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NULL CONTROLLABILITY AND STABILIZATION OF THE LINEAR KURAMOTO-SIVASHINSKY EQUATION

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Abstract. In this article, we study the boundary controllability of the linear Kuramoto-Sivashinsky equation on a bounded interval. The control acts on the first spatial derivative at the left endpoint. First, we prove that this control system is null controllable. It is done using a spectral analysis and the method of moments. Then, we introduce a boundary feedback law stabilizing to zero the solution of the closed-loop system.

1. **Introduction and main results.** The Kuramoto-Sivashinsky (KS) equation reads as

$$y_t + y_{xxxx} + \lambda y_{xx} + yy_x = 0, \tag{1}$$

where the real number $\lambda > 0$ is called the "anti-diffusion" parameter. This equation was derived independently by Kuramoto et al. in [17, 18, 16] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky in [25] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on stability of a plane flame front. This nonlinear partial differential equation describes incipient instabilities in a variety of physical and chemical systems (see, for instance, [5], [19] and [13]).

From a mathematical point of view, well-posedness and dynamical properties of KS equations have a huge literature since the pioneer articles [23, 24, 12].

We are interested in control properties of the KS equation. In this direction, one finds the papers [1, 6] by Christofides and Armaou where the stabilization via distributed scalar controls is achieved for the KS equation with periodic boundary conditions, and [21] by Liu and Krstic where they proved the stability of (1) with homogeneous Dirichlet boundary conditions for small values of the parameter λ. Moreover, in [14] by Hu and Temam, a robust boundary control problem for KS is formulated and solved.

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In this article, we address the problem of boundary controllability of the following linear KS control system

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} = 0, \\ y(t,0) = 0, \quad y(t,1) = 0, \\ y_x(t,0) = u(t), \quad y_x(t,1) = 0, \end{cases} \tag{2}$$

where the state is $y(t,\cdot) : [0,1] \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$. Given $T > 0$ and a function y_0, we wonder if there exists a control $u = u(t)$ such that the solution $y = y(t,x)$ of (2) with initial condition $y(0,x) = y_0(x)$ satisfies $y(T,x) = 0$. If this control exists for any function y_0 lying in an appropriate space, we say that (2) is null controllable in time T. We will see that (2) is null controllable if and only if λ does not belong to the following countable set

$$\mathcal{N} := \left\{ \pi^2(k^2 + l^2);\ k, l \in \mathbb{N}, 1 \le k < l, k \text{ and } l \text{ have the same parity} \right\}. \tag{3}$$

To do that, we use a problem of moments approach and a spectral analysis of the underlying spatial operator

$$A : w \in D(A) \subset L^2(0,1) \longmapsto -w'''' - \lambda w'' \in L^2(0,1),$$
$$D(A) := H^4(0,1) \cap H_0^2(0,1).$$

When λ belongs to $\mathcal{N}$, this approach is not possible. Indeed, for those values of λ, the system (2) is not null controllable with controls acting only on the first spatial derivatives at the end-points. Thus, we obtain the following result.

Theorem 1.1 (Null controllability)**.** *Let $T > 0$ and $\lambda \notin \mathcal{N}$. For any $y_0 \in L^2(0,1)$, there exists $u \in H^1(0,T)$ such that the solution $y \in C([0,T], L^2(0,1))$ of (2) with initial condition $y(0,\cdot) = y_0$, satisfies $y(T,\cdot) = 0$.*

Then, we wonder if we can steer the system asymptotically to zero with a feedback control, i.e. we address the stabilization issue. It is known from [21] that if $\lambda < 4\pi^2$, then (2) is exponentially stable in $L^2(0,1)$. On the other hand, if $\lambda \ge 4\pi^2$ the stability fails. In fact, the operator A has a finite number of positive eigenvalues. In order to stabilize this system, we design a finite-dimensional based feedback, as in [6, 1] for the KS equation with periodic boundary conditions. Similar feedback laws have been implemented for the heat and wave equation in [8, 9]. Thus, we obtain the following result.

Theorem 1.2 (Stabilization)**.** *Let $\lambda \notin \mathcal{N}$. There exist a feedback operator K and two constants $C, \nu > 0$ such that for any $y_0 \in L^2(0,1)$, the solution of (2) with control in the feedback form $u(t) := K(y(t,\cdot))$ satisfies*

$$\|y(t,\cdot)\|_{L^2(0,L)} \le Ce^{-\nu t}\|y_0\|_{L^2(0,L)}$$

Remark 1. As we have said, in the critical case $\lambda \in \mathcal{N}$, the linear system is not null controllable anymore. It is due to the behavior of some eigenfunction of the operator A. We will see that the space of non-controllable functions is finite-dimensional. To obtain the null controllability of the linear system in these cases, we have to add another control. While discussing this point later, we will see that controlling $y_x(t,0)$ and $y_x(t,1)$ does not improve the situation in the critical cases. Unlike that, the system becomes null controllable if we can act on $y(t,0)$ and $y_x(t,0)$. This result with two input controls has been proved in [20] for the case $\lambda = 0$.

Remark 2. To study the controllability of a nonlinear equation, a first approach would be to use the controllability of the linearized system, and then to prove the property for the nonlinear equation by means of a fixed point theorem. Even if the linearized control system is not null controllable ($\lambda \in \mathcal{N}$), it is not excluded that the nonlinear system is null controllable. That is the situation for the Korteweg-de Vries control system studied in [7, 3, 4], where some finite-dimensional subspaces of non-controllable functions appear for the linear system. However, the nonlinearity gives the controllability. In those papers, the analysis of the nonlinear system is based on power series expansion of second and third order (a first order expansion gives the linear system for which the controllability does not hold). A future research direction could be to apply those tools for the KS equation in the critical cases.

2. **Spectral analysis.**

It is not difficult to see that the self-adjoint operator A has a compact resolvent. Hence the spectrum $\sigma(A)$ of A consists only of eigenvalues. Furthermore the eigenvalues of A, denoted by $\{\sigma_k\}_{k\in\mathbb{N}}$, form a discrete subset of $\mathbb{R}$, satisfying $\lim_{k\to\infty}\sigma_k = -\infty$. The eigenfunctions, denoted by $\{\phi_k\}_{k\in\mathbb{N}}$, form an orthonormal basis of $L^2(0,1)$. For the work to be done here, we need very detailed informations about the asymptotic behavior of the eigen-elements of the operator A. For any $k \in \mathbb{N}$, we have

$$\begin{cases} -\lambda\phi_k'' - \phi_k'''' = \sigma_k\phi_k, \\ \phi_k(0) = 0, \quad \phi_k(1) = 0, \\ \phi_k'(0) = 0, \quad \phi_k'(1) = 0. \end{cases} \tag{4}$$

First at all, let us see that $4\sigma_k \le \lambda^2$ for any $k \in \mathbb{N}$. To do that, we multiply (4) by ϕ_k and integrate on $(0,1)$. By using $ab \le (a^2/\lambda + \lambda b^2/4)$, with $a = \|\phi_k''\|_{L^2(0,1)}$ and $b = \|\phi_k\|_{L^2(0,1)}$, we obtain

$$\sigma_k \int_0^1 |\phi_k(x)|^2 = -\int_0^1 |\phi_k''(x)|^2 - \lambda\int_0^1 \phi_k''(x)\phi_k(x) \le \frac{\lambda^2}{4}\int_0^1 |\phi_k(x)|^2,$$

which gives us the upper bound for the eigenvalues.

As we shall see in the controllability section, the following lemma is crucial.

Lemma 2.1. *Let $\mathcal{N}$ the subset of $\mathbb{R}$ defined in (3). For any $\lambda \notin \mathcal{N}$, the eigenfunctions of A satisfy*

$$\phi_k''(0) \neq 0, \quad \forall k \in \mathbb{N}. \tag{5}$$

Moreover, if $\lambda \notin \mathcal{N}$, the eigenvalues are simple.

Proof. First, we look for the eigenfunctions. Let (ϕ,σ) satisfy

$$\begin{cases} -\lambda\phi'' - \phi'''' = \sigma\phi, \\ \phi(0) = 0, \quad \phi(1) = 0, \\ \phi'(0) = 0, \quad \phi'(1) = 0. \end{cases} \tag{6}$$

We distinguish 3 cases corresponding to the sign of the eigenvalue.

Case 1: $\sigma = 0$. The solution of (6) is given by

$$\phi = C_1 - \alpha C_2 x - C_1\cos(\alpha x) + C_2\sin(\alpha x)$$

with $\alpha = \sqrt{\lambda}$ and C_1, C_2 such that they satisfy

$$C_1(1-\cos(\alpha)) + C_2(\sin(\alpha) - \alpha) = 0, \quad \text{and} \quad C_1\sin(\alpha) + C_2(\cos(\alpha) - 1) = 0.$$

These equations have a solution $(C_1, C_2) \neq 0$ if and only if α satisfies $2\cos(\alpha) = 2 - \alpha\sin(\alpha)$. This is possible for instance when $\lambda = 4n^2\pi^2$ for some $n \in \mathbb{N}$. Thus, ϕ

is given by $\phi = C(1-\cos(\alpha x))$ with C any constant. We see that $\phi''(0) = C\lambda \neq 0$. The other possibility is if we have

$$\cos(\alpha) = \frac{4-\alpha^2}{\alpha^2+4}, \quad \sin(\alpha) = \frac{4\alpha}{\alpha^2+4},$$

and hence,

$$\phi = C(\alpha - 2\alpha x - \alpha\cos(\alpha x) + 2\sin\alpha x).$$

We easily verify that $\phi''(0) = C\alpha^3 \neq 0$.

Case 2: $\sigma < 0$. This part is very similar to the spectral analysis developed in [10]. The solution of (6) is given by

$$\phi = C_1\cosh(\alpha(x-1/2)) + C_2\sinh(\alpha(x-1/2)) + C_3\cos(\beta(x-1/2)) + C_4\sin(\beta(x-1/2))$$

with

$$\alpha = \sqrt{\frac{-\lambda+\sqrt{\lambda^2-4\sigma}}{2}} > 0, \quad \beta = \sqrt{\frac{\lambda+\sqrt{\lambda^2-4\sigma}}{2}} > 0,$$

and $C_i, i = 1, \dots, 4$ such that the boundary conditions are verified. After some computations, one gets two sequences of eigenvalues $\{\sigma_{1,n}\}_{n\geq 1}$ and $\{\sigma_{2,n}\}_{n\geq 1}$:

- $\{\sigma_{1,n}\}_{n\geq 1}$ corresponds to the negative solutions of

$$\alpha\sin(\beta/2)\cosh(\alpha/2) = \beta\sinh(\alpha/2)\cos(\beta/2), \tag{7}$$

  and the eigenfunctions are

$$\phi_{1,n} = C\left[-\frac{\sin(\beta/2)}{\sinh(\alpha/2)}\sinh(\alpha(x-1/2)) + \sin(\beta(x-1/2))\right].$$

  We get that $\phi''_{1,n}(0) = C(\alpha^2+\beta^2)\sin(\beta/2)$ for any $n \in \mathbb{N}$, that is not zero from (7).

- $\{\sigma_{2,n}\}_{n\geq 1}$ corresponds to the negative solutions of

$$-\alpha\sinh(\alpha/2)\cos(\beta/2) = \beta\sin(\beta/2)\cosh(\alpha/2), \tag{8}$$

  and the eigenfunctions are

$$\phi_{2,n} = C\left[-\frac{\cos(\beta/2)}{\cosh(\alpha/2)}\cosh(\alpha(x-1/2)) + \cos(\beta(x-1/2))\right].$$

  We get that $\phi''_{2,n}(0) = -C(\alpha^2+\beta^2)\cos(\beta/2)$ for any $n \in \mathbb{N}$, that is not zero from (8).

Case 3: $\sigma > 0$. In this part we will find some values of λ for which (5) does not hold. The solution of (6) is given by

$$\phi = C_1\cos(\alpha(x-1/2)) + C_2\sin(\alpha(x-1/2)) + C_3\cos(\beta(x-1/2)) + C_4\sin(\beta(x-1/2))$$

with

$$\alpha = \sqrt{\frac{\lambda-\sqrt{\lambda^2-4\sigma}}{2}} > 0, \quad \beta = \sqrt{\frac{\lambda+\sqrt{\lambda^2-4\sigma}}{2}} > 0,$$

and $C_i, i = 1, \dots, 4$ such that the boundary conditions are verified. After some computations, one gets two set of eigenvalues $\{\hat{\sigma}_{1,n}\}_{n=1}^{m_1}$ and $\{\hat{\sigma}_{2,n}\}_{n=1}^{m_2}$ with $m_1, m_2 \in \mathbb{N}$:

- $\{\hat{\sigma}_{1,n}\}_{n=1}^{m_1}$ corresponds to the positive solutions of

$$\beta\cos(\beta/2)\sin(\alpha/2)=\alpha\cos(\alpha/2)\sin(\beta/2). \tag{9}$$

Here, we have two possibilities. If $\sin(\alpha/2)\neq 0$, the eigenfunction is

$$\hat{\phi}_{1,n}=C\left[-\frac{\sin(\beta/2)}{\sin(\alpha/2)}\sin(\alpha(x-1/2))+\sin(\beta(x-1/2))\right].$$

By using (9), one can see that $\hat{\phi}''_{1,n}(0)=C(\beta^2-\alpha^2)\sin(\beta/2)$ is not zero for any $n\in\mathbb{N}$. On the other hand, if $\sin(\alpha/2)=0$, the eigenfunction is

$$\hat{\phi}_{1,n}=C\left[-\frac{\beta\cos(\beta/2)}{\alpha\cos(\alpha/2)}\sin(\alpha(x-1/2))+\sin(\beta(x-1/2))\right].$$

From (9), we get that $\hat{\phi}''_{2,n}(0)=0$ if and only if

$$\lambda=\pi^2((2p)^2+(2q)^2)$$

with $p,q\in\mathbb{N}$, $0<p<q$. In this case, $\hat{\sigma}_{1,n}=16\pi^4p^2q^2$.

- $\{\hat{\sigma}_{2,n}\}_{n=1}^{m_2}$ corresponds to the positive solutions of

$$\beta\cos(\alpha/2)\sin(\beta/2)=\alpha\sin(\alpha/2)\cos(\beta/2). \tag{10}$$

Here, we have again two possibilities. If $\cos(\alpha/2)\neq 0$, the eigenfunction is

$$\hat{\phi}_{2,n}=C\left[-\frac{\cos(\beta/2)}{\cos(\alpha/2)}\cos(\alpha(x-1/2))+\cos(\beta(x-1/2))\right].$$

By using (10), one can see that $\hat{\phi}''_{2,n}(0)=C(\alpha^2-\beta^2)\cos(\beta/2)$ is not zero for any $n\in\mathbb{N}$. On the other hand, if $\cos(\alpha/2)=0$, the eigenfunction is

$$\hat{\phi}_{2,n}=C\left[-\frac{\beta\sin(\beta/2)}{\alpha\sin(\alpha/2)}\cos(\alpha(x-1/2))+\cos(\beta(x-1/2))\right].$$

We can see that $\hat{\phi}''_{2,n}(0)=0$ if and only if

$$\lambda=\pi^2[(2p+1)^2+(2q+1)^2]$$

with $p,q\in\mathbb{N}$, $0\leq p<q$. In this case we get $\hat{\sigma}_{2,n}=\pi^4(1+2p)^2(1+2q)^2$.

In this way, we have obtained the set of values λ for which (5) holds.

Finally, let us prove the last statement in the lemma. Let φ_1,φ_2 two eigenfunctions associated to the same eigenvalue σ. By defining $w:=\varphi_1''(0)\varphi_2-\varphi_2''(0)\varphi_1$, we see that w satisfies

$$\begin{cases} -\lambda w''-w''''=\sigma w,\\ w(0)=0, \quad w(1)=0,\\ w'(0)=0, \quad w'(1)=0,\end{cases}$$

and $w''(0)=0$. Therefore, one concludes that $w=0$, i.e. φ_1 and φ_2 are linearly dependent, which ends the proof of the lemma. □

Remark 3. From the previous lemma we see that for a fixed $\lambda\in\mathcal{N}$, the subspace formed by the eigenfunctions ϕ satisfying $\phi''(0)=0$ is finite-dimensional. For example, if $\lambda=20\pi^2$, this subspace is one-dimensional and is generated by the eigenfunction

$$\phi=2\sin(2\pi x-\pi)+\sin(4\pi x).$$

The associated eigenvalue is $\sigma=64\pi^4$.

In order to prove an asymptotic behavior of the eigen-elements of A, we have to focus in the case $\sigma < 0$ since we know that there are only a finite number of non-negatives eigenvalues. From the proof of the previous lemma, in particular from (7) and (8), one can see that the following result holds.

Lemma 2.2. *There exists some real positive constants D_i with $i = 1, 2, 3$ such that*

(i) *The real numbers $\{\sigma_k\}_{k\in\mathbb{N}}$ have the asymptotic form*

$$\sigma_k = -D_1 k^4 + O(k^3) \quad \text{as } k \to \infty.$$

(ii) *One has*

$$\lim_{k\to+\infty} \frac{|\phi_k''(0)|}{k^2} = D_2, \qquad \lim_{k\to+\infty} \frac{|\phi_k'''(0)|}{k^3} = D_3.$$

3. Controllability.

3.1. Well-posedness.

Let us first explain what we mean by a solution of the linear KS control system. If $y = y(t, x)$ is a solution of (2), then the function

$$w(t, x) = y(t, x) - (x^3 - 2x^2 + x)u(t)$$

satisfies

$$\begin{cases} w_t + w_{xxxx} + \lambda w_{xx} = F(t, x), \\ w(t, 0) = 0, \quad w(t, 1) = 0, \\ w_x(t, 0) = 0, \quad w_x(t, 1) = 0, \\ w(0, x) = y_0(x) - (x^3 - 2x^2 + x)u(0), \end{cases} \tag{11}$$

where

$$F(t, x) := -\lambda(6x - 4)u(t) - (x^3 - 2x^2 + x)\dot{u}(t).$$

From section 2 we know that the operator A, whose domain is $H^4(0, 1) \cap H_0^2(0, 1)$, generates a strongly continuous semigroup in $L^2(0, 1)$. Thus, if the initial condition $(y_0 - (x^3 - 2x^2 + x)u(0)) \in L^2(0, 1)$ and $F \in L^1(0, T, L^2(0, 1))$, then (11) has a unique solution (called mild solution) in the space $C([0, T], L^2(0, 1))$. Moreover, if $(y_0 - (x^3 - 2x^2 + x)u(0)) \in H^4(0, 1) \cap H_0^2(0, 1)$ and $F \in C^1([0, T], L^2(0, 1))$, then (11) has a unique solution (called classical solution) in the space

$$C([0, T], H^4(0, 1) \cap H_0^2(0, 1)) \cap C^1([0, T], L^2(0, 1)).$$

In this way we see that if $y_0 \in L^2(0, 1)$ and $u \in H^1(0, T)$, then there exists a unique solution $y \in C([0, T], L^2(0, 1))$ of (2). It is important to note that for any $t \in [0, T]$ we can speak of $y(t, \cdot)$ as a function lying in $L^2(0, 1)$. Furthermore, using the energy estimates developed in [14], we can see that we have in fact a more regular solution

$$y \in C([0, T], L^2(0, 1)) \cap L^2(0, T, H^2(0, 1)).$$

3.2. Null controllability.

Given $T > 0$, the system (2) is said to be null controllable in a space H if for any state $y_0 \in H$, one can find a control u such that the solution y of (2) with initial data y_0 satisfies $y(T) = 0$. Let us give the following characterization of the null-controllability property.

Lemma 3.1. *The control system (2) is null controlable in time T if and only if for any $y_0 \in L^2(0, 1)$, there exists a function $u \in H^1(0, T)$ such that for any $q_T \in L^2(0, 1)$*

$$\int_0^1 y_0(x)q(0, x)dx = -\int_0^T u(t)q_{xx}(t, 0)dt \tag{12}$$

where $q = q(t,x)$ is the solution of

$$\begin{cases} -q_t + \lambda q_{xx} + q_{xxxx} = 0, \\ q(t,0) = 0, \quad q(t,1) = 0, \\ q_x(t,0) = 0, \quad q_x(t,1) = 0, \\ q(T,x) = q_T(x). \end{cases} \tag{13}$$

Proof. Let $q_T \in L^2(0,1)$ and $q = q(t,x)$ the solution of (13). Let us multiply (2) by q and integrate by parts. We obtain

$$\int_0^1 y_0(x)q(0,x)dx - \int_0^1 y(T,x)q_T(x)dx = -\int_0^T u(t)q_{xx}(t,0)dt. \tag{14}$$

If (12) holds, then $\int_0^1 y(T)q_T dx = 0$ for any $q_T \in L^2(0,1)$ and hence $y(T) = 0$. Thus the control u steers the system from y_0 to zero. Reciprocally, if u steers the system from y_0 to zero, then from (14) we obtain (12). □

Now, we use the basis of $L^2(0,1)$ formed by the eigenfunctions of A. Any $q_T \in L^2(0,1)$ can be written as

$$q_T = \sum_{k\in\mathbb{N}} q_k \phi_k,$$

with $\{q_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ satisfying $\sum_{k\in\mathbb{N}} |q_k|^2 < \infty$. Hence, the solution of (13) is given by

$$q(t,x) = \sum_{k\in\mathbb{N}} q_k e^{(T-t)\sigma_k} \phi_k(x)$$

and we have

$$q_{xx}(t,0) = \sum_{k\in\mathbb{N}} q_k e^{(T-t)\sigma_k} \phi_k''(0).$$

Using this fact in (12), on gets the following lemma.

Lemma 3.2. *The control system (2) is null controlable in time T if and only if for any*

$$y_0 = \sum_{k\in\mathbb{N}} y_0^k \phi_k \in L^2(0,1),$$

there exists a function $f \in H^1(0,T)$ such that

$$\phi_k''(0) \int_0^T f(t) e^{\sigma_k t} dt = -y_0^k e^{\sigma_k T}, \quad \forall k \in \mathbb{N}. \tag{15}$$

The control is given by $u(t) := f(T-t)$.

Now, we consider $\lambda \notin \mathcal{N}$. Thus, the eigenfunctions ϕ_k satisfy $\phi_k''(0) \neq 0$ for any $k \in \mathbb{N}$ and we can write (15) as follows.

$$\int_0^T f(t) e^{\sigma_k t} dt = -\frac{y_0^k e^{\sigma_k T}}{\phi_k''(0)}, \quad \forall k \in \mathbb{N}. \tag{16}$$

Thanks to the behavior described in Lemma 2.2, to solve this moment problem, we can apply the general theory developed in [11] by Fattorini and Russell (see also [2] and [22]). Thus, by applying Corollary 3.2 in [11], one gets Theorem 1.1. Moreover, the results in [11] prove that one obtains not only the null controllability of (2) in the space $L^2(0,1)$ with controls in $H^1(0,T)$, but also in the spaces $H^s(0,1)$ for any $s \in \mathbb{R}$ with controls $u \in C^\infty([0,T])$ satisfying $u(0) = u(T) = 0$. In this paper, we will use the framework of the Theorem 1.1, i.e. states in $L^2(0,1)$ and controls in $H^1(0,T)$.

From (15), we clearly see that if $\phi_k''(0) = 0$ for some $k \in \mathbb{N}$, then we could not control the k-th coordinate of the solution. This is the case, as we saw in section 2, if $\lambda \in \mathcal{N}$. The system (2) is no longer null controllable since there exists a finite-dimensional subspace of $L^2(0,1)$ formed by some eigenfunctions satisfying $\phi_k''(0) = 0$ (see Remark 3). To overcome this difficulty, we can add another control. If we are allowed to control the spatial derivative at both endpoints with controls u_1 and u_2, i.e., we consider the system

$$\begin{cases} y_t + \lambda y_{xx} + y_{xxxx} = 0, \\ y(t,0) = 0, \quad y(t,1) = 0, \\ y_x(t,0) = u_1(t), \quad y_x(t,1) = u_2(t), \end{cases} \tag{17}$$

one can prove that the null-controllability is equivalent to the existence of f_1, f_2 such that

$$\phi_k''(0) \int_0^T f_1(t) e^{\sigma_k t} dt + \phi_k''(1) \int_0^T f_2(t) e^{\sigma_k t} dt = -y_0^k e^{\sigma_k T}, \quad \forall k \in \mathbb{N}. \tag{18}$$

Unfortunately, the same eigenfunctions ϕ_k for which $\phi_k''(0) = 0$, also satisfy $\phi_k''(1) = 0$. Thus, the second control does not give the null controllability. On the other hand, if we add a control acting on $y(t,0)$ we do obtain the controllability. Indeed, in this case the null controllability of the control system

$$\begin{cases} y_t + \lambda y_{xx} + y_{xxxx} = 0, \\ y(t,0) = u_1(t), \quad y(t,1) = 0, \\ y_x(t,0) = u_2(t), \quad y_x(1) = 0, \end{cases} \tag{19}$$

is equivalent to the existence of f_1, f_2 such that

$$\phi_k''(0) \int_0^T f_2(t) e^{\sigma_k t} dt + \phi_k'''(0) \int_0^T f_1(t) e^{\sigma_k t} dt = -y_0^k e^{\sigma_k T}, \quad \forall k \in \mathbb{N}. \tag{20}$$

and this problem of moments can be solved in the same way as (15) since is not possible that an eigenfunction ϕ_k satisfy both conditions $\phi_k''(0) = 0$ and $\phi_k'''(0) = 0$. Indeed, the function ϕ_k is a non-trivial solution of a fourth-order ODE such that $\phi_k(0) = \phi_k'(0) = 0$.

4. **Stabilization.** Since the eigenvalues of the operator A (see (4)) are real and satisfy $\lim_{k\to\infty} \sigma_k = -\infty$, we know that there could be at most a finite number of non-negative eigenvalues. This unstable situation actually occurs when the real parameter λ is larger or equals to $4\pi^2$ (see [21]). We also know that if $\lambda < 4\pi^2$, then the linear KS equation (and even the nonlinear KS equation) is asymptotically stable in $L^2(0,1)$. Here, we focus in the case $\lambda \geq 4\pi^2$.

In order to stabilize our linear control system, we are going to design a feedback law moving the first unstable eigenvalues to the left without moving the others. Thus, all the eigenvalues of the closed-loop system will be negative.

Now, in order to deal with an homogeneous Dirichlet problem instead of system (2), we set, as previously,

$$w(t,x) = y(t,x) - (x^3 - 2x^2 + x)u(t).$$

This leads to the equation

$$\begin{cases} w_t = Aw + b(x)\dot{u}(t) + a(x)u(t), \\ w(t,0) = 0, \quad w(t,1) = 0, \\ w_x(t,0) = 0, \quad w_x(t,1) = 0, \\ w(0,x) = y_0(x) - b(x)u(0), \end{cases} \tag{21}$$

where

$$\begin{aligned} b(x) &= -x^3 + 2x^2 - x, \\ a(x) &= -\lambda(6x - 4). \end{aligned}$$

Any solution $w = w(t,x)$ of (21) can be expanded as a series in the eigenfunctions $\{\phi_k\}_{k\in\mathbb{N}}$:

$$w(t,x) = \sum_{k=1}^{\infty} w_k(t)\phi_k(x).$$

Let $n \in \mathbb{N}$ be such that for any $k > n$, one has $\sigma_k < -1$. Our feedback is based on a finite pole shifting procedure for the first n eigenvalues. Let Π^n denote the orthogonal projection onto the subspace spanned by the n first eigenfunctions $\phi_1, \phi_2, \dots, \phi_n$.

As we have

$$\Pi^n(w_t) = \sum_{k=1}^{n} \dot{w}_k(t)\phi_k(x), \quad \text{and} \quad \Pi^n(Aw(t,x)) = \sum_{k=1}^{n} \sigma_k w_k(t)\phi_k(x),$$

we can write

$$\sum_{k=1}^{n} \dot{w}_k(t)\phi_k(x) = \sum_{k=1}^{n} \sigma_k w_k(t)\phi_k(x) + \Pi^n(b(x)\dot{u}(t) + a(x)u(t)),$$

or shortly,

$$\forall k \in \{1,\dots,n\}, \quad \dot{w}_k(t) = \sigma_k w_k(t) + b_k\dot{u}(t) + a_k u(t), \tag{22}$$

where

$$\begin{aligned} b_k &:= \int_0^1 (-x^3 + 2x^2 - x)\phi_k(x)dx, \\ a_k &:= -\lambda\int_0^1 (6x-4)\phi_k(x)dx. \end{aligned}$$

The n equations in (22) form a finite-dimensional differential system controlled by u and $\dot{u}$. Set

$$\alpha(t) = \dot{u}(t),$$

and consider now u as being part of the state and α as the control. Then the former differential system can be written in matrix-form as the finite-dimensional control system

$$\dot{X}_n(t) = A_n X_n(t) + B_n\alpha(t), \tag{23}$$

with

$$X_n(t) = \begin{pmatrix} u(t) \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}, \quad A_n = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ a_1 & \sigma_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_n & 0 & \cdots & 0 & \sigma_n \end{pmatrix}, \quad B_n = \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Let us now prove the following result.

Proposition 1. *The finite-dimensional control system (23) is controllable.*

Proof. Let us see that the Kalman condition is verified. Simple computations give

$$\det(B_n, A_nB_n, \dots, A_n^{n-1}B_n) = \prod_{k=1}^{n}(a_k + \sigma_k b_k) VdM(\sigma_1, \dots, \sigma_n),$$

where $VdM(\sigma_1, \dots, \sigma_n)$ is a Van der Monde determinant whose value is

$$\prod_{k>j}(\sigma_k - \sigma_j),$$

that is not zero since all the eigenvalues are simple. Now, let us compute by using integrations by parts,

$$\begin{aligned} a_k + \sigma_k b_k &= \lambda \int_0^1 b''(x)\phi_k(x)dx + \sigma_k \int_0^1 b(x)\phi_k(x)dx \\ &= \int_0^1 b(x)(\lambda\phi_k''(x) + \sigma_k\phi_k(x))dx \\ &= -\int_0^1 b(x)\phi_k''''(x)dx = b'(x)\phi_k''\big|_{x=0}^1 = \phi_k''(0). \end{aligned}$$

From Lemma 2.1 we see that the Kalman condition holds. Hence, the system (23) is controllable. □

Thus, this system can be stabilizable by the pole shifting method (see [15]), obtaining the following corollary.

Corollary 1. *There exists a vector $K_n = (K_n^0, K_n^1, \dots, K_n^n)$ such that the matrix $A_n + B_nK_n$ admits $n+1$ eigenvalues $\{\mu_k\}_{k=0}^n$ satisfying for any $k \in \{0, \dots, n\}$, $Re(\mu_k) < -1$.*

If we take $u'(t) = K_nX_n(t)$ such that $u(0) = 0$ in (21), we get the closed-loop system

$$\begin{cases} u' = K_nX_n, \\ w_t = Aw + b(x)K_nX_n + a(x)u, \\ w(t,0) = 0, \quad w(t,1) = 0, \\ w_x(t,0) = 0, \quad w_x(t,1) = 0, \\ w(0,x) = y_0(x), \end{cases} \tag{24}$$

where the state is $(u(t), w(t,\cdot)) \in \mathbb{R} \times L^2(0,1)$.

Let us denote by $(\mu_k, \hat{X}_k)$ with $k = 0, \dots, n$, the eigenvalues and eigenvectors of the matrix $(A_n + B_nK_n)$.

It is not difficult to see that the eigenvalues of the closed-loop system (24) are

$$E_v = \{\mu_0, \mu_1, \dots, \mu_n, \sigma_{n+1}, \sigma_{n+2}, \dots\},$$

and the corresponding eigenfunctions are given by

$$E_f = \left\{ \begin{pmatrix} \hat{X}_0 \\ g_0 \end{pmatrix}, \dots, \begin{pmatrix} \hat{X}_n \\ g_n \end{pmatrix}, \begin{pmatrix} 0 \\ P_{n+1}(\phi_{n+1}) \end{pmatrix}, \begin{pmatrix} 0 \\ P_{n+1}(\phi_{n+2}) \end{pmatrix}, \dots \right\},$$

where the functions $g_k \in span\{\phi_{n+1}, \phi_{n+2}, \dots\} \subset L^2(0,1)$ are defined by

$$(A - \mu_k)g_k = F_n(\hat{X}_k) \tag{25}$$

with $F_n : \mathbb{R}^{n+1} \longrightarrow span\{\phi_{n+1}, \phi_{n+2}, \dots\}$ the following operator

$$(y_0, \dots, y_n) \mapsto \sum_{j \ge n+1} \left[(a_j + b_jK_n^0)y_0 + b_jK_n^1y_1 + \dots + b_jK_n^ny_n\right]\phi_j.$$

Let us note that to solve (25), we have to impose the conditions

$$\mu_k \neq \sigma_j \,, \quad \forall k = 0, \ldots, n, \, \forall j \in \mathbb{N} \,.$$

If we write $F_n(\hat{X}_k) = \sum_{j \geq n+1} f_j^k \phi_j$, then the solution to (25) is given by

$$g_k = \sum_{j \geq n+1} \frac{f_j^k}{\sigma_j - \mu_k} \phi_j .$$

Finally, one can see that the subspace E_f form a basis of the space $\mathbb{R} \times L^2(0,1)$ and that all the eigenvalues of the closed-loop system are in the left half of the line $Re(z) = -1$ on the complex plane. Thus, one obtains the exponential stability of the closed loop system (24) and therefore Theorem 1.2.

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