DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **11**, Number **3**, May **2009**

doi:10.3934/dcdsb.2009.11.655

pp. **655–668**

RAPID EXPONENTIAL STABILIZATION FOR A LINEAR KORTEWEG-DE VRIES EQUATION

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(Communicated by Jerry Bona)

Abstract. We consider a control system for a Korteweg-de Vries equation with homogeneous Dirichlet boundary conditions and Neumann boundary control. We address the rapid exponential stabilization problem. More precisely, we build some feedback laws forcing the solutions of the closed-loop system to decay exponentially to zero with arbitrarily prescribed decay rates. We also perform some numerical computations in order to illustrate this theoretical result.

1. **Introduction.** In this paper we address the boundary stabilizability problem for a linear Korteweg-de Vries (KdV) equation on a bounded domain. We consider a system with homogeneous Dirichlet boundary conditions where the control acts on the Neumann boundary condition at the right endpoint. This issue has been studied in the literature firstly in the case of periodic boundary conditions, mainly by adding a damping term to the equation. For example, in [8] a damping term distributed all along the domain is considered; in [17] the authors use a damping term distributed with localized support and in [16] the authors use a boundary damping term. In all these papers, an exponential decay of the solutions is proved. In the case of homogeneous Dirichlet boundary conditions with a localized damping term, the same stability property has been proved in [12] and [14].

Here, we are interested in the case where there is no damping. In [23], Zhang considers a feedback law which allows him to prove that solutions decay exponentially to zero. In [12], Perla Menzala, Vasconcellos and Zuazua prove that the solutions decay exponentially to zero even in the case without control. It is done when the length of the domain does not belong to a countable set of *critical values* introduced by Rosier in [13]. In the case of critical domains, it is known that there exist some initial conditions such that the corresponding solutions conserve their L^2-norm.

Our main aim in this work is to prove that for any $\omega > 0$, one can build a feedback law such that the closed-loop system has an exponential decay rate ω at least. This is a big difference with the previous works, where one proves the

2000 *Mathematics Subject Classification.* Primary: 93D15, 35Q53; Secondary: 93B05.

Key words and phrases. stabilization, Korteweg-de Vries equation, controllability, feedback.

This work has been partially supported by the "Agence Nationale de la Recherche (ANR), Project C-QUID, number BLAN-3-139579.

exponential decay, but nothing is said about its rate. There exist few results of this kind for control systems of partial differential equations. Among them, one can cite the works by Slemrod in [18] for some bounded control operators (the case of distributed controls) and by Komornik in [6] for some unbounded control operators (the case of point or boundary controls). Both methods use a Gramian approach and are inspired by the one introduced independently by Kleinman in [5] and by Luke in [11] in a finite-dimensional framework. Recently, Urquiza in [22] has generalized to infinite-dimensional control systems a method called the Bass method by Russell (see [15, pages 117-118]). Actually, Urquiza was inspired on numerical computations with the Komornik approach performed by Briffaut (see [1]). These numerical results showed a decay rate twice better than the one predicted by Komornik's theorem. This faster decay is exactly the decay achieved by Urquiza's approach for a controllable system where one has an unbounded control operator and where the free-control system is defined by a skew-operator which is the infinitesimal generator of a strongly continuous group. As we will see, the main task to do, in order to be able to apply this method to our control system, is to obtain its exact controllability. After proving that, we obtain the wanted result of stabilizability. We also perform some numerical computations in order to verify in practice the exponential decay predicted by our theoretical result.

Remark 1. In order to get rapid stabilization for some partial differential equations control system, there exists another method, called backstepping method, which use neither a Gramian approach nor operator Riccati equations. We cite [20, 19] by Krstic and Smyshlyaev, [10] by Liu and the references therein.

2. **Statement of the problem and Urquiza's method.** Let $L > 0$ be fixed. Let us consider the following linear control system for the KdV equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = u(t), \end{cases} \tag{1}$$

where the state is $y(t,\cdot) : [0,L] \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$. In [23], Zhang considers the following feedback law with $\alpha \in (0,1)$ and $L = 1$

$$u(t) = \alpha y_x(t,0). \tag{2}$$

One obtains, for the closed-loop system, that the energy satisfies

$$\frac{d}{dt}\int_0^1 |y(t,x)|^2 dx = -(1-\alpha)|y_x(t,0)|^2.$$

Thus, a decay to zero of the solutions is naturally expected. In fact, Zhang proves that the closed-loop system is well posed in $L^2(0,1)$ and that there exist $\omega > 0$ and $C > 0$ such that

$$\forall y_0 \in L^2(0,1), \forall t > 0, \quad \|y(t,\cdot)\|_{L^2(0,1)} \leq Ce^{-\omega t}\|y_0\|_{L^2(0,1)}, \tag{3}$$

where y is the solution of (1)-(2) with initial data y_0 and $L = 1$. That means, the feedback law (2) stabilizes the control system (1) to the origin. Then, Rosier proves in [13] that for some values of L, called *critical values* (see the definition of the set N in (15)), there exist some initial conditions y_0 such that the corresponding solution of (1) with $u = 0$, conserves its L^2-norm. These solutions also satisfy

$$y_x(t,0) = 0,$$

and therefore a feedback law as (2) does not stabilize the system. (Note that $1 \notin N$). Later, in [12], Perla, Vasconcellos and Zuazua prove that (3) actually holds for (1) with $u = 0$, provided that the length L of the interval is not critical.

In this paper, we are interested in the design of some feedback laws

$$u(t) = \Pi(y(t, \cdot)), \tag{4}$$

such that the closed-loop system (1) and (4) has an exponential decay rate (the constant ω in (3)) as large as desired. In order to get this stabilizability property we use a method due to Urquiza [22]. Let us explain his result on the following abstract control system

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t), \\ y(0) = y_0, \end{cases} \tag{5}$$

with state $y(t)$ in a Hilbert space Y and control $u(t)$ in a Hilbert space U. Here, the initial condition $y_0 \in Y$, A is a skew-adjoint operator (i.e. $A^* = -A$) in Y whose domain is dense in Y, and B is an unbounded operator from U to Y. Let us assume that these operators satisfy the following hypothesis.

(H1) The skew-adjoint operator A is the infinitesimal generator of a strongly continuous group on Y.

(H2) The operator $B : U \to D(A)'$ is linear continuous.

(H3) *Regularity property.* For every $0 < T < \infty$ there exists $C_T > 0$ such that

$$\int_0^T \|B^* e^{-tA^*} y\|_U^2 dt \le C_T \|y\|_Y^2, \quad y \in D(A^*).$$

(H4) *Controllability property.* There exist $T > 0$ and $c_T > 0$ such that

$$\int_0^T \|B^* e^{-tA^*} y\|_U^2 dt \ge c_T \|y\|_Y^2, \quad y \in D(A^*).$$

Then, one has the following result whose proof mainly relies on general results about the algebraic Riccati equation associated with the linear quadratic regulator problem (see [3]).

Theorem 2.1. *(see* [22, Theorem 2.1]*) Consider operators A and B under assumptions* (H1)-(H4)*. For any $\omega > 0$, we have*

(i) The symmetric positive operator Λ_ω defined by

$$(\Lambda_\omega x, z)_Y = \int_0^\infty \left(B^* e^{-\tau(A+\omega I)^*} x, B^* e^{-\tau(A+\omega I)^*} z \right)_U d\tau, \quad \forall x, z \in Y,$$

is coercive and is an isomorphism on Y.

(ii) Let $F_\omega := -B^* \Lambda_\omega^{-1}$. The operator $A + BF_\omega$ with $D(A + BF_\omega) = \Lambda_\omega(D(A^*))$ is the infinitesimal generator of a strongly continuous semigroup on Y.

(iii) The closed-loop system (system (5) with the feedback law $u = F_\omega(y)$) is exponentially stable with a decay rate equals to 2ω, that is,

$$\exists C > 0, \forall x \in Y, \quad \|e^{t(A+BF_\omega)} x\|_Y \le C e^{-2\omega t} \|x\|_Y.$$

As one can see, the feedback operator is built in an explicit way. This fact and the free choice of the parameter ω are the main advantages of this method. The first point to check in order to be able to apply this theorem to our linear KdV control system is (H1). As we easily see, hypothesis (H1) holds if we take as control, the function v defined by

$$u(t) = y_x(t, 0) + v(t).$$

Hence our system becomes

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) - y_x(t,0) = v(t). \end{cases} \tag{6}$$

We can rewrite (6) in the abstract form (5) by defining the operators A and B as follows

$$D(A) := \left\{ w \in H^3(0,L); w(0) = w(L) = 0, w'(0) = w'(L) \right\},$$
$$Aw := -w' - w''',$$
$$B : s \in \mathbb{R} \mapsto L_s \in D(A^*)',$$
$$L_s : z \in D(A^*) \mapsto s z_x(L) \in \mathbb{R}.$$

It is not difficult to see that $A^* = -A$ and that

$$(Aw, w)_{L^2(0,L)} = 0, \quad \forall w \in D(A).$$

Hence, from classical semigroup results, one sees that the operator A satisfies (H1). We also see that (H2) holds for the operator B. Hypothesis (H3) and (H4) are more delicate to show and will be proved in section 3. As our operator B stands for a boundary control, we will see that assumption (H3) is a sharp trace regularity. Concerning (H4), it implies an exact controllability result that will be stated below. Then, in section 4, by applying Theorem 2.1, a feedback law for our control system is given in an explicit way and the rapid stabilizability is asserted in a precise way. Finally, in section 5 we check the performance of our feedback laws on some numerical simulations.

3. **Proof of (H3) and (H4).** In this section we first study the asymptotic behavior of the eigenvalues of the operator A. Then, we apply a classical Ingham's inequality to prove that (H3) and (H4) hold for our control system. An exact boundary controllability result is also stated.

3.1. **Spectral properties of the operator A.** It is not difficult to see that the skew-adjoint operator A has a compact resolvent. Hence the spectrum $\sigma(A)$ of A consists only of eigenvalues. Furthermore the spectrum of A is a discrete subset of $i\mathbb{R}$ and the eigenfunctions form an orthonormal basis of $L^2(0,L)$. For the work to be done here, we require very detailed informations about the asymptotic behavior of the eigen-elements of the operator A. Let us denote by $(i\lambda_k)_{k\in\mathbb{Z}}$ the eigenvalues of A and by $(\phi_k)_{k\in\mathbb{Z}}$ its eigenfunctions.

Proposition 1. *The real numbers $(\lambda_k)_{k\in\mathbb{Z}}$ have the asymptotic form*

$$\lambda_k = \frac{8\pi^3 k^3}{L^3} + O(k^2) \quad \text{as } k \to \pm\infty.$$

Proof. The eigenvalue problem to be considered is

$$\begin{cases} -\phi' - \phi''' = i\lambda\phi, \\ \phi(0) = \phi(L) = 0, \\ \phi'(0) = \phi'(L). \end{cases}$$

To each λ corresponds at least a real a such that $\lambda = 2a(4a^2 - 1)$. Thus, the three solutions of

$$z^3 + z + i\lambda = 0$$

read as

$$z_0 = \sqrt{|3a^2 - 1|} - ai, \quad z_1 = -\sqrt{|3a^2 - 1|} - ai, \quad z_2 = 2ai.$$

We distinguish 3 cases.

1. If $3a^2 - 1 < 0$.

In this case, it is easy to see that the eigenfunction ϕ of A associated to the eigenvalue $\lambda = 2a(4a^2 - 1)$ may be written

$$\phi(x) = e^{-iax}(\alpha \cos(\sqrt{1 - 3a^2}x) + \beta \sin(\sqrt{1 - 3a^2}x)) + \gamma e^{2iax},$$

where α, β and γ are some constants such that $\phi(0) = \phi(L) = 0$ and $\phi'(0) = \phi'(L)$. That means, such that

$$\alpha + \gamma = 0,$$

$$e^{-iaL}(\alpha \cos(\sqrt{1 - 3a^2}L) + \beta \sin(\sqrt{1 - 3a^2}L)) - \alpha e^{2iaL} = 0, \tag{7}$$

$$\begin{aligned} - ia\alpha + \beta\sqrt{1 - 3a^2} + 2ia\gamma &= -iae^{-iaL}(\alpha \cos(\sqrt{1 - 3a^2}L) + \beta \sin(\sqrt{1 - 3a^2}L)) \\ &2ia\gamma e^{2iaL} + e^{-iaL}\sqrt{1 - 3a^2}(-\alpha \sin(\sqrt{1 - 3a^2}L) + \beta \cos(\sqrt{1 - 3a^2}L)). \end{aligned} \tag{8}$$

From (7), one obtains

$$\beta = \alpha \frac{e^{3iaL} - \cos(\sqrt{1 - 3a^2}L)}{\sin(\sqrt{1 - 3a^2}L)}.$$

Taking the real part of equation (8), one obtains that a must satisfy

$$\begin{aligned} \sqrt{1 - 3a^2} \cos(2aL) = 3a \sin(aL) \sin(\sqrt{1 - 3a^2}L) \\ + \sqrt{1 - 3a^2} \cos(aL) \cos(\sqrt{1 - 3a^2}L). \end{aligned} \tag{9}$$

The number of parameters a satisfying (9) is finite and depends on L. As if a satisfies (9), then $(-a)$ so, we find in this case $2N_L$ eigenvalues

$$\{\lambda_{-N_L}, ..., \lambda_{-1}, \lambda_1, ..., \lambda_{N_L}\}$$

2. If $3a^2 - 1 = 0$.

We don't find any eigenfunction in this case. In fact, here

$$z_0 = z_1 = \frac{i\sqrt{3}}{3},\ z_2 = \frac{-2i\sqrt{3}}{3} \quad \text{or} \quad z_0 = z_1 = \frac{-i\sqrt{3}}{3},\ z_2 = \frac{2i\sqrt{3}}{3}$$

and the candidate function to be an eigenfunction cannot satisfy the boundary conditions.

3. If $3a^2 - 1 > 0$.

In this case, it is easy to see that the eigenfunction ϕ of A associated to the eigenvalue $\lambda = 2a(4a^2 - 1)$ may be written as

$$\phi(x) = e^{-iax}(\alpha \cosh(\sqrt{3a^2 - 1}x) + \beta \sinh(\sqrt{3a^2 - 1}x)) + \gamma e^{2iax}$$

where α, β and γ are some constants such that $\phi(0) = \phi(L) = 0$ and $\phi'(0) = \phi'(L)$. That means, such that

$$\alpha + \gamma = 0,$$

$$e^{-iaL}(\alpha \cosh(\sqrt{3a^2 - 1}L) + \beta \sinh(\sqrt{3a^2 - 1}L)) - \alpha e^{2iaL} = 0, \tag{10}$$

$$\begin{aligned}
&-ia\alpha+2ia\gamma+\beta\sqrt{3a^2-1}=\\
&\quad -iae^{-iaL}(\alpha\cosh(\sqrt{3a^2-1}L)+\beta\sinh(\sqrt{3a^2-1}L))+2ia\gamma e^{2iaL}\\
&\qquad +e^{-iaL}\sqrt{3a^2-1}(\alpha\sinh(\sqrt{3a^2-1}L)+\beta\cosh(\sqrt{3a^2-1}L)) \qquad (11)
\end{aligned}$$

We deduce from (10)-(11) that

$$\beta=\alpha\frac{e^{3iaL}-\cosh(\sqrt{3a^2-1}L)}{\sinh(\sqrt{3a^2-1}L)},$$

and

$$\begin{aligned}
&-3a+\Im(\beta)\sqrt{3a^2-1}=-3a\cos(2aL)\\
&\quad +\sqrt{3a^2-1}\sin(-aL)(\sinh(\sqrt{3a^2-1}L)+\Re(\beta)\cosh(\sqrt{3a^2-1}L))\\
&\qquad +\sqrt{3a^2-1}\cos(aL)\Im(\beta)\cosh(\sqrt{3a^2-1}L).
\end{aligned}$$

From these equations, one obtains that a satisfies the following one

$$\begin{aligned}
&\sqrt{3a^2-1}\cos(2aL)-3a\sin(aL)\sinh(\sqrt{3a^2-1}L)\\
&\qquad -\sqrt{3a^2-1}\cos(aL)\cosh(\sqrt{3a^2-1}L)=0. \qquad (12)
\end{aligned}$$

If one neglects the terms $e^{-L\sqrt{3a^2-1}}$ as $a\to\pm\infty$, one gets

$$e^{L\sqrt{3a^2-1}}=\frac{\cos(2aL)}{2\cos(aL-\pi/3)}$$

and hence there exists, for $k\in\mathbb{N}$ large enough, a unique solution a_{k+N_L} (respectively a_{-k-N_L}) in the interval

$$\left[k\frac{\pi}{L},(k+1)\frac{\pi}{L}\right],\quad \left(\text{respectively } \left[-(k+1)\frac{\pi}{L},-k\frac{\pi}{L}\right]\right)$$

defined by equation (12) and given asymptotically by

$$a_k=\frac{5\pi}{6L}+\frac{k\pi}{L}+O(\frac{1}{k})\quad \left(\text{respectively } a_{-k}=-\frac{5\pi}{6L}-\frac{k\pi}{L}+O(\frac{1}{k})\right). \qquad (13)$$

The associated eigenfunction, ϕ_k is

$$\begin{aligned}
\phi_k(x)=\alpha_k\Bigg[e^{-ia_kx}\Bigg(&\cosh(\sqrt{3a_k^2-1}x)\\
&+\frac{e^{3ia_kL}-\cosh(\sqrt{3a_k^2-1}L)}{\sinh(\sqrt{3a_k^2-1}L)}\sinh(\sqrt{3a_k^2-1}x)\Bigg)-e^{2ia_kx}\Bigg],
\end{aligned}$$

where α_k is chosen in such a way that $\|\phi_k\|_{L^2(0,L)}=1$. Asymptotically, one sees that (α_k) converge to $1/\sqrt{L}$ as k goes to ∞.

Thus, from (13), one deduces the asymptotic behavior of the eigenvalues and therefore the proof of this proposition is complete. □

Remark 2. We easily deduce from equations (9) and (12) that

$$\forall k\in\mathbb{Z},\quad a_k=-a_{-k}\quad\text{and}\quad \lambda_k=-\lambda_{-k}.$$

Remark 3. Similar asymptotical behaviors have been found out in [16] and [23].

From the proof of the last proposition, we deduce the following lemma.

Lemma 3.1. *There exists a constant $C > 0$ such that*

$$\lim_{k\to\pm\infty} \frac{|\phi_k'(L)|}{|k|} = C$$

Proof. By using the formulae for the eigenfunctions ϕ_k, we get

$$\phi_k'(L) = \alpha_k \left[-3ia_k e^{2ia_k L} + \frac{e^{-ia_k L}\sqrt{3a_k^2 - 1}}{\sinh(L\sqrt{3a_k^2-1})} \left(e^{3ia_k L}\cosh(L\sqrt{3a_k^2-1}) - 1 \right) \right]$$

This fact together with (13) allows us to find that

$$\lim_{k\to\pm\infty} \frac{|\phi_k'(L)|}{|k|} = \frac{2\pi\sqrt{3}}{L^{3/2}} > 0\,.$$

□

3.2. **Ingham's inequality.** Given the asymptotic behavior of the eigenvalues of A, we have to modify the choice of the state space $L^2(0, L)$ in order to prove (H3) and (H4). From the previous section, we know that $\{\phi_k\}_{k\in\mathbb{Z}}$ is a basis of $L^2(0, L)$. Thus, for any $f \in L^2(0, L)$ there exists a unique sequence $\{f_k\}_{k\in\mathbb{Z}}$ with $\sum_{k\in\mathbb{Z}} |f_k|^2 < \infty$ such that

$$f = \sum_{k\in\mathbb{Z}} f_k\phi_k \quad \text{and} \quad \|f\|_{L^2(0,L)} = \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{1/2}.$$

Let us now define some useful spaces.

Definition 3.2. Let us denote by Z the linear hull of the basis functions $\{\phi_k\}_{k\in\mathbb{Z}}$. Then Z is a dense subspace of $L^2(0, L)$. For any $s \in \mathbb{R}$ we define the space H_s as the completion of Z with respect to the norm defined by

$$\Big\| \sum_{k\in\mathbb{Z}} c_k\phi_k \Big\|_s := \Big(\sum_{k\in\mathbb{Z}} (1 + |\lambda_k|)^{\frac{2}{3}s} |c_k|^2 \Big)^{1/2}.$$

In each space H_s, one has the orthonormal basis $\{(1 + |\lambda_k|)^{-\frac{s}{3}}\phi_k\}_{k\in\mathbb{Z}}$.

With this definition we can state the following well-posedness result whose proof is direct from the previous analysis.

Proposition 2. *For any $z_0 = \sum_{k\in\mathbb{Z}} z_0^k\phi_k \in H_s$, there exists a unique solution of the homogeneous problem*

$$\begin{cases} z_t + z_x + z_{xxx} = 0, z(0, \cdot) = z_0, \\ z(t, 0) = z(t, L) = 0, z_x(t, L) - z_x(t, 0) = 0, \end{cases}$$

which belongs to $C(\mathbb{R}, H_s)$ and is given by

$$z(t, x) = \sum_{k\in\mathbb{Z}} e^{i\lambda_k t} z_0^k \phi_k(x).$$

Moreover, as $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, one has that

$$\forall t \in \mathbb{R}, \quad \|z(t, \cdot)\|_s = \|z_0\|_s.$$

Now, we are interested in the regularity needed to obtain $z_x(\cdot, L) \in L^2(0, T)$ for any $T > 0$. As one has at least formally,

$$z_x(t, L) = \sum_{k\in\mathbb{Z}} e^{i\lambda_k t} z_0^k \phi_k'(L),$$

one sees the importance of Lemma 3.1 which gives us the asymptotic behavior of $\phi_k'(L)$ as k tends to $\pm\infty$. In order to find the regularity needed, we use the following classical result mainly due to Ingham (see [4] and [7]).

Lemma 3.3 (Ingham's inequality). *Let $T > 0$. Let $\{\beta_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence of pairwise distinct real numbers such that*

$$\lim_{|k|\to+\infty} \beta_{k+1} - \beta_k = +\infty.$$

Then there exist two strictly positive constants C_1 and C_2 such that for any sequence $\{\gamma_k\}_{k\in\mathbb{Z}}$ satisfying $\sum_{k\in\mathbb{Z}} \gamma_k^2 < \infty$, the series $f(t) = \sum_{k\in\mathbb{Z}} \gamma_k e^{i\beta_k t}$ converges in $L^2(0,T)$ and satisfies

$$C_1 \sum_{k\in\mathbb{Z}} \gamma_k^2 \leq \int_0^T |f(t)|^2 dt \leq C_2 \sum_{k\in\mathbb{Z}} \gamma_k^2.$$

Let us apply this lemma. Let $z_0 = \sum_{k\in\mathbb{Z}} z_0^k \phi_k \in H_s$ for some $s \geq 0$. We want to take $\beta_k = \lambda_k$ and $\gamma_k = z_0^k \phi_k'(L)$. From the asymptotic behavior, we have that if $s \geq 1$, then

$$\sum_{k\in\mathbb{Z}} |z_0^k|^2 |\phi_k'(L)|^2 < \infty.$$

This together with Proposition 1 allow us to apply the Ingham's inequality and get the existence of two constants C_1, $C_2 > 0$ such that for any $z_0 \in H_s$, with $s \geq 1$,

$$C_1 \sum_{k\in\mathbb{Z}} |z_0^k|^2 |\phi_k'(L)|^2 \leq \int_0^T |z_x(t,L)|^2 dt \leq C_2 \sum_{k\in\mathbb{Z}} |z_0^k|^2 |\phi_k'(L)|^2 . \tag{14}$$

We can estimate by above the right-hand side in terms of the H_1-norm of z_0, and consequently, in terms of any H_s-norm with $s \geq 1$. To get inequalities (H3) and (H4) in the space H_1, we need to estimate by below the left-hand side in terms of the H_1-norm of z_0. In order to do that we can not lose any coefficient z_0^k. Thus, the condition

$$\forall k \in \mathbb{Z}, \quad \phi_k'(L) \neq 0,$$

is required. From the work of Rosier in [13], we know that if L satisfies

$$L \notin N := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}};\ k,l \in \mathbb{N}^* \right\}, \tag{15}$$

then, there exist no $\mu \in \mathbb{C}$, $\varphi \in H^3(0,L)\backslash\{0\}$ satisfying

$$\begin{cases} \mu\varphi + \varphi' + \varphi''' = 0, \\ \varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = 0. \end{cases}$$

In particular, this implies that $\phi_k'(L) \neq 0$ for any $k \in \mathbb{Z}$ and therefore from (14), we get the existence of positive constants c_T and C_T such that

$$c_T \|z_0\|_{H_1}^2 \leq \int_0^T |z_x(t,L)|^2 dt \leq C_T \|z_0\|_{H_1}^2, \quad \forall z_0 \in H_1. \tag{16}$$

The left-hand inequality in (16) is called an observability inequality and as we will see below it implies an exact controllability result. This ends the proof of (H3) and (H4).

3.3. **Controllability.** A direct consequence of (16) is the exact boundary controllability of our control system (6). Let us define what we mean by a solution of this system.

Definition 3.4. Let $T > 0$ be fixed. Let $y_0 \in H_{-1}$ and $v \in L^2(0,T)$. A solution of the Cauchy problem

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ y(0,.) = y_0, \\ y(t,0) = y(t,L) = 0, \ y_x(t,L) - y_x(t,0) = v(t), \end{cases} \tag{17}$$

is a function $y \in C([0,T], H_{-1})$ satisfying $y(0) = y_0$ and

$$\forall \tau \in [0,T], \forall z_0 \in H_1, \quad \langle y(\tau), z(\tau)\rangle_{H_{-1},H_1} = \langle y_0, z_0\rangle_{H_{-1},H_1} + \int_0^\tau z_x(t,L)v(t)dt,$$

where $z \in C([0,T], H_1)$ is the solution of the Cauchy problem

$$\begin{cases} z_t + z_x + z_{xxx} = 0, z(0,\cdot) = z_0, \\ z(t,0) = z(t,L) = 0, z_x(t,L) - z_x(t,0) = 0. \end{cases}$$

With this definition, we obtain the following result whose proof is classical and hence omitted here (see for example [7, page 13]).

Proposition 3. *Let $T > 0$. Let $y_0 \in H_{-1}$ and $v \in L^2(0,T)$. Then, the problem (17) has a unique solution.*

Let us now focus our attention on the controllability problem. It is a classical result, that the observability inequality previously proved in this paper implies the following theorem.

Theorem 3.5 (Exact controllability)**.** *Let $T > 0$ and $L > 0$ be such that $L \notin N$. Let $y_0, y_T \in H_{-1}$. Then there exists a control $v \in L^2(0,T)$ such that the solution of (17) satisfies $y(T,\cdot) = y_T$.*

Remark 4. Using the Hilbert Uniqueness Method (see [9]) one can choose a control $v \in L^2(0,T)$ of minimal L^2-norm among all the controls driving the system from y_0 at $t = 0$ to y_T at $t = T$.

Remark 5. In order to have a controllability result in more regular spaces, one may apply the method used in [21] and [2]. This method mainly consists in considering more regular controls allowing us to derive the equation. Applying that, one could get Theorem 3.5 in the space H_2 with controls in $H^1(0,T)$.

4. **Rapid stabilization.** In this section, we apply Urquiza's method to our linear control system (6). Let us design the feedback laws allowing us to get the rapid stabilization result. We first define, for any q_0 and $\psi_0 \in H_1$, the bilinear form

$$a_\omega(q_0, \psi_0) := \int_0^\infty e^{-2\omega\tau} q_x(\tau, L)\psi_x(\tau, L)d\tau,$$

where q and ψ are the respective solutions of

$$\begin{cases} q_\tau + q_x + q_{xxx} = 0, \ q(0,.) = q_0, \\ q(\tau,0) = q(\tau,L) = 0, \ q_x(\tau,L) - q_x(\tau,0) = 0 \end{cases}$$

and

$$\begin{cases} \psi_\tau + \psi_x + \psi_{xxx} = 0, \ \psi(0,.) = \psi_0, \\ \psi(\tau,0) = \psi(\tau,L) = 0, \ \psi_x(\tau,L) - \psi_x(\tau,0) = 0. \end{cases}$$

We then define the operator $\Lambda_\omega : H_1 \longrightarrow H_{-1}$ assumed to satisfy

$$< \Lambda_\omega q_0, \psi_0 >_{H_{-1},H_1} = a_\omega(q_0, \psi_0), \quad \forall q_0, \psi_0 \in H_1.$$

Finally, we define the following operator

$$\begin{array}{rcl} F_\omega : H_1 & \longrightarrow & \mathbb{R} \\ z & \longrightarrow & F_\omega(z) := -q_0'(L), \end{array}$$

where q_0 is the solution of the following Lax-Milgram problem

$$a_\omega(q_0, \psi_0) = < z, \psi_0 >_{H_{-1},H_1}, \quad \forall \psi_0 \in H_1\,.$$

From section 3 and Theorem 2.1 one easily gets the following result.

Theorem 4.1. *Let $\omega > 0$. The closed-loop system*

$$\left\{ \begin{array}{l} y_t + y_x + y_{xxx} = 0,\ y(0,.) = y_0, \\ y(t,0) = y(t,L) = 0,\ y_x(t,L) - y_x(t,0) = F_\omega(y(t)), \end{array} \right.$$

is globally well posed in H_1. Moreover, the solutions decay to zero with an exponential rate of 2ω, i.e.,

$$\exists C > 0, \forall y_0 \in H_1, \quad \|y(t,\cdot)\|_{H_1} \le Ce^{-2\omega t}\|y_0\|_{H_1}.$$

5. **Numerical simulations.** Let $\omega > 0$ be fixed. We use the Galerkin method and an approximation by modal superposition to decompose our solutions as in [1]. Let $(i\lambda_k, \phi_k)$ be the eigenmodes of

$$\left\{ \begin{array}{l} -\phi_k' - \phi_k''' = i\lambda_k\phi_k, \\ \phi_k(0) = \phi_k(L) = 0, \\ \phi_k(L) - \phi_k(0) = 0. \end{array} \right.$$

Let us define

$$V_N = Span\{\phi_{-N}, ..., \phi_{-1}, \phi_1, ..., \phi_N\}, \quad \forall N \in \mathbb{N}^*.$$

For any $z_0 \in H_1$ let $q_N^0 \in V_N$ be the unique solution of the variational equation

$$a_\omega(q_N^0, \psi_N^0) = \int_0^\infty e^{-2\omega\tau} q_{N\,x}(\tau, L)\psi_{N\,x}(\tau, L)d\tau = \int_0^L z_0(x)\psi_N^0(x)dx, \quad \forall \psi_N^0 \in V_N,$$

where q_N and ψ_N are the respective solutions of

$$\left\{ \begin{array}{l} q_{N\,\tau} + q_{N\,x} + q_{N\,xxx} = 0, \\ q_N(\tau,0) = q_N(\tau,L) = 0, \\ q_{N\,x}(\tau,L) - q_{N\,x}(\tau,0) = 0, \\ q_N(0,.) = q_N^0 \end{array} \right. \tag{18}$$

and

$$\left\{ \begin{array}{l} \psi_{N\,\tau} + \psi_{N\,x} + \psi_{N\,xxx} = 0, \\ \psi_N(\tau,0) = \psi_N(\tau,L) = 0, \\ \psi_{N\,x}(\tau,L) - \psi_{N\,x}(\tau,0) = 0, \\ \psi_N(0,.) = \psi_N^0. \end{array} \right. \tag{19}$$

We define the discrete operator,

$$L_N : z_0 \mapsto q_N^0.$$

As $(q_N^0, \psi_N^0) \in V_N \times V_N$ we can write

$$q_N^0 = \sum_{k=-N}^{N} \gamma_k^0 \phi_k(x)$$

and

$$\psi_N^0 = \sum_{k=-N}^{N} \hat{\gamma}_k^0 \phi_k(x)$$

and we easily deduce from (18) and (19) that

$$q_N(\tau, x) = \sum_{k=-N}^{N} e^{i\lambda_k \tau} \gamma_k^0 \phi_k(x),$$

$$\psi_N(\tau, x) = \sum_{k=-N}^{N} e^{i\lambda_k \tau} \hat{\gamma}_k^0 \phi_k(x).$$

Let us define $m_k = \phi_k'(L)$ for $k \in \mathbb{N}^*$. Then

$$a_\omega(q_N^0, \psi_N^0) = \sum_{k,j=-N}^{N} \frac{\gamma_k^0 \hat{\gamma}_j^0 m_k m_j}{2\omega - i\lambda_k - i\lambda_j}.$$

In order to solve the stabilization problem we write the problem in a weak form where the boundary term appears. We multiply (6) by $w \in D(A)$ and get by integration by parts

$$\int_0^L y_t w dx - \int_0^L y(w_x + w_{xxx}) dx = v(t) w_x(L).$$

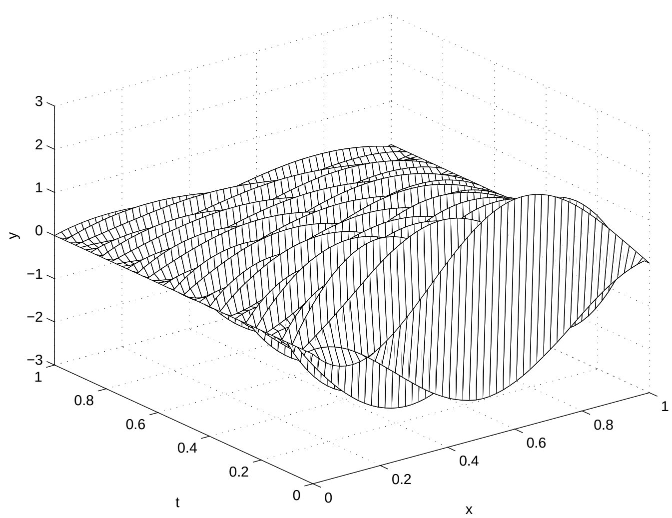

FIGURE 1. Evolution of the solution y with $\omega = 2$.

We take as an approximation of the controlled solution, $y_N : [0, \infty] \to V_N$ solution of

$$\int_0^L y_{N\,t}(t,x) w dx - \int_0^L y_N(t,x)(w_x + w_{xxx}) dx = v_N(t) w_x(L),\ \forall w \in D(A), \quad (20)$$

where the stabilizing control is chosen as

$$v_N(t) = \frac{\partial}{\partial x} L_N(y_N(t,.))$$

with initial data, $y_N(0) = P_N(y_0)$ (P_N is the orthogonal projection on V_N). Let $q_N^0(t) = L_N(y_N(t,.))$. It satisfies

$$\begin{aligned}
\sum_{k,j=-N}^{N} \frac{\gamma_k^0 \hat{\gamma}_j^0 m_k m_j}{2\omega - i\lambda_k - i\lambda_j} &= \int_0^L \sum_{-N}^{N} y_k(t)\phi_k(x) \sum_{-N}^{N} \hat{\gamma}_l^0 \phi_l(x) \\
&= \sum_{k,l=-N}^{N} y_k(t)\gamma_l^0 \int_0^L \phi_k(x)\phi_l(x) \\
&= \sum_{k=-N}^{N} y_k(t)\gamma_k^0
\end{aligned}$$

Let A_ω be the matrix with coefficients

$$(A_\omega)_{l,j} = \frac{m_l m_j}{2\omega - i\lambda_l - i\lambda_j}.$$

Let J be the one anti-diagonal matrix, Γ_N^0 the vector of coefficients γ_k^0 and $Y_N(t)$ the vector of coefficients $y_k(t)$. Then we get

$$\Gamma_N^0 = A_\omega^{-1} J Y_N(t)$$

and the control is

$$v_N(t) = \sum_{k=-N}^{N} [A_\omega^{-1} J Y_N(t)]_k m_k.$$

So with (20), choosing $w = \phi_{-k}$ for $k = -N, ..., -1, 1, ..., N$, we get

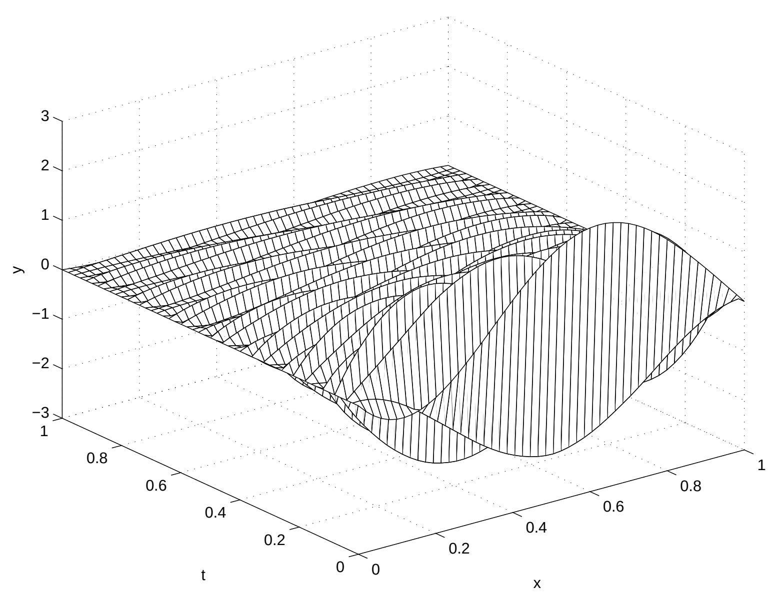

FIGURE 2. Evolution of the solution y with $\omega = 3$.

$$Y_N'(t) - iD(\lambda_{-N}, \ldots, \lambda_N)Y_N(t) = KY_N(t) \tag{21}$$

where $D(\lambda_{-N}, \ldots, \lambda_N)$ is a diagonal matrix with coefficients $\lambda_{-N}, \ldots, \lambda_N$ and K is the matrix with coefficients

$$(K)_{l,j} = m_{-l} \sum_{k=-N}^{N} [A_\omega^{-1} J]_{kj} m_k.$$

The matrix equation (21) can easily be solved. Results are drawn on some figures for $N = 10$ and the initial condition given by $Y_N^0(k) = 1$, for $k = -N, ..., -1, 1, ...N$. On Figure 1 and Figure 2, we show the evolution of the solution for $\omega = 2$ and $\omega = 3$ respectively. Note particularly, on Figure 3, the excellent agreement between theoretical and numerical results for the time-evolution of the H_1-norm.

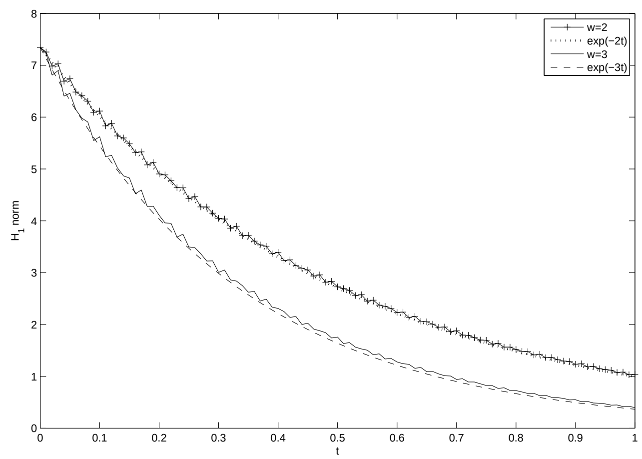

FIGURE 3. Time-evolution of the norm $\|y\|_{H_1}$ compared with $e^{-\omega t}\|y_0\|_{H_1}$ for $\omega = 2$ and $\omega = 3$.

Acknowledgements. We would like to thank Jean-Michel Coron for having attracted our attention to Urquiza's work and for fruitful discussions.

REFERENCES

[1] Jean-Séverin Briffaut, "Méthodes Numériques Pour le contrôle et la Stabilisation Rapide de Grandes Structures Flexibles," Ph.D. thesis, Ecole Nationale des Ponts et Chaussées, 1999.

[2] Emmanuelle Crépeau, *Exact controllability of the Boussinesq equation on a bounded domain*, Differential and Integral Equations, **16** (2003), 303–326.

[3] Franco Flandoli, Irena Lasiecka and Roberto Triggiani, *Algebraic Riccati equations with non-smoothing observation arising in hyperbolic and Euler-Bernoulli boundary control problems*, Ann. Mat. Pura Appl., **153** (1988), 307–382.

[4] Albert Edward Ingham, *Some trigonometrical inequalities with applications to the theory of series*, Math. Z., **41** (1936), 367–379.

[5] David L. Kleinman, *An easy way to stabilize a linear control system*, IEEE Trans. Automatic Control, **15** (1970), 692.

[6] Vilmos Komornik, *Rapid boundary stabilization of linear distributed systems*, SIAM J. Control Optim., **35** (1997), 1591–1613.

[7] Vilmos Komornik and Paola Loreti, "Fourier Series in Control Theory," Springer Monographs in Mathematics, Springer-Verlag, New York, 2005.

[8] Vilmos Komornik, David L. Russell and Bing-Yu Zhang, *Stabilisation de l'équation de Korteweg-de Vries*, C. R. Acad. Sci. Paris Sér. I Math., **312** (1991), 841–843.

[9] Jacques-Louis Lions, "Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués," Masson, Paris, 1988.

[10] Weijiu Liu, *Boundary feedback stabilization of an unstable heat equation*, SIAM J. Control Optim., **42** (2003), 1033–1043.

[11] Dahlard L. Lukes, *Stabilizability and optimal control*, Funkcial. Ekvac., **11** (1968), 39–50.

[12] G. Perla Menzala, C. F. Vasconcellos and E. Zuazua, *Stabilization of the Korteweg-de Vries equation with localized damping*, Quart. Appl. Math., **60** (2002), 111–129.

[13] Lionel Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var., **2** (1997), 33–55 (electronic).

[14] Lionel Rosier and Bing-Yu Zhang, *Global stabilization of the generalized Korteweg-de Vries equation posed on a finite domain*, SIAM J. Control Optim., **45** (2006), 927–956 (electronic).

[15] David L. Russell, "Mathematics of Finite-Dimensional Control Systems," Lecture Notes in Pure and Applied Mathematics, Marcel Dekker Inc., New York, 1979.

[16] David L. Russell and Bing-Yu Zhang, *Smoothing and decay properties of solutions of the Korteweg-de Vries equation on a periodic domain with point dissipation*, J. Math. Anal. Appl., **190** (1995), 449–488.

[17] ———, *Exact controllability and stabilizability of the Korteweg-de Vries equation*, Trans. Amer. Math. Soc., **348** (1996), 3643–3672.

[18] Marshall Slemrod, *A note on complete controllability and stabilizability for linear control systems in Hilbert space*, SIAM J. Control, **12** (1974), 500–508.

[19] Andrey Smyshlyaev and Miroslav Krstic, *Backstepping boundary control for first order hyperbolic PDEs and application to systems with actuator and sensor delays*, Systems and Control Letters, **57** (2008), 750–758.

[20] ———, *Closed-form boundary state feedbacks for a class od 1-D partial integro-differential equations*, IEEE Trans. on Automatic Control, **49** (2004), 2185–2202.

[21] Jose Manuel Urquiza, "Contrôle D'équations des Ondes Linéaires et Quasilinéaires," Ph.D. thesis, Université Paris VI, 2000.

[22] ———, *Rapid exponential feedback stabilization with unbounded control operators*, SIAM J. Control Optim., **43** (2005), 2233–2244 (electronic).

[23] Bing-Yu Zhang, *Boundary stabilization of the Korteweg-de Vries equation*, in "Control and Estimation of Distributed Parameter Systems: Nonlinear Phenomena," Internat. Ser. Numer. Math., vol. **118**, Birkhäuser, Basel, 1994, 371–389.

Received February 2008; revised November 2008.

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