
+ \int_s^T q_1(0,t)\varphi_{1,x}^2(0,t)dt + \sum_{i=2}^{N+1} \int_s^T q_i(l_i,t)\varphi_{i,x}^2(l_i,t)dt &- \sum_{i=2}^{N+1} \int_s^T q_i(0,t)\varphi_{i,x}^2(0,t)dt.
\end{aligned} \tag{20}$$

• Let us first choose $q(x,t) = t$ and $s = 0$ in (20). Then we obtain

$$\begin{aligned}
T\int_{\mathcal{R}} \varphi^2(x,T)dx &= \int_0^T \int_{\mathcal{R}} \varphi^2 dxdt - \int_0^T t\varphi_1^2(0,t)dt + \sum_{i=2}^{N+1} \int_0^T t\varphi_i^2(0,t)dt \\
&- 2\int_0^T t\varphi_1(0,t)\varphi_{1,xx}(0,t)dt + 2\sum_{i=2}^{N+1} \int_0^T t\varphi_i(0,t)\varphi_{i,xx}(0,t)dt \\
&+ \int_0^T t\varphi_{1,x}^2(0,t)dt + \sum_{i=2}^{N+1} \int_0^T t\varphi_{i,x}^2(l_i,t)dt - \sum_{i=2}^{N+1} \int_0^T t\varphi_{i,x}^2(0,t)dt,
\end{aligned}$$

and using the boundary condition of (18) at the internal node 0, we have

$$\begin{aligned}
T\int_{\mathcal{R}} \varphi^2(x,T)dx &= \int_0^T \int_{\mathcal{R}} \varphi^2 dxdt + \sum_{i=2}^{N+1} \int_0^T t\varphi_{i,x}^2(l_i,t)dt \\
&+ \left(1 - \sum_{i=2}^{N+1} \frac{1}{\alpha_i^2}\right) \int_0^T t\varphi_1^2(0,t)dt + \left(1 - \sum_{i=2}^{N+1} \beta_i^2\right) \int_0^T t\varphi_{1,x}^2(0,t)dt.
\end{aligned} \tag{21}$$

By using Poincaré inequality and the estimation of the trace of the function, we have,

$$\int_0^T \int_{\mathcal{R}} \varphi^2 dxdt \leq \frac{L^2}{\pi^2} \int_0^T \int_{\mathcal{R}} \varphi_x^2 dxdt \quad \text{and} \quad \int_0^T \varphi_1^2(0,t)dt \leq L \int_0^T \int_{\mathcal{R}} \varphi_x^2 dxdt.$$

As we can not estimate the trace of $\varphi_x(0,t)$, we need to use the strong hypothesis in (3), i.e. $\sum_{i=2}^{N+1} \beta_i^2 = 1$.

Then, from (21) we get

$$T \int_{\mathcal{R}} \varphi^2(x,T)dx \\ \leq \left(\frac{L^2}{\pi^2} + \left(1 - \sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} \right) TL \right) \int_0^T \int_{\mathcal{R}} \varphi_x^2 dxdt + T \sum_{i=2}^{N+1} \int_0^T \varphi_{i,x}^2(l_i,t)dt. \quad (22)$$

• Taking now $q(x,t) = 1$ and $s = 0$ in (20), we obtain

$$\int_{\mathcal{R}} \varphi^2(x,T)dx - \int_{\mathcal{R}} \varphi^2(x,0)dx = -\int_0^T \varphi_1^2(0,t)dt + \sum_{i=2}^{N+1} \int_0^T \varphi_i^2(0,t)dt \\ - 2\int_0^T \varphi_1(0,t)\varphi_{1,xx}(0,t)dt + 2\sum_{i=2}^{N+1} \int_0^T \varphi_i(0,t)\varphi_{i,xx}(0,t)dt \\ + \int_0^T \varphi_{1,x}^2(0,t)dt + \sum_{i=2}^{N+1} \int_0^T \varphi_{i,x}^2(l_i,t)dt - \sum_{i=2}^{N+1} \int_0^T \varphi_{i,x}^2(0,t)dt.$$

Using again the boundary condition of (18) at the internal node 0, we have

$$\sum_{i=2}^{N+1} \int_0^T \varphi_{i,x}^2(l_i,t)dt = \int_{\mathcal{R}} \varphi^2(x,T)dx - \int_{\mathcal{R}} \varphi^2(x,0)dx \\ + \left(\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} - 1 \right) \int_0^T \varphi_1^2(0,t)dt + \left(\sum_{i=2}^{N+1} \beta_i^2 - 1 \right) \int_0^T \varphi_{1,x}^2(0,t)dt,$$

which implies by (3) that

$$\sum_{i=2}^{N+1} \int_0^T \varphi_{i,x}^2(l_i,t)dt + \int_{\mathcal{R}} \varphi^2(x,0)dx \leq \int_{\mathcal{R}} \varphi_T^2(x)dx. \quad (23)$$

• Picking $s = 0$, $q_1(x,t) = x$ and $q_i(x,t) = \frac{\alpha_i}{N\beta_i} x$ in (20), we obtain

$$\int_{\mathcal{R}} q(x,T)\varphi^2(x,T)dx - \int_{\mathcal{R}} q(x,0)\varphi^2(x,0)dx = \int_0^T \int_{\mathcal{R}} q_x \varphi^2 dxdt \\ - 3\int_0^T \int_{\mathcal{R}} q_x \varphi_x^2 dxdt + 2\int_0^T \varphi_1(0,t)\varphi_{1,x}(0,t)dt \\ - 2\sum_{i=2}^{N+1} \int_0^T \frac{\alpha_i}{N\beta_i} \varphi_i(0,t)\varphi_{i,x}(0,t)dt + \sum_{i=2}^{N+1} \int_0^T \frac{\alpha_i l_i}{N\beta_i} \varphi_{i,x}^2(l_i,t)dt.$$

Using again the boundary condition of (18) at the internal node 0, we have

$$3\int_0^T\int_{\mathcal{R}} q_x\varphi_x^2 dxdt = \int_{\mathcal{R}} q(x,0)\varphi^2(x,0)dx - \int_{\mathcal{R}} q(x,T)\varphi^2(x,T)dx + \int_0^T\int_{\mathcal{R}} q_x\varphi^2 dxdt + \sum_{i=2}^{N+1}\int_0^T \frac{\alpha_i l_i}{N\beta_i}\varphi_{i,x}^2(l_i,t)dt. \quad (24)$$

Then, from (24) we get, with Poincaré inequality and the fact that the operator A^* is dissipative,

$$\left(3\min\left(1,\frac{\alpha_i}{N\beta_i}\right) - \max\left(1,\frac{\alpha_i}{N\beta_i}\right)\frac{L^2}{\pi^2}\right)\int_0^T\int_{\mathcal{R}}\varphi_x^2 dxdt \leq 2\max\left(1,\frac{\alpha_i}{N\beta_i}\right)L\int_{\mathcal{R}}\varphi^2(x,T)dx + \max\left(1,\frac{\alpha_i}{N\beta_i}\right)L\sum_{i=2}^{N+1}\int_0^T\varphi_{i,x}^2(l_i,t)dt. \quad (25)$$

Gathering (22) and (25), we have

$$\left(T - 2\frac{\delta_T}{\Gamma}\right)\int_{\mathcal{R}}\varphi_T^2(x)dx \leq \left(T + \frac{\delta_T}{\Gamma}\right)\sum_{i=2}^{N+1}\int_0^T\varphi_{i,x}^2(l_i,t)dt, \quad (26)$$

where we used the notation

$$\delta_T = \left(\frac{L^2}{\pi^2} + \left(1 - \sum_{i=2}^{N+1}\frac{1}{\alpha_i^2}\right)TL\right)L\max\left(1,\frac{\alpha_i}{N\beta_i}\right)$$

and

$$\Gamma = 3\min\left(1,\frac{\alpha_i}{N\beta_i}\right) - \frac{L^2}{\pi^2}\max\left(1,\frac{\alpha_i}{N\beta_i}\right).$$

Note that $\Gamma > 0$ under the condition

$$L < \sqrt{3}\pi\left(\frac{\min(1,\frac{\alpha_i}{N\beta_i})}{\max(1,\frac{\alpha_i}{N\beta_i})}\right)^{1/2}$$

which is weaker than the hypothesis (2).

In order to have the observability inequality (19) from (26), we have to impose

$$T > \frac{2\delta_T}{\Gamma},$$

that leads us to

$$T > \frac{2\max\left(1,\frac{\alpha_i}{N\beta_i}\right)L^3}{\pi^2\Big[\Gamma - 2\max\left(1,\frac{\alpha_i}{N\beta_i}\right)L^2\left(1 - \sum_{i=2}^{N+1}\frac{1}{\alpha_i^2}\right)\Big]}. \quad (27)$$

This condition on time T makes also appear other condition on L:

$$\Gamma - 2\max\left(1,\frac{\alpha_i}{N\beta_i}\right)L^2\left(1 - \sum_{i=2}^{N+1}\frac{1}{\alpha_i^2}\right) > 0,$$

which is equivalent to

$$L < \sqrt{3}\pi \left(\frac{\min(1, \frac{\alpha_i}{N\beta_i})}{\max(1, \frac{\alpha_i}{N\beta_i})} \right)^{1/2} \frac{1}{\sqrt{2\pi^2 \left(1 - \sum_{i=2}^{N+1} \frac{1}{\alpha_i^2}\right) + 1}}$$

that is exactly hypothesis (2). This finishes the proof of Theorem 3.1 where the existence of time T_{min} is given by condition (27) and the observability constant is

$$C = \frac{T\Gamma + \delta_T}{T\Gamma - 2\delta_T}.$$

□

Remark 3. From previous proof we can deduce that if we have the condition $\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} = 1$, then we have to ask

$$L < \sqrt{3}\pi \left(\frac{\min(1, \frac{\alpha_i}{N\beta_i})}{\max(1, \frac{\alpha_i}{N\beta_i})} \right)^{1/2}.$$

Moreover, if $\alpha_i = \sqrt{N}$ and $\beta_i = \frac{1}{\sqrt{N}}$, then we have to ask $L < \sqrt{3}\pi$.

Once the observability inequality is established as in Theorem 3.1, then the exact controllability result of the linear system (4) is obtained by duality and the Hilbert Uniqueness Method (HUM). Thus, the following is true.

Theorem 3.2. *Let $l_i > 0$ satisfying (2) and assume that (3) holds. There exists a positive constant $T_{\min}$ such that the linear system (4) is exactly controllable in time $T > T_{\min}$. More precisely, for any states $y_0 \in L^2(\mathcal{R})$ and $y_T \in L^2(\mathcal{R})$, there exist N Neumann boundary controls $h_i \in L^2(0,T)$ such that the solution y to (4) on the tree-shaped network of $N+1$ edges satisfies $y(\cdot,0) = y_0$ and $y(\cdot,T) = y_T$.*

3.2. **Nonlinear system.** We now prove the main result of this paper, i.e., Theorem 1.1. We do it by a fixed point argument, following for instance [9].

Proof of Theorem 1.1. Let $y_0 \in L^2(\mathcal{R})$ and $y_T \in L^2(\mathcal{R})$ such that

$$\|y_0\|_{L^2(\mathcal{R})} < r \quad \text{and} \quad \|y_T\|_{L^2(\mathcal{R})} < r,$$

with $r > 0$ chosen later. We consider the map

$$\Psi : z \in \mathcal{B} \mapsto y^1 + y^2 + y^3,$$

where y^1, y^2, y^3 are the solutions of

$$\begin{cases} y^1_t(x,t) + y^1_x(x,t) + y^1_{xxx}(x,t) = 0, & x \in \mathcal{R},\ t>0, \\ y^1_1(-l_1,t) = 0, & t>0, \\ \left.\begin{array}{l} y^1_i(l_i,t) = 0, \\ y^1_{i,x}(l_i,t) = 0, \\ y^1_1(0,t) = \alpha_i y^1_i(0,t), \end{array}\right\} \forall i \in \{2,\cdots,N+1\}, & t>0, \\ y^1_{1,x}(0,t) = \displaystyle\sum_{i=2}^{N+1} \beta_i y^1_{i,x}(0,t), & t>0, \\ y^1_{1,xx}(0,t) = \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i} y^1_{i,xx}(0,t), & t>0, \\ y^1(x,0) = y_0(x), & x \in \mathcal{R}, \end{cases} \tag{28}$$

$$\begin{cases} y^2_t(x,t) + y^2_x(x,t) + y^2_{xxx}(x,t) = -z(x,t)z_x(x,t), & x \in \mathcal{R},\ t>0, \\ y^2_1(-l_1,t) = 0, & t>0, \\ \left.\begin{array}{l} y^2_i(l_i,t) = 0, \\ y^2_{i,x}(l_i,t) = 0, \\ y^2_1(0,t) = \alpha_i y^2_i(0,t), \end{array}\right\} \forall i \in \{2,\cdots,N+1\}, & t>0, \\ y^2_{1,x}(0,t) = \displaystyle\sum_{i=2}^{N+1} \beta_i y^2_{i,x}(0,t), & t>0, \\ y^2_{1,xx}(0,t) = \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i} y^2_{i,xx}(0,t), & t>0, \\ y^2(x,0) = 0, & x \in \mathcal{R}, \end{cases} \tag{29}$$

and

$$\begin{cases} y^3_t(x,t) + y^3_x(x,t) + y^3_{xxx}(x,t) = 0, & x \in \mathcal{R},\ t>0, \\ y^3_1(-l_1,t) = 0, & t>0, \\ \left.\begin{array}{l} y^3_i(l_i,t) = 0, \\ y^3_{i,x}(l_i,t) = h_i(t), \\ y^3_1(0,t) = \alpha_i y^3_i(0,t), \end{array}\right\} \forall i \in \{2,\cdots,N+1\}, & t>0, \\ y^3_{1,x}(0,t) = \displaystyle\sum_{i=2}^{N+1} \beta_i y^3_{i,x}(0,t), & t>0, \\ y^3_{1,xx}(0,t) = \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i} y^3_{i,xx}(0,t), & t>0, \\ y^3(x,0) = 0, & x \in \mathcal{R}. \end{cases} \tag{30}$$

We see that y^1 is the solution of the homogeneous linear equation (4) (without control, without source term but with non null initial data), y^2 is the solution of the linear equation (4) with null initial data, right hand side $f = -zz_x$ and without control, and y^3 is the solution of the linear equation (4) with null initial data, without source term and with N controls h_i. We take the N controls $h_i \in L^2(0,T)$ ($i \in \{2,\cdots,N+1\}$) such that

$$y^3(\cdot,T) = y_T - y^1(\cdot,T) - y^2(\cdot,T).$$

These controls exist thanks to Theorem 3.2, assuming (2) and (3). Note also that the control operator $y_T \mapsto (h_2,\cdots,h_{N+1})$ mapping the final state to the control driving the linear system to that state is continuous.

We will prove that the map Ψ has a fixed point, using the Banach fixed point theorem. To do that, we consider Ψ restricted to the closed ball

$\overline{B(0,R)} = \{y \in \mathcal{B}, \|y\|_{\mathcal{B}} \leq R\}$ with $R > 0$ to be chosen later. To apply the Banach fixed point theorem, it suffices to show that $\Psi(\overline{B(0,R)}) \subset \overline{B(0,R)}$ and for any z, $\tilde{z} \in \mathcal{B}$, $\|\Psi(z) - \Psi(\tilde{z})\|_{\mathcal{B}} \leq C \|z - \tilde{z}\|_{\mathcal{B}}$ with $C < 1$. First, using (17), Proposition 5 and the continuity of the control operator, we have

$$\begin{aligned}
\|\Psi(z)\|_{\mathcal{B}} &\leq \left\|y^1\right\|_{\mathcal{B}} + \left\|y^2\right\|_{\mathcal{B}} + \left\|y^3\right\|_{\mathcal{B}} \\
&\leq C\left(\|y_0\|_{L^2(\mathcal{R})} + \int_0^T \|zz_x(t)\|_{L^2(\mathcal{R})}dt + \sum_{i=2}^{N+1} \|h_i\|_{L^2(0,T)}\right) \\
&\leq C_1 \|y_0\|_{L^2(\mathcal{R})} + C_2 \|y_T\|_{L^2(\mathcal{R})} + C_3 \|z\|_{\mathcal{B}}^2 \\
&\leq (C_1 + C_2)r + C_3 R^2,
\end{aligned}$$

and we get the first condition $(C_1 + C_2)r + C_3 R^2 \leq R$. Second, we have, using the same arguments,

$$\begin{aligned}
\|\Psi(z) - \Psi(\tilde{z})\|_{\mathcal{B}} &\leq \left\|y^2 - \tilde{y}^2\right\|_{\mathcal{B}} + \left\|y^3 - \tilde{y}^3\right\|_{\mathcal{B}} \\
&\leq C\left(\int_0^T \|zz_x(t) - \tilde{z}\tilde{z}_x(t)\|_{L^2(\mathcal{R})}dt + \sum_{i=2}^{N+1} \left\|h_i - \tilde{h}_i\right\|_{L^2(0,T)}\right) \\
&\leq CK\left(\|z\|_{\mathcal{B}} + \|\tilde{z}\|_{\mathcal{B}}\right)\|z - \tilde{z}\|_{\mathcal{B}} + C_4 \left\|y^2(\cdot,T) - \tilde{y}^2(\cdot,T)\right\|_{L^2(\mathcal{R})} \\
&\leq CK\left(\|z\|_{\mathcal{B}} + \|\tilde{z}\|_{\mathcal{B}}\right)\|z - \tilde{z}\|_{\mathcal{B}} + C_4 \left\|y^2 - \tilde{y}^2\right\|_{\mathcal{B}} \\
&\leq C_5\left(\|z\|_{\mathcal{B}} + \|\tilde{z}\|_{\mathcal{B}}\right)\|z - \tilde{z}\|_{\mathcal{B}} \\
&\leq 2C_5 R \|z - \tilde{z}\|_{\mathcal{B}},
\end{aligned}$$

that impose the second condition $2C_5R < 1$. These conditions are satisfied for instance if we choose r and R such that

$$R < \min\left\{\frac{1}{2C_5}, \frac{1}{2C_3}\right\}, \qquad r < \frac{R}{2(C_1 + C_2)},$$

that ends the proof of Theorem 1.1. □

REFERENCES

[1] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials, *J. Math. Pures Appl. (9)*, **96** (2011), 555–590, URL https://doi-org.usm.idm.oclc.org/10.1016/j.matpur.2011.06.005.

[2] K. Ammari and E. Crépeau, Feedback stabilization and boundary controllability of the Korteweg–de Vries equation on a star-shaped network, *SIAM J. Control Optim.*, **56** (2018), 1620–1639, URL https://doi.org/10.1137/17M113959X.

[3] S. Avdonin and M. Tucsnak, Simultaneous controllability in sharp time for two elastic strings, *ESAIM Control Optim. Calc. Var.*, **6** (2001), 259–273, URL https://doi-org.usm.idm.oclc.org/10.1051/cocv:2001110.

[4] L. Baudouin, E. Crépeau and J. Valein, Global Carleman estimate on a network for the wave equation and application to an inverse problem, *Math. Control Relat. Fields*, **1** (2011), 307–330, URL https://doi-org.usm.idm.oclc.org/10.3934/mcrf.2011.1.307.

[5] A. Benabdallah, F. Boyer, M. González-Burgos and G. Olive, Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null controllability in cylindrical domains, *SIAM J. Control Optim.*, **52** (2014), 2970–3001, URL https://doi-org.usm.idm.oclc.org/10.1137/130929680.

[6] M. Cavalcante, The Korteweg–de Vries equation on a metric star graph, *Z. Angew. Math. Phys.*, **69** (2018), Art. 124, 22, URL https://doi.org/10.1007/s00033-018-1018-6.

[7] C. M. Cazacu, L. I. Ignat and A. F. Pazoto, Null-controllability of the linear Kuramoto-Sivashinsky equation on star-shaped trees, *SIAM J. Control Optim.*, **56** (2018), 2921–2958, URL https://doi-org.usm.idm.oclc.org/10.1137/16M1103348.

[8] E. Cerpa, Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain, *SIAM J. Control Optim.*, **46** (2007), 877–899, URL https://doi.org/10.1137/06065369X.

[9] E. Cerpa, Control of a Korteweg-de Vries equation: a tutorial, *Math. Control Relat. Fields*, **4** (2014), 45–99, URL https://doi.org/10.3934/mcrf.2014.4.45.

[10] E. Cerpa and E. Crépeau, Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **26** (2009), 457–475, URL https://doi.org/10.1016/j.anihpc.2007.11.003.

[11] E. Cerpa, E. Crépeau and C. Moreno, On the boundary controllability of the korteweg-de vries equation on a star-shaped network, *IMA Journal of Math. Control and Information*.

[12] J.-M. Coron and E. Crépeau, Exact boundary controllability of a nonlinear KdV equation with critical lengths, *J. Eur. Math. Soc. (JEMS)*, **6** (2004), 367–398, URL http://link.springer.de/cgi/linkref?issn=1435-9855&year=04&volume=6&page=367.

[13] J.-M. Coron, O. Glass and Z. Wang, Exact boundary controllability for 1-D quasilinear hyperbolic systems with a vanishing characteristic speed, *SIAM J. Control Optim.*, **48** (2009/10), 3105–3122, URL https://doi-org.usm.idm.oclc.org/10.1137/090749268.

[14] R. Dáger and E. Zuazua, *Wave propagation, observation and control in* 1*-d flexible multi-structures*, vol. 50 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer-Verlag, Berlin, 2006, URL https://doi-org.usm.idm.oclc.org/10.1007/3-540-37726-3.

[15] B. Dekoninck and S. Nicaise, Control of networks of Euler-Bernoulli beams, *ESAIM Control Optim. Calc. Var.*, **4** (1999), 57–81, URL https://doi-org.usm.idm.oclc.org/10.1051/cocv:1999103.

[16] E. Fernández-Cara, M. González-Burgos and L. de Teresa, Boundary controllability of parabolic coupled equations, *J. Funct. Anal.*, **259** (2010), 1720–1758, URL https://doi-org.usm.idm.oclc.org/10.1016/j.jfa.2010.06.003.

[17] O. Glass and S. Guerrero, Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit, *Asymptot. Anal.*, **60** (2008), 61–100.

[18] G. R. Leugering and E. J. P. G. Schmidt, On exact controllability of networks of nonlinear elastic strings in 3-dimensional space, *Chin. Ann. Math. Ser. B*, **33** (2012), 33–60, URL https://doi-org.usm.idm.oclc.org/10.1007/s11401-011-0693-9.

[19] T. Li, *Controllability and observability for quasilinear hyperbolic systems*, vol. 3 of AIMS Series on Applied Mathematics, American Institute of Mathematical Sciences (AIMS), Springfield, MO; Higher Education Press, Beijing, 2010.

[20] D. Mugnolo, D. Noja and C. Seifert, Airy-type evolution equations on star graphs, *Anal. PDE*, **11** (2018), 1625–1652, URL https://doi.org/10.2140/apde.2018.11.1625.

[21] S. Nicaise, Control and stabilization of 2×2 hyperbolic systems on graphs, *Math. Control Relat. Fields*, **7** (2017), 53–72, URL https://doi-org.usm.idm.oclc.org/10.3934/mcrf.2017004.

[22] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of Applied Mathematical Sciences, Springer-Verlag, New York, 1983, URL https://doi.org/10.1007/978-1-4612-5561-1.

[23] L. Rosier, Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain, *ESAIM Control Optim. Calc. Var.*, **2** (1997), 33–55, URL https://doi.org/10.1051/cocv:1997102.

[24] L. Rosier and B.-Y. Zhang, Control and stabilization of the Korteweg-de Vries equation: recent progresses, *J. Syst. Sci. Complex.*, **22** (2009), 647–682, URL https://doi.org/10.1007/s11424-009-9194-2.

[25] E. J. P. G. Schmidt, On the modelling and exact controllability of networks of vibrating strings, *SIAM J. Control Optim.*, **30** (1992), 229–245, URL https://doi-org.usm.idm.oclc.org/10.1137/0330015.

[26] Z. A. Sobirov, H. Uecker and M. I. Akhmedov, Exact solutions of the Cauchy problem for the linearized KdV equation on metric star graphs, *Uzbek. Mat. Zh.*, 143–154.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: eduardo.cerpa@usm.cl
E-mail address: emmanuelle.crepeau@univ-grenoble-alpes.fr
E-mail address: julie.valein@univ-lorraine.fr