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On the boundary controllability of the Korteweg-de Vries equation on a star-shaped network

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A system of N Korteweg-de Vries equations coupled by the boundary conditions is considered in this paper. The configuration studied here is the one called star-shaped network, where the boundary inputs can act on a central node and on the N external nodes. In the literature, there is a recent result proving the exact controllability of this system by using $(N+1)$ controls. We succeed to remove the input acting on the central node and consequently we obtain the exact controllability with N inputs.

Keywords: Korteweg-de Vries equation, star-shaped network, controllability.

1. Introduction

Since the publication of the pioneer work Korteweg & de Vries (1895), the Korteweg-de Vries (KdV) equation has appeared in different contexts to describe propagation phenomena. Thus, it has attracted the interest of a number of researchers motivated by its applicability and by nice mathematical tools introduced to deal with it.

Concerning the study of the control properties of this equation, the first results are in Russell & Zhang (1993), Russell & Zhang (1993) and Sun (1996), all of them in the periodic domain framework. After that, the important case of bounded domain was considered in Rosier (1997) and in a number of other articles. See Cerpa (2014) and Rosier & Zhang (2009) for a complete bibliographical review.

After studying the control properties of a single KdV equation, it is very natural and physically motivated to consider systems of coupled KdV equations. This case is developed in Micu *et al.* (2009), Cerpa & Pazoto (2011), Capistrano-Filho *et al.* (2016) and Capistrano-Filho *et al.* (2017) where controllability is studied. Let us also mention Araruna *et al.* (2016) where the authors study a dispersive system consisting in a KdV equation coupled to a Schrödinger equation. In all these papers the coupling is given by internal terms but it is not the only interesting case. Indeed, the case of boundary couplings can represent transmission conditions when an equation is posed on a network.

There is a huge literature studying partial differential equations on networks from different viewpoints. Let us mention some references on stability properties for networks: Chitour *et al* (2017) for transport equations; Bastin *et al* (2007) and Suzuki *et al* (2013) for conservation laws; and Valein & Zuazua (2009) and Gugat & Sigalotti (2010) for wave equations.

In the recent paper Ammari & Crépeau (2018), the authors obtained stabilization and controllability results for a KdV system posed on a star-shaped network. They considered a system formed by N KdV equations. Denoting u_j each solution to the j-th equation posed on a bounded interval $(0,l_j)$, they study the system

$$\begin{cases} (\partial_t u_j + \partial_x u_j + u_j\partial_x u_j + \partial_x^3 u_j)(t,x) = 0, & j=1,\cdots,N,\ x\in(0,l_j),\ t>0,\\ u_j(t,0) = u_k(t,0), & j,k=1,\cdots,N,\ t>0,\\ \displaystyle\sum_{j=1}^{N}\partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2 + f_0(t), & t>0,\\ u_j(t,l_j) = 0, \quad \partial_x u_j(t,l_j) = g_j(t), & j=1,\cdots,N,\ t>0,\\ u_j(0,x) = u_j^0(x), & j=1,\cdots,N,\ x\in(0,l_j), \end{cases} \tag{1.1}$$

where $\alpha > N/2$. The state of the system is $(u_1,u_2,\cdots,u_N)$, the initial state is $(u_1^0,u_2^0,\cdots,u_N^0)$, and the boundary controls are $f_0,g_1,\cdots,g_N$. We call f_0 the control on the central node and $(g_1,\cdots,g_N)$ the controls on the external nodes. The main topics under study in Ammari & Crépeau (2018) are the well-posedness and the stabilization of (1.1). At the end they state the controllability results they are able to prove. In particular, under some conditions on the lengths l_j related to some critical phenomena (see for example Rosier (1997)), they say that (1.1) is locally exactly controllable by using the $(N+1)$ controls $g_0,g_1,\cdots,g_N$. Their result is based on an observability inequality proven by using a compactness-uniqueness argument.

In this paper we improve the controllability results in Ammari & Crépeau (2018) in two directions. We prove that system (1.1) is exactly controllable with only N controls $g_1,\cdots,g_N$ and we are able to consider the cases $\alpha \geqslant N/2$ and not only $\alpha > N/2$ as in Ammari & Crépeau (2018). The star-shaped network is represented in Figure 1, with black nodes where we put control and white node where there is no control.

More precisely, our main result is the following.

THEOREM 1.1 Let $(l_j)_{j=1,\cdots,N} \in (0,+\infty)^N$ and $\alpha \geqslant N/2$. There exist $L_0, T_{\min} > 0$ such that if

$$L := \max_{j=1,..,N} l_j < L_0 \quad \text{and} \quad T > T_{\min}, \tag{1.2}$$

then the nonlinear control system (1.1) is locally exactly controllable with $g_0 = 0$.

Our proof uses a multiplier approach in a direct way. That means, we avoid the use of a contradiction argument. A drawback of this method is that we obtain non sharp conditions on the lengths l_j and on the time of control but we get an explicit constant of observability and thus en explicit characterization of the controllability.

This paper is structured as follows. In section 2 we state the well-posedness results we need for our system of N coupled Korteweg-de Vries equations on a finite star-shaped network. Linear and nonlinear cases are included. In section 3 we prove that both linear and nonlinear systems are exactly controllable by using only N external inputs, under some conditions on the lengths of each interval and on the time of control. The linear case is studied in section 3.1 by using a duality approach and proving the desired observability inequality. The result for the nonlinear system is obtained in section 3.2 by applying a fixed-point argument.

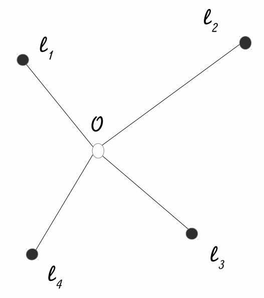

FIG. 1. Star-Shaped Network for $N = 4$

2. Well-posedness framework

In this section we state the regularity framework and the well-posedness results we need in this paper for the linear system

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x^3 u_j)(t,x) = f_j(t,x), & j = 1,\cdots,N,\ x \in (0,l_j),\ t > 0, \\ u_j(t,0) = u_k(t,0), & j,k = 1,\cdots,N,\ t > 0, \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) + f_0(t), & t > 0, \\ u_j(t,l_j) = 0, \quad \partial_x u_j(t,l_j) = g_j(t), & j = 1,\cdots,N,\ t > 0, \\ u_j(0,x) = u_j^0(x), & j = 1,\cdots,N,\ x \in (0,l_j), \end{cases} \tag{2.1}$$

and the nonlinear one

$$\begin{cases} (\partial_t u_j + \partial_x u_j + u_j \partial_x u_j + \partial_x^3 u_j)(t,x) = 0, & j = 1,\cdots,N,\ x \in (0,l_j),\ t > 0, \\ u_j(t,0) = u_k(t,0), & j,k = 1,\cdots,N,\ t > 0, \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2, & t > 0, \\ u_j(t,l_j) = 0, \quad \partial_x u_j(t,l_j) = g_j(t), & j = 1,\cdots,N,\ t > 0, \\ u_j(0,x) = u_j^0(x), & j = 1,\cdots,N,\ x \in (0,l_j). \end{cases} \tag{2.2}$$

REMARK 2.1 It is important to notice that in this paper we are not using f_0 in (2.1) as a control. However, we have to consider it as a source term in well-posedness results for the linear system in order to deal with the boundary nonlinearity in (2.2). The same role is played by the source terms f_j in (2.1) with $j = 1,\cdots,N$.

Let us define the spaces

$$\mathbb{L}^2(\mathcal{T}) = \prod_{j=1}^{N} L^2(0,l_j), \quad L^1(0,T;\mathbb{L}^2(\mathcal{T})) = \prod_{j=1}^{N} L^1(0,T;L^2(0,l_j)),$$

$$L^2(0,T;\mathbb{L}^2(\mathcal{T})) = \prod_{j=1}^{N} L^2(0,T;L^2(0,l_j)),$$

$$H_r^s(0,l_j) = \left\{ v \in H^s(0,l_j) \Big/ \ v^{(i-1)}(l_j) = 0 \text{ for any } 1 \leqslant i \leqslant s \right\},$$

$$\mathbb{H}_e^s(\mathcal{T}) = \left\{ u = (u_1, \cdots, u_N) \in \prod_{j=1}^{N} H_r^s(0,l_j) \Big/ \ u_j(0) = u_k(0), \forall j,k = 1, \cdots, N \right\},$$

and

$$\mathbb{B} := C([0,T],\mathbb{L}^2(\mathcal{T})) \cap L^2(0,T;\mathbb{H}_e^1(\mathcal{T})).$$

We also consider the spatial operator

$$A : D(A) \subset \mathbb{L}^2(\mathcal{T}) \to \mathbb{L}^2(\mathcal{T}),$$

with

$$D(A) = \left\{ u \in \mathbb{H}_e^2(\mathcal{T}) \bigcap \prod_{j=1}^{N} H^3(0,l_j) \Big/ \ \sum_{j=1}^{N} \frac{d^2 u_j}{dx^2}(0) = -\alpha u_1(0) \right\},$$

and defined by

$$Au := A(u_1, \cdots, u_N) = \Big(-\partial_x u_1 - \partial_x^3 u_1, \cdots, -\partial_x u_N - \partial_x^3 u_N \Big).$$

The operator A and its adjoint A^* are easily proven to be dissipative, see for example Proposition 2.1 in Ammari & Crépeau (2018). Therefore, by using semigroups theory, we see that the operator A generates a strongly continuous semigroup of contractions on $\mathbb{L}^2(\mathcal{T})$. Using this and a density argument, the following result is obtained.

THEOREM 2.1 Let $u^0 = (u_1^0, \cdots, u_N^0) \in \mathbb{L}^2(\mathcal{T})$, $g = (g_1, \cdots, g_N) \in L^2(0,T)^N$, $f_0 \in L^2(0,T)$ and $f = (f_1, \cdots, f_N) \in L^1(0,T;\mathbb{L}^2(\mathcal{T}))$. Then, there exists a unique mild solution $u = (u_1, \cdots, u_N) \in \mathbb{B}$ of system (2.1). Furthermore, we obtain the existence of positive constants C_1, C_2, C_3 such that

$$||u||_{\mathbb{B}}^2 \leqslant C_1 \Big(\ ||u^0||_{\mathbb{L}^2(\mathcal{T})}^2 + ||g||_{L^2(0,T)^N}^2 + ||f_0||_{L^2(0,T)}^2 + \|f\|_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))}^2 \Big),$$

$$||\partial_x u_j(\cdot,0)||_{L^2(0,T)}^2 \leqslant C_2 \Big(\ ||u^0||_{\mathbb{L}^2(\mathcal{T})}^2 + ||g||_{L^2(0,T)^N}^2 + ||f_0||_{L^2(0,T)}^2 + \|f\|_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))}^2 \Big),$$

and

$$||u_1(\cdot,0)||_{L^2(0,T)}^2 \leqslant C_3 \Big(\ ||u^0||_{\mathbb{L}^2(\mathcal{T})}^2 + ||g||_{L^2(0,T)^N}^2 + ||f_0||_{L^2(0,T)}^2 + \|f\|_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))}^2 \Big).$$

REMARK 2.2 As we autorize in this work the case $\alpha = \frac{N}{2}$ we can not use directly the result Ammari & Crépeau (2018) [Propositions 2.3] where the condition $\alpha > \frac{N}{2}$ was imposed.

Proof. In a first time, we suppose that $u^0 \in D(A)$, $(g, f_0) \in C_0^2([0,T])^{N+1}$, where $C_0^2([0,T]) := \{\varphi \in C^2([0,T]), \varphi(0) = 0\}$ and $f = 0$. We can prove as in Ammari & Crépeau (2018) [Propositions 2.2] that there exists a unique solution $u \in C([0,T],D(A)) \cap C^1([0,T],\mathbb{L}^2(\mathcal{T}))$ of system (2.1). Let $q = (q_1, \cdots, q_N) \in C^\infty([0,T] \times [0,l_j];\mathbb{R})^N$, such that $q_i(.,0) = q_k(.,0)$. Then by multiplying each first equation of system (2.1) by $q_j u_j$, integrating on $[0,s]\times[0,l_j]$ with $s\in[0,T]$ and using some integrations by parts, we obtain the following equation,

$$\begin{aligned}
&\sum_{j=1}^{N}\int_0^{l_j}|u_j(s,x)|^2 q_j(s,x)dx-\int_0^s\sum_{j=1}^{N}|u_j(t,0)|^2\partial_x^2 q_j(t,0)dt\\
&\qquad+(2\alpha-N)\int_0^s q_1(t,0)|u_1(t,0)|^2dt=\int_0^s\sum_{j=1}^{N}\int_0^{l_j}(\partial_t q_j+\partial_x q_j+\partial_x^3 q_j)|u_j|^2dxdt\\
&\qquad-3\int_0^s\sum_{j=1}^{N}\int_0^{l_j}|\partial_x u_j|^2\partial_x q_j dxdt-\int_0^s\sum_{j=1}^{N}(q_j|\partial_x u_j|^2+2\partial_x q_j u_j\partial_x u_j)(t,0)dt\\
&\qquad+\sum_{j=1}^{N}\int_0^{l_j}|u_j(0,x)|^2q_j(0,x)dx+\int_0^s\sum_{j=1}^{N}|g_j(t)|^2q_j(t,l_j)dt+2\int_0^s q_1(t,0)u_1(t,0)f_0(t)dt. \qquad (2.3)
\end{aligned}$$

By choosing first $q_j(t,x)=1$, and integrating (2.3) in time on $[0,T]$, we obtain

$$\begin{aligned}
&\|u\|^2_{L^2(0,T,\mathbb{L}^2(\mathcal{T}))}+\|\partial_x u(.,0)\|^2_{L^2(0,T)}\\
&\qquad\leqslant T\Big(2\int_0^T|f_0(t)||u_1(t,0)|dt+\|(g_1,\dots,g_N)\|^2_{L^2(0,T)}+\|u^0\|^2_{L^2(\mathcal{T})}\Big). \qquad (2.4)
\end{aligned}$$

Then we choose $s=T$ and $q_j(t,x)=\frac{x(2l_j-x)}{l_j^2}$ for $j=1,\dots,N$. We see that:

1. $q_j(.,0)=0$,
2. $\forall(t,x)\in[0,T]\times[0,l_j]$, $0\leqslant q_j(t,x)\leqslant 1$,
3. $\forall(t,x)\in[0,T]\times[0,l_j]$, $0\leqslant \partial_x q_j(t,x)\leqslant \frac{2}{l_j}$,
4. $\forall(t,x)\in[0,T]\times[0,l_j]$, $\partial_x^2 q_j(t,x)=-\frac{2}{l_j^2}$.

Then (2.3) gives us

$$\begin{aligned}
&\frac{N}{L^2}\|u_1(.,0)\|^2_{L^2(0,T)}\\
&\leqslant\frac{2}{l}\|u\|^2_{L^2(0,T,\mathbb{L}^2(\mathcal{T}))}+\|\partial_x u(.,0)\|^2_{L^2(0,T)}+\|(g_1,\dots g_N)\|^2_{L^2(0,T)^N}+\|u^0\|^2_{\mathbb{L}^2(\mathcal{T})}+\frac{L^2}{N}\int_0^T|f_0(t)|^2dt, \qquad (2.5)
\end{aligned}$$

where $L=\max_{j=1\dots N} l_j$ and $l=\min_{j=1\dots N} l_j$. Using this with (2.4) we get for some $C>0$ that

$$\begin{aligned}
\|u_1(.,0)\|^2_{L^2(0,T)}&\leqslant C\Big(\int_0^T|f_0(t)||u_1(t,0)|dt+\|(g_1,\dots,g_N)\|^2_{L^2(0,T)^N}+\|u^0\|^2_{\mathbb{L}^2(\mathcal{T})}+\|f_0\|^2_{L^2(0,T)}\Big)\\
&\leqslant C\Big(\|g\|^2_{L^2(0,T)^N}+\|f_0\|^2_{L^2(0,T)}+\|u^0\|^2_{\mathbb{L}^2(\mathcal{T})}\Big).
\end{aligned} \qquad (2.6)$$

By the density of $D(A)$ in $\mathbb{L}^2(\mathcal{T})$ and of $C_0^2([0,T])$ in $L^2(0,T)$ we easily obtain the desired three estimates in the case $f=0$.

When we have a source term, $f \in L^1(0,T,\mathbb{L}^2(\mathcal{T}))$, by using the previous results, we can suppose that $u^0 = 0$, $f_0 = 0$ and $g = 0$. Then by using standard semi-group theory see Pazy (2012), we get that if $f \in L^1(0,T,\mathbb{L}^2(\mathcal{T}))$ then $u \in C([0,T],\mathbb{L}^2(\mathcal{T}))$ and verifies $\|u\|_{C([0,T],\mathbb{L}^2(\mathcal{T}))} \leqslant C\|f\|_{L^1(0,T,\mathbb{L}^2(\mathcal{T}))}$. Thus we easily get the three desired estimates.
□

For the nonlinear system we can use the previous linear result and a fixed point argument similarly as in Ammari & Crépeau (2018) where the case with no control was studied. Thus, we obtain the following.

THEOREM 2.2 There exist $\varepsilon > 0$ and $C > 0$ such that for $u^0 = (u_1^0, \cdots, u_N^0) \in \mathbb{L}^2(\mathcal{T})$ and $(g_1, \cdots, g_N) \in L^2(0,T)^N$ with

$$\|u^0\|_{\mathbb{L}^2(\mathcal{T})} + \|(g_1, \cdots, g_N)\|_{L^2(0,T)^N} \leqslant \varepsilon,$$

there exists a unique solution $u = (u_1, \cdots, u_N) \in \mathbb{B}$ of the nonlinear system (2.2) which satisfies

$$||u||_{\mathbb{B}} \leqslant C\Big(\|u^0\|_{\mathbb{L}^2(\mathcal{T})} + ||(g_1, \cdots, g_N)||_{L^2(0,T)^N}\Big).$$

3. Controllability results

Since now the control in the central node f_0 is turn off, what means that $f_0 = 0$. This section is split into two subsections. The first one deals with the exact controllability of the linear system (2.1) by using a duality argument and the multiplier method in order to prove the observability inequality giving the result. In the second subsection, the nonlinear system (2.2) is considered and the local exact controllability is obtained by means of a fixed point theorem.

3.1 *Linear System*

Due to the linearity of system (2.1) we can consider the case of null initial data, that means taking $u_1^0 = \cdots = u_N^0 = 0$ in (2.1). It can be easily seen that the exact controllability of (2.1) is equivalent to the surjectivity of the operator

$$\Lambda : (g_1, \cdots, g_N) \in L^2(0,T)^N \mapsto (u_1(T,\cdot), \cdots u_N(T,\cdot)) \in \mathbb{L}^2(\mathcal{T}),$$

where $u = (u_1, \cdots, u_N)$ is the solution of (2.1) when controls $(g_1, \cdots, g_N)$ are chosen. From the well-posedness results we know that this operator is linear and continuous. It is known (see Brezis (1999) [Théorème II.19]) that the surjectivity of this operator is equivalent to an observability inequality for the adjoint operator of Λ, which is given by

$$\Lambda^* : (\varphi_1^T, \cdots \varphi_N^T) \in \mathbb{L}^2(\mathcal{T}) \longrightarrow (\partial_x \varphi_1(\cdot, l_1), \cdots, \partial_x \varphi_N(\cdot, l_N)) \in L^2(0,T)^N. \tag{3.1}$$

where $\varphi = (\varphi_1, \cdots, \varphi_N)$ is the solution of the backward adjoint system

$$\begin{cases} (\partial_t \varphi_j + \partial_x \varphi_j + \partial_x^3 \varphi_j)(t,x) = 0, & j = 1, \cdots, N,\ x \in (0, l_j),\ t > 0, \\ \varphi_j(t,0) = \varphi_k(t,0), & j,k = 1, \cdots, N,\ t > 0, \\ \partial_x \varphi_j(t,0) = 0, & j = 1, \cdots, N,\ t > 0, \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 \varphi_j(t,0) = (\alpha - N)\varphi_1(t,0), & t > 0, \\ \varphi_j(t, l_j) = 0, & j = 1, \cdots, N,\ t > 0, \\ \varphi_j(T,x) = \varphi_j^T(x), & j = 1, \cdots, N,\ x \in (0, l_j). \end{cases} \tag{3.2}$$

The desired observability inequality giving the exact controllability of the linear system is stated and proven in the following theorem.

THEOREM 3.1 Let $(l_j)_{j=1,\cdots,N} \in (0,+\infty)^N$ and $\alpha \geqslant N/2$. There exist $L_0, T_{\min} > 0$ such that if

$$L = \max_{j=1,..,N} l_j < L_0 \quad \text{and} \quad T > T_{\min}, \tag{3.3}$$

then we have

$$\|\varphi^T\|^2_{\mathbb{L}^2(\mathcal{T})} \leqslant C \sum_{j=1}^{N} \|\partial_x \varphi_j(t,l_j)\|^2_{L^2(0,T)}, \quad \forall \varphi^T \in \mathbb{L}^2(\mathcal{T}), \tag{3.4}$$

where $\varphi = (\varphi_1, \cdots, \varphi_N)$ is the solution of (3.2) with final condition $\varphi^T = (\varphi_1^T, \cdots, \varphi_N^T)$ and C is a positive constant.

Proof. By multiplying each equation of (3.2) by $q_j \varphi_j$ and integrating by parts on $[s,T] \times [0,l_j]$ with $s \in [0,T]$, we get after some computations

$$\begin{aligned}
&\sum_{j=1}^{N} \int_0^{l_j} |\varphi_j(T,x)|^2 q_j(T,x)dx - \sum_{j=1}^{N} \int_0^{l_j} |\varphi_j(s,x)|^2 q_j(s,x)dx = \\
&\qquad \int_s^T \sum_{j=1}^{N} \int_0^{l_j} (\partial_t q_j + \partial_x q_j + \partial_x^3 q_j)|\varphi_j|^2 dxdt - 3\int_s^T \sum_{j=1}^{N} \int_0^{l_j} |\partial_x \varphi_j|^2 \partial_x q_j dxdt \\
&\qquad + \int_s^T \sum_{j=1}^{N} |\varphi_j(t,0)|^2 \partial_x^2 q_j(t,0)dt + \int_s^T \sum_{j=1}^{N} |\varphi_j(t,0)|^2 q_j(t,0)dt \\
&\qquad + \int_s^T \sum_{j=1}^{N} |\partial_x \varphi_j(t,l_j)|^2 q_j(t,l_j)dt + 2\int_s^T \sum_{j=1}^{N} q_j(t,0)\partial_x^2 \varphi_j(t,0)\varphi_j(t,0)dt.
\end{aligned} \tag{3.5}$$

By choosing $q_j(t,x) = t$ and $s = 0$, we obtain

$$\begin{aligned}
\sum_{j=1}^{N} \int_0^{l_j} T|\varphi_j(T,x)|^2 dx &= \int_0^T \sum_{j=1}^{N} \int_0^{l_j} |\varphi_j(t,x)|^2 dxdt \\
&\quad + \int_0^T \sum_{j=1}^{N} t|\partial_x \varphi_j(t,l_j)|^2 dt, +2\int_0^T \sum_{j=1}^{N} t\partial_x^2 \varphi_j(t,0)\varphi_j(t,0)dt \\
&\quad + \int_0^T \sum_{j=1}^{N} t|\varphi_j(t,0)|^2 dt,
\end{aligned} \tag{3.6}$$

from where we deduce with the boundary condition in (3.2)

$$\begin{aligned}
\sum_{j=1}^{N} \int_0^{l_j} T|\varphi_j(T,x)|^2 dx &\leqslant \int_0^T \sum_{j=1}^{N} \int_0^{l_j} |\varphi_j(t,x)|^2 dxdt \\
&\quad + T\int_0^T \sum_{j=1}^{N} |\partial_x \varphi_j(t,l_j)|^2 dt + T(2\alpha - N)\int_0^T |\varphi_1(t,0)|^2 dt.
\end{aligned} \tag{3.7}$$

By choosing $q_j(t,x)=1$, we get that

$$\sum_{j=1}^{N}\int_0^{l_j}|\varphi_j(T,x)|^2dx-\sum_{j=1}^{N}\int_0^{l_j}|\varphi_j(s,x)|^2dx=\int_s^T\sum_{j=1}^{N}|\partial_x\varphi_j(t,l_j)|^2dt$$
$$+(2\alpha-N)\int_s^T|\varphi_1(t,0)|^2dt, \quad (3.8)$$

from where we obtain

$$\int_0^T\sum_{j=1}^{N}\int_0^{l_j}|\varphi_j(t,x)|^2dxdt\leqslant T\sum_{j=1}^{N}\int_0^{l_j}|\varphi_j(T,x)|^2dx. \quad (3.9)$$

By picking $s=0$ in (3.5) and $q_j(t,x)=\frac{(2l_j-x)(l_j-x)}{2l_j^2}$ which satisfy

1. $0\leqslant q_j(t,x)\leqslant 1$, for all $(t,x)\in[0,T]\times[0,l_j]$,
2. $\frac{-3}{2l_j}\leqslant\partial_x q_j(t,x)\leqslant\frac{-1}{2l_j}$, for all $(t,x)\in[0,T]\times[0,l_j]$,
3. $\partial_x^2 q_j(t,x)=\frac{1}{l_j^2}>0$, for all $(t,x)\in[0,T]\times[0,l_j]$,

we obtain

$$\int_0^T\sum_{j=1}^{N}\frac{1}{l_j^2}|\varphi_j(t,0)|^2dt+\int_0^T\sum_{j=1}^{N}\frac{3}{2l_j}\int_0^{l_j}|\partial_x\varphi_j(t,x)|^2dxdt$$
$$\leqslant\sum_{j=1}^{N}\int_0^{l_j}|\varphi_j(T,x)|^2dx+\int_0^T\sum_{j=1}^{N}\frac{3}{2l_j}\int_0^{l_j}|\varphi_j(t,x)|^2dxdt$$
$$-(2\alpha-N)\int_0^T|\varphi_1(t,0)|^2dt. \quad (3.10)$$

Let us recall

$$L=\max_{j=1,..,N}l_j \quad\text{and}\quad l=\min_{j=1,..,N}l_j.$$

Thanks to (3.10) and 3.9 we have

$$\Big((2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}\Big)\int_0^T|\varphi_1(t,0)|^2dt+\int_0^T\sum_{j=1}^{N}\frac{3}{2l_j}\int_0^{l_j}|\partial_x\varphi_j(t,x)|^2dxdt$$
$$\leqslant(1+\frac{3T}{2l})\sum_{j=1}^{N}\int_0^{l_j}|\varphi_j(T,x)|^2dx. \quad (3.11)$$

From Poincaré's inequality (with uniform constant L/π) and equation (3.7), we can write

$$\sum_{j=1}^{N}\int_0^{l_j} T|\varphi_j(T,x)|^2dx \leqslant \frac{L^2}{\pi^2}\Big(\frac{2L}{3}+\frac{LT}{l}\Big)\sum_{j=1}^{N}\int_0^{l_j}|\varphi_j(T,x)|^2dx + \frac{T(1+\frac{3T}{2l})(2\alpha-N)}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\sum_{j=1}^{N}\int_0^{l_j}|\varphi_j(T,x)|^2dx + T\int_0^T\sum_{j=1}^{N}|\partial_x\varphi_j(t,l_j)|^2dt,$$

and then

$$\Big\{T-\frac{L^2}{\pi^2}\Big(\frac{2L}{3}+\frac{LT}{l}\Big)-\frac{T(1+\frac{3T}{2l})(2\alpha-N)}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\Big\}\|\varphi(T,x)\|^2_{L^2(\mathcal{T})} \leqslant T\sum_{j=1}^{N}\|\partial_x\varphi_j(t,l_j)\|^2_{L^2(0,T)}.$$

Thus, we are led to study the sign of the constant

$$T-\frac{L^2}{\pi^2}\Big(\frac{2L}{3}+\frac{LT}{l}\Big)-\frac{T(1+\frac{3T}{2l})(2\alpha-N)}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}},$$

that can be written as

$$T\Big(1-\frac{L^3}{l\pi^2}-\frac{2\alpha-N}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\Big)-T^2\frac{\frac{3}{2l}(2\alpha-N)}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}-\frac{2L^3}{3\pi^2}.$$

The previous expression can be seen as a quadratic equation in variable T. Thus, in order to force this expression to be positive we have to impose that

$$\Delta=\Big(1-\frac{L^3}{l\pi^2}-\frac{2\alpha-N}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\Big)^2-4\frac{\frac{3}{2l}(2\alpha-N)}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\Big(\frac{2L^3}{3\pi^2}\Big)$$

is positive. This holds if and only if

$$\Big(1-\frac{L^3}{l\pi^2}-\frac{2\alpha-N}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\Big)^2>4\frac{(2\alpha-N)}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\Big(\frac{L^3}{l\pi^2}\Big).$$

Note that the previous inequality is true for L sufficiently small. In consequence, we obtain the observability inequality (3.4) with this direct proof if L is small enough and $T_{\min}<T<T_{\max}$ where $T_{\min}$ and $T_{\max}$ are the roots

$$\Big(1-\frac{L^3}{l\pi^2}-\frac{2\alpha-N}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\Big)\Big(\frac{\frac{3}{l}(2\alpha-N)}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\Big)^{-1}\pm\Delta^{1/2}\Big(\frac{\frac{3}{l}(2\alpha-N)}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\Big)^{-1}.$$

Once we have proved the observability inequality for $T < T_{\max}$, we observe that the inequality still holds for $T \geqslant T_{\max}$. This ends the proof of this theorem.
□

REMARK 3.1 In the limit case $\alpha = \frac{N}{2}$, then we obtain the simpler observability inequality

$$\left\{T\left(1-\frac{L^3}{l\pi^2}\right)-\frac{2L^3}{3\pi^2}\right\}\|\varphi(T,x)\|^2_{\mathbb{L}^2(\mathcal{T})} \leqslant T\sum_{j=1}^{N}\|\partial_x\varphi_j(t,l_j)\|^2_{L^2(0,T)}$$

under the conditions

$$\frac{L^3}{l\pi^2} < 1 \quad \text{and} \quad T > \frac{\frac{2L^3}{3\pi^2}}{1-\frac{L^3}{l\pi^2}}.$$

As by duality a direct consequence of the observability inequality is the controllability of the linear system see Lions (1988) and Theorem 2.42 in Coron (2007).

THEOREM 3.2 Let $(l_j)_{j=1,\cdots,N} \in (0,+\infty)^N$ and $\alpha \geqslant N/2$. There exist $L_0, T_{\min} > 0$ such that if

$$L = \max_{j=1,..,N} l_j < L_0 \quad \text{and} \quad T > T_{\min}, \tag{3.12}$$

then the linear control system (2.1) is exactly controllable. This means that for any states $u^0 = (u_1^0,\cdots,u_N^0) \in \mathbb{L}^2(\mathcal{T})$ and $u^T = (u_1^T,\cdots,u_N^T) \in \mathbb{L}^2(\mathcal{T})$, there exists some controls $g = (g_1,\cdots,g_N) \in L^2(0,T)^N$ such that the solution $u = (u_1,\cdots,u_N) \in \mathbb{B}$ of (2.1) satisfies

$$u_1(T,\cdot) = u_1^T, \quad u_2(T,\cdot) = u_2^T, \cdots, u_N(T,\cdot) = u_N^T.$$

3.2 *Nonlinear System*

We study in this section the local exact controllability for the nonlinear system (2.2) which we rewrite here:

$$\begin{cases} (\partial_t u_j + \partial_x u_j + u_j\partial_x u_j + \partial_x^3 u_j)(t,x) = 0, & j = 1,\cdots,N,\ x \in (0,l_j),\ t > 0, \\ u_j(t,0) = u_k(t,0), & j,k = 1,\cdots,N,\ t > 0, \\ \displaystyle\sum_{j=1}^{N}\partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2, & t > 0, \\ u_j(t,l_j) = 0, \quad \partial_x u_j(t,l_j) = g_j(t), & j = 1,\cdots,N,\ t > 0, \\ u_j(0,x) = u_j^0(x), & j = 1,\cdots,N,\ x \in (0,l_j). \end{cases} \tag{3.13}$$

We have already mentioned in the introduction but let us be more precise in the statement of our main result.

THEOREM 3.3 Let $(l_j)_{j=1,\cdots,N} \in (0,+\infty)^N$ and $\alpha \geqslant N/2$. There exist $L_0, T_{\min} > 0$ such that if

$$L = \max_{j=1,..,N} l_j < L_0 \quad \text{and} \quad T > T_{\min}, \tag{3.14}$$

then the nonlinear control system (3.13) is locally exactly controllable. This means that there exists $\varepsilon > 0$ such that for any states $u^0 = (u_1^0,\cdots,u_N^0) \in \mathbb{L}^2(\mathcal{T})$ and $u^T = (u_1^T,\cdots,u_N^T) \in \mathbb{L}^2(\mathcal{T})$ with

$$\|u^0\|_{\mathbb{L}^2(\mathcal{T})} < \varepsilon \quad \text{and} \quad \|u^T\|_{\mathbb{L}^2(\mathcal{T})} < \varepsilon$$

there exists some controls $g=(g_1,\cdots,g_N)\in L^2(0,T)^N$ such that the solution $u=(u_1,\cdots,u_N)\in\mathbb{B}$ of (3.13) satisfies

$$u_1(T,\cdot)=u_1^T,\quad u_2(T,\cdot)=u_2^T,\ \cdots,u_N(T,\cdot)=u_N^T.$$

Proof. Let $u^0,u^T\in\mathbb{L}^2(\mathcal{T})$ such that $\|u^0\|_{\mathbb{L}^2(\mathcal{T})}<\varepsilon$ and $\|u^T\|_{\mathbb{L}^2(\mathcal{T})}<\varepsilon$ for some $\varepsilon>0$ to be chosen later.

We consider the map

$$\Pi: v\in\mathbb{B}\longrightarrow u^1+u^2+u^3\in\mathbb{B},$$

where u^1,u^2,u^3 are the solutions of

$$\begin{cases}(\partial_t u_j^1+\partial_x u_j^1+\partial_x^3 u_j^1)(t,x)=0, & j=1,\cdots,N,\ x\in(0,l_j),\ t>0,\\ u_j^1(t,0)=u_k^1(t,0), & j,k=1,\cdots,N,\ t>0,\\ \displaystyle\sum_{j=1}^N \partial_x^2 u_j^1(t,0)=-\alpha u_1^1(t,0), & t>0,\\ u_j^1(t,l_j)=0, & j=1,\cdots,N,\ t>0,\\ \partial_x u_j^1(t,l_j)=0, & j=1,\cdots,N,\ t>0,\\ u_j^1(0,x)=u_j^0(x), & j=1,\cdots,N,\ x\in(0,l_j),\end{cases}\tag{3.15}$$

$$\begin{cases}(\partial_t u_j^2+\partial_x u_j^2+\partial_x^3 u_j^2)(t,x)=-v_j\partial_x v_j, & j=1,\cdots,N,\ x\in(0,l_j),\ t>0,\\ u_j^2(t,0)=u_k^2(t,0), & j,k=1,\cdots,N,\ t>0,\\ \displaystyle\sum_{j=1}^N \partial_x^2 u_j^2(t,0)=-\alpha u_1^2(t,0)-\frac{N}{3}(v_1(t,0))^2, & t>0,\\ u_j^2(t,l_j)=0, & j=1,\cdots,N,\ t>0,\\ \partial_x u_j^2(t,l_j)=0, & j=1,\cdots,N,\ t>0,\\ u_j^2(0,x)=0, & j=1,\cdots,N,\ x\in(0,l_j),\end{cases}\tag{3.16}$$

and

$$\begin{cases}(\partial_t u_j^3+\partial_x u_j^3+\partial_x^3 u_j^3)(t,x)=0, & j=1,\cdots,N,\ x\in(0,l_j),\ t>0,\\ u_j^3(t,0)=u_k^3(t,0), & j,k=1,\cdots,N,\ t>0,\\ \displaystyle\sum_{j=1}^N \partial_x^2 u_j^3(t,0)=-\alpha u_1^3(t,0), & t>0,\\ u_j^3(t,l_j)=0, & j=1,\cdots,N,\ t>0,\\ \partial_x u_j^3(t,l_j)=g_j(t), & j=1,\cdots,N,\ t>0,\\ u_j^3(0,x)=0, & j=1,\cdots,N,\ x\in(0,l_j),\end{cases}\tag{3.17}$$

where $g=(g_1,\cdots,g_N)\in L^2(0,T)^N$ is a control such that

$$u^3(T,\cdot)=u^T-u^1(T,\cdot)-u^2(T,\cdot).$$

This control exists thanks to Theorem 3.2. It is important to notice that the control operator (mapping a final state to the respective control driving the linear system to that final state) is continuous. In this part we use the assumptions on L and T to guarantee the controllability of the linear system.

It is easy to see that this proof ends if we are able to find a fixed point $u\in\mathbb{B}$ of the operator Π. To do that, we will apply the Banach fixed point theorem. Let $R>0$ and define

$$B(0,R)=\left\{u\in L^2(0,T,\mathbb{H}_e^1(\mathcal{T}))\Big/\|u\|_{L^2(0,T,\mathbb{H}_e^1(\mathcal{T}))}\leqslant R\right\}.$$

By using the estimates in Theorem 2.1 and the continuity of the control operator, we obtain

$$\begin{aligned}\|\Pi(v)\|_B &\leqslant C_1\|u^0\|_{\mathbb{L}^2(\mathcal{T})} + C_2\left(\|vv_x\|_{L^1(0,T;\mathbb{L}^2\mathcal{T})} + \|v_1(.,0)\|_{L^2(0,T)}\right) + C_3\|u^T\|_{\mathbb{L}^2(\mathcal{T})}\\ &\leqslant C_1\|u^0\|_{\mathbb{L}^2(\mathcal{T})} + C_2'\|v\|^2_{L^2(0,T;\mathbb{L}^2\mathcal{T})} + C_3\|u^T\|_{\mathbb{L}^2(\mathcal{T})}\end{aligned}$$

Thus for $v \in B(0,R)$, we have

$$\|\Pi(v)\|_B \leqslant (C_1 + C_3)\varepsilon + C_2'R^2$$

with R and ε small enough so that $(C_1 + C_3)\varepsilon + C_2'R^2 < R$, we get that

$$\Pi(B(0,R)) \subset B(0,R)$$

Furthermore, $\forall u, v \in B(0,R)$,

$$\begin{aligned}\|\Pi(u) - \Pi(v)\|_{\mathbb{B}} &\leqslant C_4(\|uu_x - vv_x\|_{L^1(0,T;\mathbb{L}^2\mathcal{T})} + \|u_1(.,0) - v_1(.,0)\|_{L^2(0,T)})\\ &\leqslant C_4'R\|u - v\|_{L^2(0,T,\mathbb{H}^1_e(\mathcal{T}))}\end{aligned}$$

and then for R, ε small enough, $C_4'R \in (0,1)$. Thus, we obtain that Π is a contraction in $B(0,R) \subset \mathbb{B}$, which ends the proof of Theorem 3.3.
□

4. Concluding remarks

Applying duality and a multiplier approach we proved the controllability of a Korteweg-de Vries equation on a star-shaped network by means of inputs acting on the external nodes. Thus, we improved a previous result by Ammari & Crépeau (2018) where the use of an additional input acting on the central node was crucial. Moreover, unlike Ammari & Crépeau (2018) we are able to deal with the case $\alpha = N/2$ which represents a system conserving the energy. Our approach imposes conditions on the lengths of the intervals involved and on the time of control. Future research should be directed to get more general results.

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