Boundary controllability of a cascade system coupling fourth- and second-order parabolic equations

Nicolás Carreño†, Eduardo Cerpa† and Alberto Mercado†

Abstract

A control system coupling fourth- and second-order parabolic equations is considered in this paper. The main topic is the study of the control properties of this system when we only control the second-order partial differential equation through a boundary condition. Depending on the choice of the diffusion coefficients, we obtain positive and negative results for approximate- and null-controllability. In particular, we prove that for any given positive time $T_0$, we can find some diffusion coefficients such that the system is null-controllable in time $T$ if $T > T_0$ and is not null-controllable if $T < T_0$.

Keywords: parabolic system, boundary controllability, moment method.

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1 Introduction

Parabolic partial differential equations are used to model several phenomena as population dynamics, chemical processes, phase transitions in fluids, temperature evolution, and so on. Once we have an appropriate model to describe a phenomenon, we can wonder if we can influence the evolution of the system. For instance, we can ask if we can drive to a desired target the temperature profile of a rod by means of manipulating some heat source acting at the end of the rod. This and other analogous questions concerning a given mathematical model are the kind of questions addressed by control theory. In this work we deal with related issues concerning some parabolic systems. The specific property of steering a system from some initial state to a final state in finite time is called exact-controllability and when the final state is the rest, we call it null-controllability. The control property can be boundary or distributed depending on the localization of the source that we can manipulate.

Concerning one-dimensional parabolic equations, the first boundary null-controllability result was proved for the heat equation by Fattorini and Russell in [16] by using the moment method. Later on, Lebeau and Robbiano in [21] (using local Carleman estimates) and Fursikov and Imanuvilov in [18] (using global Carleman estimates) obtained the distributed null-controllability of the heat equation in higher dimensions. Fourth-order parabolic equations (bilaplacian type) have also been studied recently. Some null-controllability results for the one-dimensional case can be found in [9] (boundary control, moment method), [10] (boundary control, global Carleman estimates) and [25]...
(distributed control, global Carleman estimates). In higher dimensions, we find [19] (distributed control, global Carleman estimates) and [20] (distributed control, local Carleman estimates).

An interesting extension is the study of control properties for systems of coupled partial differential equations, specially if the number of control inputs is less than the number of equations. In this case, we have to control a part of the system indirectly using the coupling with other equations which are directly under the influence of the control. Regarding control of parabolic systems, many works have been devoted to study different type of couplings of second-order equations (see the survey [2] and reference therein). Some Kalman type conditions naturally appear to characterize good couplings (see [24], [17] and [1]) and a very useful tool to prove distributed controllability of parabolic systems with less controls than equations is the Carleman estimates approach ([4]). Unfortunately, it is very hard to use it in the boundary control case. In the framework of second-order operators, other recent results we can cite are [22] (considering nonlocal terms in the equations), [13, 15] (algebraic methods to eliminate controls), [5] (approximate controllability), [14] (control and coupling region do not intersect).

Regarding systems involving fourth-order parabolic equations, we find controllability results for the stabilized Kuramoto-Sivashinsky system, which is a non-linear system coupling a bi-laplacian equation with a heat equation. Using a Carleman estimates approach, the boundary [11] and distributed null-controllability [12, 7] has been obtained. In [12] the case of one distributed control acting on the fourth-order equation is considered while in [7] the control acts on the second-order equation.

In this paper, we study the boundary controllability properties of a cascade system coupling a bi-laplacian operator to a heat equation, which reads as

\[
\begin{align*}
  u_t(t,x) + u_{xxxx}(t,x) &= v(t,x), & t > 0, & x \in (0,L), \\
  v_t(t,x) - dv_{xx}(t,x) &= 0, & t > 0, & x \in (0,L), \\
  u(t,0) &= u_{xx}(t,0) = 0, & t > 0, \\
  u(t,L) &= u_{xx}(t,L) = 0, & t > 0, \\
  v(t,0) &= h(t), & v(t,L) = 0.
\end{align*}
\]

(1.1)

where the state is given by \((u,v)\) and the control is \(h\), which only acts on the heat equation. The parameter \(d > 0\) is the diffusion of the heat equation and it will play a crucial role in order to define which kind of control properties we are able to obtain.

To study the null-controllability for (1.1), we will prove that this system enters in the general framework introduced in [3], where the authors consider the existence of minimal time of control for parabolic systems. See also [17] and [23] for some results for second-order operators.

Now we state the precise definitions of the properties we are interested in.

**Definition 1.1.** Let \(T > 0\). System (1.1) is said to be approximate-controllable in time \(T\) if for any \(\varepsilon > 0\) and for any states \((u_0,v_0),(u_T,v_T)\) \(\in L^2(0,L) \times H^{-1}(0,L)\), there exists a control \(h \in L^2(0,T)\) such that the solution of (1.1) with initial condition

\[ u(0,\cdot) = u_0 \quad \text{and} \quad v(0,\cdot) = v_0 \]

satisfies

\[ \|u(T,\cdot) - u_T\|_{L^2(0,L)} + \|v(T,\cdot) - v_T\|_{H^{-1}(0,L)} \leq \varepsilon. \]

**Definition 1.2.** Let \(T > 0\). System (1.1) is said to be null-controllable in time \(T\) if for any state \((u_0,v_0)\) \(\in L^2(0,L) \times H^{-1}(0,L)\), there exists a control \(h \in L^2(0,T)\) such that the solution of (1.1) with initial condition

\[ u(0,\cdot) = u_0 \quad \text{and} \quad v(0,\cdot) = v_0 \]

satisfies

\[ \|u(T,\cdot)\|_{L^2(0,L)} + \|v(T,\cdot)\|_{H^{-1}(0,L)} \leq \varepsilon. \]
satisfies
\[ u(T, \cdot) = 0 \quad \text{and} \quad v(T, \cdot) = 0. \]

The main result we obtain concerning approximate-controllability is the following one.

**Theorem 1.3.** System (1.1) is approximate-controllable if and only if \( \sqrt{d} \) is integer or irrational.

The proof to this theorem is given in Section 2. It is based on a duality approach leading us to study a unique continuation property for the adjoint system.

Regarding null-controllability, the results that we obtain also depend on the coefficient \( d \), and there are cases when the system is null-controllable for all time \( T > 0 \) or only when \( T \) is larger than a given \( T_0 > 0 \). It turns out that the key property of the coefficient \( d \) is how closely \( \sqrt{d} \) can be approximated by rational numbers. In the literature we find the following measure of this property.

**Definition 1.4.** The Liouville-Roth constant of a real number \( x \) is the least upper bound of the set of real numbers \( \mu \) such that
\[ 0 < \left| \frac{x}{q} \right| < \frac{1}{q^\mu} \]
is satisfied by an infinite number of integer pairs \((p, q)\) with \( q > 0 \).

**Remark 1.5.** The Liouville-Roth constant is also called irrationality measure. It is known that it is 1 for rational numbers, it is no less than 2 for irrational numbers and it is exactly 2 for irrational algebraic numbers (see [6]).

Using this definition we can state our main results on null-controllability.

**Theorem 1.6.** If \( \sqrt{d} \) has finite Liouville-Roth constant, then system (1.1) is null-controllable in time \( T > 0 \).

**Theorem 1.7.** For any \( T_0 > 0 \), there exists \( d > 0 \) such that system (1.1) is null-controllable in time \( T \) if \( T > T_0 \) and is not null-controllable if \( T < T_0 \).

**Theorem 1.8.** There exists \( d > 0 \) such that system (1.1) is not null-controllable in time \( T \) for any time of control \( T > 0 \).

These theorems will be proved in Section 3. To do that we write our control system (1.1) in the general form used in [3], which reduces the proof to the computation of what is call the condensation index. The authors of (1.1) use the moment method and a duality approach.

**Remark 1.9.** The moment method has been proved to be useful to study boundary controllability of systems with less controls than equations. Very precise results can be obtained concerning the existence of minimal time of control. The drawback is that precise expressions for spectrum data are needed. Thus, this method is hard to be applied when non-constant coefficients appear in the partial differential equations forming the system.

## 2 Approximate-controllability

Let us start by saying a few words on the well-posedness of system (1.1). We notice that, given the cascade structure of the system, we can first solve the heat equation. This is possible in the framework \( h \in L^2(0, T) \) and initial condition \( v(0, \cdot) = v_0 \) with \( v_0 \in H^{-1}(0, L) \) (see [3]). Once
we have the solution $v \in C(0,T;H^{-1}(0,L)) \cap L^2(0,T;L^2(0,L))$ for the heat equation, we plug it into the right-hand side of the fourth-order equation. Using [8, Proposition 2.1] we obtain that the solution of the fourth-order equation with initial condition $u(0,\cdot) = u_0 \in L^2(0,L)$ satisfies $u \in C(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L))$.

With this regularity framework we can start the proof of Theorem 1.3.

**Proof:** Let us introduce the operator

$$\Lambda : h \in L^2(0,T) \mapsto (u(T,\cdot), v(T, \cdot)) \in L^2(0,L) \times H^{-1}(0,L),$$

where $(u, v)$ is the solution to (1.1) with control $h$ and initial data $u(0, \cdot) = 0$ and $v(0, \cdot) = 0$.

Thus, the approximate-controllability of (1.1) is equivalent to the property that the range of the operator $\Lambda$ is dense in $L^2(0,L) \times H^{-1}(0,L)$. It is already a classical duality approach to see that the latter is equivalent to the injectivity of the operator $\Lambda^\ast$. After simple computations, we find

$$\Lambda^\ast : (\varphi_T, \psi_T) \in L^2(0,L) \times H_0^1(0,L) \mapsto \psi_x(\cdot,0) \in L^2(0,T),$$

where $(\varphi, \psi)$ is the solution of

$$
\left\{
\begin{array}{l}
\varphi_t(t,x) + \varphi_{xxxx}(t,x) = 0, \\
\psi_t(t,x) - d\psi_{xx}(t,x) = \varphi(t,x), \\
\varphi(0) = \varphi_{xx}(0) = 0, \\
\psi(0) = \psi_{xx}(0) = 0,
\end{array}
\right.
$$

with initial data $\varphi(T, \cdot) = \varphi_T$ and $\psi(T, \cdot) = \psi_T$.

In order to study the injectivity of $\Lambda^\ast$, we write

$$\varphi_T = \sum_{k \in \mathbb{N}} b_k \varphi_k(x), \quad \text{and} \quad \psi_T = \sum_{k \in \mathbb{N}} a_k \varphi_k(x),$$

where for any $k \in \mathbb{N}$ we denote by $\varphi_k$ the normalized eigenfunctions associated to the laplacian operator with homogeneous Dirichlet boundary conditions:

$$\varphi_k(x) = \sqrt{\frac{2}{L}} \sin \left(\frac{k\pi x}{L}\right).$$

Then, the solution of system (2.1) can be explicitly given. We have

$$\varphi(t,x) = \sum_{k \in \mathbb{N}} b_k e^{-k^4(T-t)} \varphi_k(x),$$

and $\psi$ depends on the nature of $d$. We deal with the three different cases.

- **Case 1:** $d \notin \mathbb{Q}$.

  We have $k^4 - dk^2 \neq 0$ for any integer $k$ and then we get

  $$
  \begin{align*}
  \psi(t,x) &= \sum_{k \in \mathbb{N}} a_k e^{-dk^2(T-t)} \varphi_k(x) + \left(e^{-dk^2(T-t)} - e^{-k^4(T-t)}\right) \frac{b_k}{k^4 - dk^2} \varphi_k(x),
  \end{align*}
  $$

  from where

  $$
  \psi_x(t,0) = \sum_{k \in \mathbb{N}} k a_k e^{-dk^2(T-t)} + \frac{b_k}{k^3 - dk} \left(e^{-dk^2(T-t)} - e^{-k^4(T-t)}\right).$$

- **Case 2:** $d = \frac{p}{q}$, with $p, q \in \mathbb{N}$.

  We have $k^4 - dk^2 \neq 0$ for any integer $k$ and then we get

  $$
  \begin{align*}
  \psi(t,x) &= \sum_{k \in \mathbb{N}} a_k e^{-dk^2(T-t)} \varphi_k(x) + \left(e^{-dk^2(T-t)} - e^{-k^4(T-t)}\right) \frac{b_k}{k^4 - dk^2} \varphi_k(x),
  \end{align*}
  $$

  from where

  $$
  \psi_x(t,0) = \sum_{k \in \mathbb{N}} k a_k e^{-dk^2(T-t)} + \frac{b_k}{k^3 - dk} \left(e^{-dk^2(T-t)} - e^{-k^4(T-t)}\right).$$

- **Case 3:** $d \in \mathbb{Q}$.

  We have $k^4 - dk^2 = 0$ for any integer $k$ and then we get

  $$
  \begin{align*}
  \psi(t,x) &= \sum_{k \in \mathbb{N}} a_k e^{-dk^2(T-t)} \varphi_k(x) + \left(e^{-dk^2(T-t)} - e^{-k^4(T-t)}\right) \frac{b_k}{k^4 - dk^2} \varphi_k(x),
  \end{align*}
  $$

  from where

  $$
  \psi_x(t,0) = \sum_{k \in \mathbb{N}} k a_k e^{-dk^2(T-t)} + \frac{b_k}{k^3 - dk} \left(e^{-dk^2(T-t)} - e^{-k^4(T-t)}\right).$$

- **Case 4:** $d \in \mathbb{Q}$ and $k^4 - dk^2 \neq 0$ for any integer $k$.

  We have $k^4 - dk^2 = 0$ for any integer $k$ and then we get

  $$
  \begin{align*}
  \psi(t,x) &= \sum_{k \in \mathbb{N}} a_k e^{-dk^2(T-t)} \varphi_k(x) + \left(e^{-dk^2(T-t)} - e^{-k^4(T-t)}\right) \frac{b_k}{k^4 - dk^2} \varphi_k(x),
  \end{align*}
  $$

  from where

  $$
  \psi_x(t,0) = \sum_{k \in \mathbb{N}} k a_k e^{-dk^2(T-t)} + \frac{b_k}{k^3 - dk} \left(e^{-dk^2(T-t)} - e^{-k^4(T-t)}\right).$$
Therefore, if \( \psi_x(\cdot,0) = 0 \) we get for all \( k \in \mathbb{N} \)

\[
ka_k + \frac{b_k}{k^3 - dk} = 0 \quad \text{and} \quad \frac{b_k}{k^3 - dk} = 0.
\]

Thus, \( a_k = b_k = 0 \) for all \( k \in \mathbb{N} \), and we get the injectivity of \( \Lambda^* \).

- **Case 2**: \( \sqrt{d} \in \mathbb{N} \).

Introducing \( m := \sqrt{d} \) we can write

\[
\psi(t,x) = \sum_{k \neq m} a_k e^{-dk^2(T-t)} \varphi_k(x) + \left( e^{-dk^2(T-t)} - e^{-k^4(T-t)} \right) \frac{b_k}{k^4 - dk^2} \varphi_k(x) \\
+ b_m(T-t)e^{-m^4(T-t)} \varphi_m(x) + a_m e^{-dm^2(T-t)} \varphi_m(x).
\]

Hence, taking into account that \( d = m^2 \), we have

\[
\psi_x(t,0) = \sum_{k \neq m} ka_k e^{-dk^2(T-t)} + \frac{b_k}{k^3 - dk} \left( e^{-dk^2(T-t)} - e^{-k^4(T-t)} \right) \\
+ mb_m(T-t)e^{-m^4(T-t)} + ma_m e^{-m^4(T-t)}.
\]

Therefore, if \( \psi_x(\cdot,0) = 0 \) we get

\[
mb_m(T-t) + ma_m = 0 \quad \forall t \in [0,T],
\]

\[
\frac{b_k}{k^3 - dk} = 0 \quad \forall k \neq m, \text{ and}
\]

\[
ka_k + \frac{b_k}{k^3 - dk} = 0, \quad \forall k \neq m.
\]

Thus, \( a_k = b_k = 0 \) for all \( k \in \mathbb{N} \), and we get the injectivity of \( \Lambda^* \).

- **Case 3**: \( \sqrt{d} \in \mathbb{Q} \setminus \mathbb{N} \).

The solution \( \psi \) is given by expression (2.4) and hence (2.5) is satisfied.

Notice that we can write \( \sqrt{d} = p^2/q \), with \( p, q \in \mathbb{N} \). Let us consider the initial conditions \( \varphi_T \) and \( \psi_T \) given by the coefficients \( a_p = -p^{-1}, b_p = p^3 - dp, a_q = q^{-1}, b_q = 0 \) and \( a_k = b_k = 0 \) for any \( k \notin \{p,q\} \). Then, is not difficult to see that \( \psi_x(\cdot,0) = 0 \) and \( \psi \neq 0 \) and \( \varphi \neq 0 \). The operator \( \Lambda^* \) is not injective in this case.

Recalling that the injectivity of \( \Lambda^* \) is equivalent to the approximate-controllability of system (1.1), we conclude the proof of Theorem 1.3.

\[\Box\]

### 3 Null-controllability

#### 3.1 Abstract control system

We will state some results for abstract control systems obtained in [3]. Given an unbounded operator \( \mathcal{A} \) in the complex Hilbert space \( \mathcal{X} \), we consider the system

\[
\begin{align*}
\begin{cases}
y' &= \mathcal{A}y + \mathcal{B}h, \\
y(0) &= y_0,
\end{cases}
\end{align*}
\]

(3.1)
with \( y_0 \in \mathbb{X} \) and \( h \in L^2(0, T) \), where \( \mathcal{B} \) is the control operator. We introduce some assumptions and notation in order to state that result.

We will assume that \( \mathbb{X} \) has a Riesz basis given by the eigenfunctions of operator \( \mathcal{A} \) denoted by \( \{ \phi_k \}_{k \in \mathbb{N}} \) with eigenvalues \( \{ \lambda_k \}_{k \in \mathbb{N}} \). We denote by \( \{ \psi_k \}_{k \in \mathbb{N}} \) the corresponding sequence of biorthogonal functions to \( \{ \phi_k \}_{k \in \mathbb{N}} \). Then \( \mathcal{A} \) can be characterized by

\[
\mathcal{A} = - \sum_{k=1}^{\infty} \lambda_k (\cdot, \varphi_k) \phi_k. 
\]

Furthermore, we denote by \( \mathbb{X}_{-1} \) the completion of \( \mathbb{X} \) with respect to the norm

\[
\| y \|_{-1} := \left( \sum_{k=1}^{\infty} \frac{|(y, \psi_k)|^2}{|\lambda_k|^2} \right)^{1/2}. 
\]

In [3] it is established a characterization of controllability of system (3.1) in terms of the sequence \( \{ \lambda_k \} \) and the control operator \( \mathcal{B} \). The result is stated in terms of a holomorphic complex function \( E(z) \) which vanishes at \( z = \lambda_k \), for any \( k \in \mathbb{N} \). One way to define such a function is given by the infinite product

\[
E(z) = \prod_{k \in \mathbb{N}} \left( 1 - \frac{z^2}{\lambda_k^2} \right), \quad z \in \mathbb{C}. 
\]

The function \( E(z) \) converges uniformly and absolutely in compact subsets of \( \mathbb{C} \) under the hypothesis of the theorem below, where are stated the controllability properties of system (3.1).

**Theorem 3.1** (Theorem 2.5 in [3]). Assume that \( \mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1}) \) is an admissible control operator for the semigroup \( \{ e^{tA} \}_{t>0} \) generated by \( \mathcal{A} \), and \( \Lambda = \{ \lambda_k \}_{k \in \mathbb{N}} \) is a complex sequence satisfying

\[
\lambda_j \neq \lambda_k, \forall j \neq k; \quad \Re(\lambda_k) \geq \delta |\lambda_k| > 0; \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{|\lambda_k|} < \infty. 
\]

Let us suppose in addition that

\[
b_k := \mathcal{B}^* \psi_k \neq 0 \quad \forall k \in \mathbb{N}. 
\]

We introduce

\[
T_0 := \limsup_{k \to \infty} \left( \frac{\ln |b_k|}{\Re(\lambda_k)} + \frac{\ln \frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)} \right), 
\]

where \( E \) is defined in (3.3). Then:

1. System (3.1) is null-controllable if \( T > T_0 \).
2. System (3.1) is not null-controllable if \( T < T_0 \).

**Remark 3.2.** The limit of condensation \( c(\Lambda) \) of a sequence \( \Lambda = \{ \lambda_k \}_{k \in \mathbb{N}} \) is defined by

\[
c(\Lambda) := \limsup_{k \to \infty} \frac{\ln \frac{1}{|E'(\lambda_k)|}}{\lambda_k}. 
\]

Under the hypothesis of Theorem 3.1, we have \( c(\Lambda) \in [0, \infty] \) (see Theorem 4.8 in [3]). Also, if

\[
\lim_{k \to \infty} \frac{\ln |b_k|}{\Re(\lambda_k)} = 0, 
\]

it can be easily proved that \( T_0 = c(\Lambda) \).
3.2 Null-controllability of System 1.1

We write the control system (1.1) as a first order control system of the form (3.1). Let us consider the Dirichlet Laplacian

\[ A := -\Delta : D(A) := H^2 \cap H^1_0(0, L) \subset L^2(0, L) \rightarrow L^2(0, L), \]

which is a self-adjoint operator. We still denote by \( A \) its self-adjoint extension to the spaces \( H^{-1}(0, L) \) with domain \( H^1_0(0, L) \), and \( (H^2 \cap H^1_0(0, L))^\prime \) with domain \( L^2(0, L) \), respectively.

We define \( X = L^2(0, L) \times H^{-1}(0, L) \) and \( A : D(A) \subset X \rightarrow X \) by

\[ D(A) = \{ u \in D(A) : u'' \in D(A) \} \times H^1_0(0, L), \]

and

\[ A = \begin{pmatrix} -A^2 & 1 \\ 0 & -dA \end{pmatrix}. \]  

(3.8)

The operator \( B \in \mathcal{L}(\mathbb{R}, (H^2 \cap H^1_0(0, L))^2)^\prime \) is defined by

\[ (Bv) \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) = vd\phi_2'(0) \quad \forall v \in \mathbb{R}, \quad \forall \phi_1, \phi_2 \in H^2 \cap H^1_0(0, L). \]

Hence, system (1.1) can be equivalently reformulated as system (3.1) with

\[ y = \left( \begin{array}{c} u \\ v \end{array} \right). \]

Let us check that our system satisfies hypothesis of Theorem 3.1. We have from the well-posedness of system (1.1) that \( B \) is an admissible control operator for \( A \). On the other hand, the family of eigenfunctions of \( A \) in \( L^2(0, L) \) is given by \( \{ \varphi_k \}_{k \in \mathbb{N}} \), defined in (2.2). It is well known that \( \{ \varphi_k \}_{k \in \mathbb{N}} \) is an orthonormal basis for \( L^2(0, L) \), and \( \{ k\varphi_k \}_{k \in \mathbb{N}} \) is an orthonormal basis for \( H^{-1}(0, L) \).

We can explicitly compute the eigenfunctions of the operator \( A \). Indeed, if we set

\[ \Phi_{1,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k, \quad \Phi_{2,k} = k \begin{pmatrix} \frac{1}{k^2-dk^2} \\ 1 \end{pmatrix} \varphi_k, \]  

(3.9)

then, it can be shown that \( \{ \Phi_{j,k} / j = 1, 2 \text{ and } k \in \mathbb{N} \} \) is a Riesz basis of \( X \), and its biorthogonal basis is given by

\[ \Psi_{1,k} = \begin{pmatrix} 1 \\ \frac{1}{dk^2-k^4} \end{pmatrix} \varphi_k, \quad \Psi_{2,k} = k \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k. \]  

(3.10)

From the definition of \( B \) we get that \( B^* \in ((H^2 \cap H^1_0(0, L))^2)^\prime \) is given by

\[ B^* \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) = -d\phi_2'(0), \]

and hence we have that

\[ B^*\Psi_{1,k} = -\frac{dck}{dk^2-k^4} \]

and

\[ B^*\Psi_{2,k} = -ck^2, \]

where \( c = \frac{\sqrt{2\pi}}{L^{\pi/2}} \). We directly get that

\[ \lim_{k \to \infty} \frac{\ln |B^*\Psi_{1,k}|}{k^4} = \lim_{k \to \infty} \frac{\ln |B^*\Psi_{2,k}|}{dk^2} = 0. \]

Hence, from Theorem 3.1 and Remark 3.2, we have the following result.
Proposition 3.3. If $c(\Lambda_d)$ is the limit of condensation of $\Lambda_d = \{dk^2, k^4\}_{k \in \mathbb{N}}$, then we have:

1. System (1.1) is null-controllable if $T > c(\Lambda_d)$.
2. System (1.1) is not null-controllable if $T < c(\Lambda_d)$.

Given that our sequence $\Lambda_d$ has two branches of eigenvalues, we will use the following characterization of $c(\Lambda_d)$.

Proposition 3.4. We have

$$c(\Lambda_d) = \max\{c_1, c_2\},$$

where

$$c_1 := \limsup_{k \to \infty} -\ln \left| \sin\left(\pi \sqrt{k \sqrt{d}}\right) \right| dk^2$$
and

$$c_2 := \limsup_{k \to \infty} -\ln \left| \sin\left(\frac{\pi k^2}{\sqrt{d}}\right) \right| k^4.$$  \hspace{1cm} (3.11)

Proof:
To begin, observe that the product (3.3) corresponding to the sequence $\Lambda_d = \{dk^2, k^4\}_{k \in \mathbb{N}}$ is given by

$$E(z) = \prod_{k \in \mathbb{N}} \left(1 - \frac{z^2}{d^2k^4}\right)\left(1 - \frac{z^2}{k^2}\right), \quad z \in \mathbb{C}.$$  

According to definition (3.7), the condensation index associated to the sequence $\Lambda_d = \{dk^2, k^4\}_{k \in \mathbb{N}}$ is given by

$$c(\Lambda_d) = \max \left\{ \limsup_{k \to \infty} -\ln \left| E'(dk^2) \right| dk^2, \limsup_{k \to \infty} -\ln \left| E'(k^4) \right| k^4 \right\}.$$  \hspace{1cm} (3.12)

Notice that using the identities

$$\sin(\pi z) = \pi z \prod_{k \in \mathbb{N}} \left(1 - \frac{z^2}{k^2}\right), \quad z \in \mathbb{C},$$
and

$$\sinh(\pi z) = \pi z \prod_{k \in \mathbb{N}} \left(1 + \frac{z^2}{k^2}\right), \quad z \in \mathbb{C},$$
we find that

$$E(z) = -i \frac{d}{\pi^6} \sin \left(\frac{\pi \sqrt{z}}{\sqrt{d}}\right) \sin(\pi \sqrt{d}) F(z), \quad z \in \mathbb{C},$$

where

$$F(z) = \frac{\sinh\left(\frac{\pi \sqrt{z}}{\sqrt{d}}\right) \sin(\pi \sqrt{d}) \sin(\pi \sqrt{z}) \sinh(\pi \sqrt{z})}{\pi \sqrt{z}^2}.$$  

It is not difficult to check that

$$E'(z) = -i \frac{d}{\pi^6} \left[\left(\frac{\pi \sqrt{z}}{2\sqrt{d}\sqrt{z}}\right) \cos\left(\frac{\pi \sqrt{z}}{\sqrt{d}}\right) \sin(\pi \sqrt{d}) + \frac{\pi}{4\sqrt{z}^3} \sin\left(\frac{\pi \sqrt{z}}{\sqrt{d}}\right) \cos(\pi \sqrt{z}) \right] F(z)$$

$$+ \sin\left(\frac{\pi \sqrt{z}}{\sqrt{d}}\right) \sin(\pi \sqrt{d}) F'(z).$$

Hence we obtain

$$|E'(dk^2)| = \left| \sin(\pi \sqrt{d} \sqrt{k}) \right| \left| \frac{F(dk^2)}{2\pi^6 k} \right|,$$  \hspace{1cm} (3.13)
and

$$|E'(k^4)| = \frac{d}{4\pi^5 k^3} \left| \frac{\pi}{\sqrt{d}} \right| |F(k^4)|.$$  \hfill (3.14)

The idea now is to find lower and upper bounds for $|F(dk^2)|$ and $|F(k^4)|$. Notice that

$$|F(dk^2)| = \frac{1}{d^2 k^4} \sinh(\pi k) \sinh(\pi \sqrt{d}) |\sin(\pi \sqrt{d}k)||\sin(\pi \sqrt{d}k)|,$$

and

$$\sinh(\pi) \sinh(\pi \sqrt{d}) \leq \sinh(\pi k) \sinh(\pi \sqrt{d}k) \leq \frac{e^{\pi k} e^{\sqrt{d} \sqrt{k}}}{2}.$$

For the other terms, we use the identities

$$\sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y),$$
$$\sinh(x + iy) = \sinh(x) \cos(y) + i \cosh(x) \sin(y),$$

which hold for every $x, y \in \mathbb{R}$. Since $\sqrt{i} = 1/\sqrt{2} + i/\sqrt{2}$, we have

$$\left| \sin \left( \frac{\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} + i \frac{\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} \right) \right| = \left| \sinh \left( \frac{\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} + i \frac{\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} \right) \right|$$
$$= \left( \sin^2 \left( \frac{\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} \right) \cosh^2 \left( \frac{\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} \right) + \cos^2 \left( \frac{\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} \right) \sinh^2 \left( \frac{\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} \right) \right)^{1/2},$$

from where we obtain

$$\sinh^2 \left( \frac{\pi \sqrt{d}}{\sqrt{2}} \right) \leq \left| \sin \left( \frac{\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} \right) \right| \left| \sinh \left( \frac{\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} \right) \right| \leq \exp \left( \frac{2\pi \sqrt{d} \sqrt{k}}{\sqrt{2}} \right),$$

and therefore

$$\sinh(\pi) \sinh(\pi \sqrt{d}) \sinh^2 \left( \frac{\pi \sqrt{d}}{\sqrt{2}} \right) \leq |F(dk^2)| \leq \frac{e^{\pi k} e^{\sqrt{d} \sqrt{k}} e^{\sqrt{2} \pi \sqrt{d} \sqrt{k}}}{4 d^2 k^4}. \hfill (3.15)$$

Now, we have that

$$|F(k^4)| = \frac{1}{k^3} \sinh \left( \frac{\pi k^2}{\sqrt{d}} \right) \sinh(\pi k) |\sin(\pi \sqrt{d}k)||\sinh(\pi \sqrt{d}k)|.$$

We can use the same estimates as before to obtain

$$\sinh \left( \frac{\pi}{\sqrt{d}} \right) \sinh(\pi) \sinh^2 \left( \frac{\pi}{\sqrt{2}} \right) \leq |F(k^4)| \leq \frac{e^{\pi k} e^{\frac{\pi}{\sqrt{2}}} \sqrt{k} e^{\pi \sqrt{d} \sqrt{k}}}{4 k^3}.$$

(3.16)

In order to finish the proof, it suffices to combine estimates (3.15) and (3.16) with (3.13) and (3.14) in the expression (3.12).
3.3 Proof of Theorem 1.6

Notice that from Propositions 3.3 and 3.4, it suffices to show that \( c_1 = c_2 = 0 \). From Definition 1.4, we directly get that, if the irrational number \( x \) has a Liouville-Roth constant \( \mu \) and \( n > \mu \), then there exists a constant \( C \) such that
\[
| x - \frac{p}{q} | \geq \frac{C}{q^n},
\] (3.17)
for any integers \( p \) and \( q \) with \( q > 0 \).

Let us first prove that \( c_1 = 0 \). For each \( k \in \mathbb{N} \), let \( h_k \in \mathbb{N} \) be such that
\[
| \sqrt{k} \sqrt{d} - h_k | \leq 1/2.
\] (3.18)
From (3.17) applied to \( x = \sqrt{d} \) with \( p = h_k \) and \( q = k \), there exist \( n \in \mathbb{N} \) and \( C > 0 \) such that
\[
| \sqrt{d} - \frac{h_k^2}{k} | \geq \frac{C}{k^n}.
\] (3.19)
Hence, from (3.18) and (3.19) we have
\[
\frac{1}{2} \geq | \sqrt{k} \sqrt{d} - h_k | = \left| \frac{k \sqrt{d} - h_k^2}{k \sqrt{d} + h_k} \right| \geq \frac{C}{k^{n-1}(\sqrt{k} \sqrt{d} + h_k)}.
\] (3.20)
It is not difficult to check from (3.20) that
\[
| \sin(\pi(\sqrt{k} \sqrt{d})) | = | \sin(\pi(\sqrt{k} \sqrt{d} - h_k)) | = | \sin(\pi(\sqrt{k} \sqrt{d} - h_k)) | 
\geq \sin\left( \frac{C \pi}{k^{n-1}(\sqrt{k} \sqrt{d} + h_k)} \right),
\] (3.21)
where the inequality comes from the fact that \( \sin x \) is increasing in the interval \( [0, \pi/2] \). Then
\[
-\ln | \sin(\pi(\sqrt{k} \sqrt{d})) | \leq -\ln | \sin(\pi k^{1-n}(\sqrt{k} \sqrt{d} + h_k)^{-1}) |.
\] (3.22)
From (3.18), we check that
\[
\limsup_{k \to \infty} - \ln \frac{| \sin(\pi k^{1-n}(\sqrt{k} \sqrt{d} + h_k)^{-1}) |}{dk^2} = \limsup_{k \to \infty} - \ln \frac{| \sin(\pi k^{1-n}(\sqrt{k} \sqrt{d} + h_k)^{-1}) |}{dk^2} = 0.
\]
Then, thanks to this last identity, (3.22) and (3.11), we conclude that \( c_1 = 0 \).

In a similar way, we prove that \( c_2 = 0 \). Indeed, for \( k \in \mathbb{N} \), let \( h_k \in \mathbb{N} \) be such that
\[
| \sqrt{d} - h_k | \leq 1/2.
\]
From (3.17) with \( x = 1/\sqrt{d} \), \( p = h_k \) and \( q = k^2 \), there exist \( n \in \mathbb{N} \) and \( C > 0 \) such that
\[
| \frac{1}{\sqrt{d}} - \frac{h_k}{k^2} | \geq \frac{C}{k^{2n}}.
\]
Then,
\[
\frac{1}{2} \geq | \frac{k^2}{\sqrt{d}} - h_k | \geq \frac{C \pi}{k^{2n-2}},
\]
and therefore
\[
| \sin(\pi(\frac{k^2}{\sqrt{d}})) | = | \sin(\pi(\frac{k^2}{\sqrt{d}} - h_k)) | = | \sin(\pi(\frac{k^2}{\sqrt{d}} - h_k)) | \geq \sin\left( \frac{C \pi}{k^{2n-2}} \right).
\] (3.23)
Using the same argument as for \( c_1 \), we obtain from (3.23) and (3.11) that \( c_2 = 0 \). This ends the proof of Theorem 1.6.
3.4 Proof of Theorem 1.7

We shall prove that, for any given $\lambda_0 > 0$, there exists $d > 0$ such that $c_1 = c_2 = \lambda_0$. Once this has been achieved, we conclude the proof of Theorem 1.7 thanks to Propositions 3.3 and 3.4.

The following result, coming from approximation of irrational numbers by continued fractions, is the main part of the proof.

**Lemma 3.5.** Let $\lambda_0$ be any positive real number. There exist an irrational number $d > 0$ and a sequence of rational numbers $\{p_k/q_k\}_{k \in \mathbb{N}}$ such that $p_k$ and $q_k$ are co-prime positive integers, the sequences $\{p_k\}_{k \in \mathbb{N}}$ and $\{q_k\}_{k \in \mathbb{N}}$ are strictly increasing, and

$$\lim_{k \to \infty} e^{\lambda_0 p_k^4} \left| \sqrt[d]{d} - \frac{p_k}{q_k} \right| = 1. \quad (3.24)$$

**Proof:** The proof is similar to [3, Lemma 6.22] and therefore omitted here.

Let $\lambda_0$ be any positive number and let $d > 0$ given by Lemma 3.5. First, we prove that $c_1, c_2 \geq \lambda_0$. From the definition of $c_1$ in (3.11), and since $\{q_k^2\}_{k \in \mathbb{N}} \subset \{k\}_{k \in \mathbb{N}}$, we have

$$c_1 \geq \limsup_{k \to \infty} -\ln \left| \frac{\sin(\pi (q_k \sqrt[d]{d} - p_k))}{dq_k^4} \right|, \quad (3.25)$$

where $\{p_k\}_{k \in \mathbb{N}}$ and $\{q_k\}_{k \in \mathbb{N}}$ are the sequences given by Lemma 3.5. From (3.24), we see that

$$\lim_{k \to +\infty} \frac{p_k}{q_k} = \sqrt[d]{d} \quad (3.26)$$

and furthermore, since $\{p_k\}_{k \in \mathbb{N}}$ and $\{q_k\}_{k \in \mathbb{N}}$ are increasing,

$$\lim_{k \to +\infty} (q_k \sqrt[d]{d} - p_k) = 0. \quad (3.27)$$

Then,

$$\limsup_{k \to \infty} \frac{-\ln \left| \sin(\pi (q_k \sqrt[d]{d} - p_k)) \right|}{dq_k^4} = \limsup_{k \to \infty} \frac{-\ln \left| \pi (q_k \sqrt[d]{d} - p_k) \right|}{dq_k^4} = \limsup_{k \to \infty} \frac{-\ln \left| \pi q_k e^{-\lambda_0 p_k^4} \right|}{dq_k^4} = \lambda_0,$$

which together with (3.25) gives $c_1 \geq \lambda_0$.

Following a similar argument, we have that

$$c_2 \geq \limsup_{k \to \infty} \frac{-\ln \left| \sin \left( \frac{\pi}{\sqrt[d]{d}}(p_k^2 - q_k^2 \sqrt[d]{d}) \right) \right|}{p_k^4} = \limsup_{k \to \infty} \frac{-\ln \left| \frac{\pi}{\sqrt[d]{d}} q_k^2 (p_k \sqrt[d]{d} + \sqrt[d]{d}) e^{-\lambda_0 p_k^4} \right|}{p_k^4} = \lambda_0.$$

We now prove $c_1, c_2 \leq \lambda_0$. For each $k \in \mathbb{N}$, there exists $h_k \in \mathbb{N}$ such that $|\sqrt[k]{d} \sqrt[d]{d} - h_k| \leq 1/2$, and then

$$|\sin(\pi \sqrt[k]{d} \sqrt[d]{d})| = |\sin(\pi (\sqrt[k]{d} \sqrt[d]{d} - h_k))| = |\sin(\pi (\sqrt[k]{d} \sqrt[d]{d} - h_k))|.$$

As we have already proved that $c(\Lambda_d) \geq \lambda_0 > 0$, we deduce from Theorem 1.6 that $\sqrt[d]{d}$ is not an algebraic number. Therefore, for each $k \in \mathbb{N}$ we have $|\sqrt[k]{d} \sqrt[d]{d} - h_k| > 0$, and therefore there exists $n_k \in \mathbb{N}$ such that

$$|q_{n_k} \sqrt[d]{d} - p_{n_k}| < |\sqrt[k]{d} \sqrt[d]{d} - h_k|, \quad \text{and} \quad q_{n_k} \geq \sqrt[k]{k}, \quad (3.29)$$
where \( \{p_k\}_{k \in \mathbb{N}} \) and \( \{q_k\}_{k \in \mathbb{N}} \) are the sequences given by Lemma 3.5. From (3.28) and (3.29) we conclude, using the argument to obtain (3.21)-(3.22) that

\[
c_1 \leq \limsup_{k \to \infty} -\frac{\ln |\sin(\pi (q_n \sqrt{d} - p_n))|}{dq_n^4} = \limsup_{k \to \infty} -\frac{\ln |\pi (q_n \sqrt{d} - p_n)|}{dq_n^4} = \lambda_0.
\]

Similarly, taking into account that \((\sqrt{d})^{-1}\) is not an algebraic number, for each \(k \in \mathbb{N}\) there exists \(g_k \in \mathbb{N}\) such that

\[
0 < \left| \frac{k^2}{\sqrt{d}} - g_k \right| \leq \frac{1}{2},
\]

and then there exists \(n_k \in \mathbb{N}\) such that

\[
\left| q_n \sqrt{d} - p_n \right| < \left| \frac{k^2}{\sqrt{d}} - g_k \right| \text{ and } p_n \geq k. \tag{3.30}
\]

Hence

\[
c_2 = \limsup_{k \to \infty} -\frac{\ln |\sin(\pi (q_n \sqrt{d} - p_n))|}{p_n^4} \leq \limsup_{k \to \infty} -\frac{\ln |\pi (q_n \sqrt{d} - p_n)|}{p_n^4} = \lambda_0.
\]

Therefore, \(c_1, c_2 \leq \lambda_0\), thus \(c(\Lambda_d) = \lambda_0\). This ends the proof of Theorem 1.7 taking

\[
T_0 := c(\Lambda_d) = \lambda_0.
\]

### 3.5 Proof of Theorem 1.8

From Propositions 3.3 and 3.4, it is enough to prove that \(c(\Lambda_d) = +\infty\) for some \(d > 0\). Such a parameter will be given by the following result:

**Lemma 3.6.** There exist an irrational number \(d > 0\) and a sequence of rational numbers \(\{p_k/q_k\}_{k \in \mathbb{N}}\) such that \(p_k\) and \(q_k\) are co-prime positive integers, the sequences \(\{p_k\}_{k \in \mathbb{N}}\) and \(\{q_k\}_{k \in \mathbb{N}}\) are strictly increasing, and

\[
\lim_{k \to \infty} e^{p_k} \left| \sqrt{d} - \frac{p_k}{q_k} \right| = 0. \tag{3.31}
\]

**Proof:** The proof is similar to [3, Lemma 6.22] and therefore omitted here.

Let \(d > 0\), \(\{p_k\}_{k \in \mathbb{N}}\) and \(\{q_k\}_{k \in \mathbb{N}}\) be given by Lemma 3.6. We will check that \(c_1 = +\infty\), which gives directly that \(c(\Lambda_d) = +\infty\).

Since \(\{q_k^2\}_{k \in \mathbb{N}}\) is a subsequence of the natural numbers, from (3.11) we have that

\[
c_1 \geq \limsup_{k \to \infty} -\frac{\ln |\sin(\pi (q_k \sqrt{d} - p_k))|}{dq_k^4}. \tag{3.32}
\]

From (3.31), we can prove that the sequences \(\{p_k\}_{k \in \mathbb{N}}\) and \(\{q_k\}_{k \in \mathbb{N}}\) satisfy (3.26), (3.27), and, furthermore, that there exist \(C > 0\) and \(k_0 \in \mathbb{N}\) such that

\[
|q_k \sqrt{d} - p_k| \leq Ce^{-p_k^2}, \quad \forall k \geq k_0.
\]
Therefore,
\[
\limsup_{k \to \infty} -\ln \left( \frac{\sin(\pi(q_k \sqrt{d} - p_k))}{dq_k^4} \right) = \limsup_{k \to \infty} -\ln \left( \frac{\pi(q_k \sqrt{d} - p_k)}{dq_k^4} \right) \geq \limsup_{k \to \infty} -\ln \left( \pi C q_k e^{-p_k^5} \right).
\]
This inequality, (3.26) and (3.32) yield
\[
c_1 \geq \limsup_{k \to \infty} \frac{p_k^5}{dq_k^4} = +\infty,
\]
since \( \{p_k\}_{k \in \mathbb{N}} \) is a strictly increasing sequence. This ends the proof of Theorem 1.8.

References


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