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Local controllability of the stabilized Kuramoto–Sivashinsky system by a single control acting on the heat equation

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ABSTRACT

In this paper we consider the so called stabilized Kuramoto–Sivashinsky system which couples a fourth order and a second order parabolic equations. We prove that this system is locally controllable to the trajectories by a single distributed control acting only on the heat equation. The main novelty is a new Carleman inequality for the solutions of a linear Kuramoto–Sivashinsky equation with nonhomogeneous boundary conditions.

RÉSUMÉ

Dans cet article on considère le système de Kuramoto–Sivashinsky stabilisé couplant des équations paraboliques de deuxième et quatrième ordres. On démontre que ce système est localement contrôlable aux trajectoires en utilisant un seul contrôle distribué agissant sur l'équation de la chaleur. La principale nouveauté est une nouvelle inégalité de Carleman pour les solutions de l'équation de Kuramoto–Sivashinsky linéaire avec des conditions aux bords nonhomogènes.

1. Introduction

Let T and L be two positive real numbers and $Q := (0, L) \times (0, T)$. We consider the stabilized Kuramoto–Sivashinsky (KS) system

$$\begin{cases} y_t + \gamma y_{xxxx} + y_{xxx} + a y_{xx} + y y_x = z_x & \text{in } Q, \\ z_t - \Gamma z_{xx} + c z_x = y_x + v \mathbb{1}_\omega & \text{in } Q, \\ y(0,t) = y_x(0,t) = 0, \quad y(L,t) = y_x(L,t) = 0 & \text{in } (0,T), \\ z(0,t) = 0, \quad z(L,t) = 0 & \text{in } (0,T), \\ y(x,0) = y_0(x), \quad z(x,0) = z_0(x) & \text{in } (0,L), \end{cases} \tag{1.1}$$

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where $\gamma, a, \Gamma > 0$ and $c \in \mathbb{R}$. Here, (y, z) is the state of the system and v is the control which acts on a small subset ω of $(0, L)$ called the *control domain*.

As it is stated in [25], system (1.1) serves as a one-dimensional model for turbulence and wave propagation in reaction–diffusion systems combining dissipative and dispersive features.

The controllability of this system has been studied in the context of boundary controls in [8]. We refer to [6,7,27] and [19,23] for controllability results concerning the KS and heat equations, respectively. Regarding systems coupling heat equations, there exists by now a vast literature. We refer for instance to [2,26] and the references therein.

Recently, in [9] the authors have considered system (1.1) with a distributed control located only on the fourth order equation and proved a local null controllability result. The present paper is meant to fill the gap in what concerns the controllability of system (1.1) with only one distributed control.

In this article we deal with the *controllability to the trajectories*, i.e., given $(\bar{y}, \bar{z})$ a solution of

$$\begin{cases} \bar{y}_t + \gamma \bar{y}_{xxxx} + \bar{y}_{xxx} + a\bar{y}_{xx} + \bar{y}\bar{y}_x = \bar{z}_x & \text{in } Q, \\ \bar{z}_t - \Gamma \bar{z}_{xx} + c\bar{z}_x = \bar{y}_x & \text{in } Q, \\ \bar{y}(0,t) = \bar{y}_x(0,t) = 0, \quad \bar{y}(L,t) = \bar{y}_x(L,t) = 0 & \text{in } (0,T), \\ \bar{z}(0,t) = 0, \quad \bar{z}(L,t) = 0 & \text{in } (0,T), \end{cases} \tag{1.2}$$

the goal is to find a control v steering the solution of system (1.1) to $(\bar{y}, \bar{z})$ at time T. More precisely, we prove the following result:

Theorem 1.1. *Let $T > 0$, $\emptyset \neq \omega \subset (0, L)$ and a solution $(\bar{y}, \bar{z})$ of system (1.2) such that $\bar{y} \in L^\infty(Q)$. Then, there exists $\delta > 0$ such that for any initial conditions $y_0 \in H^{-2}(0, L)$ and $z_0 \in H^{-1}(0, L)$ verifying*

$$\|y_0 - \bar{y}(\cdot, 0)\|_{H^{-2}(0,L)} + \|z_0 - \bar{z}(\cdot, 0)\|_{H^{-1}(0,L)} \leq \delta,$$

there exists a control $v \in L^2(\omega \times (0, L))$ such that the solution $(y, z) \in L^2(Q)^2 \cap \mathcal{C}([0, T]; H^{-2}(0, L) \times H^{-1}(0, L))$ of system (1.1) satisfies

$$y(\cdot, T) = \bar{y}(\cdot, T) \text{ and } z(\cdot, T) = \bar{z}(\cdot, T) \text{ in } (0, L).$$

Remark 1.2. The regularity $L^\infty(Q)$ of $\bar{y}$ improves the one in [7], where $\bar{y}$ is asked to belong to $L^\infty(0, T; H_0^2(0, L))$, in the sense that this is the minimum regularity of $\bar{y}$ for which we are able to prove Theorem 1.1.

As it is usual for this kind of problems, we first prove a null controllability result for the linearized system around $(\bar{y}, \bar{z})$, which is given by

$$\begin{cases} y_t + \gamma y_{xxxx} + y_{xxx} + a y_{xx} + \bar{y} y_x + \bar{y}_x y = f_0 + z_x & \text{in } Q, \\ z_t - \Gamma z_{xx} + c z_x = f_1 + y_x + v \mathbb{1}_\omega & \text{in } Q, \\ y(0,t) = y_x(0,t) = 0, \quad y(L,t) = y_x(L,t) = 0 & \text{in } (0,T), \\ z(0,t) = 0, \quad z(L,t) = 0 & \text{in } (0,T), \\ y(x,0) = y_0(x), \quad z(x,0) = z_0(x) & \text{in } (0,L), \end{cases} \tag{1.3}$$

where f_0 and f_1 satisfy decreasing properties near $t = T$. The main tool to obtain this result is an appropriate *Carleman estimate* for the solutions of the nonhomogeneous adjoint system

$$\begin{cases} -\varphi_t + \gamma\varphi_{xxxx} - \varphi_{xxx} + a\varphi_{xx} - \bar{y}\varphi_x = -\psi_x + g_0 & \text{in } Q, \\ -\psi_t - \Gamma\psi_{xx} - c\psi_x = -\varphi_x + g_1 & \text{in } Q, \\ \varphi(0,t) = \varphi_x(0,t) = 0, \quad \varphi(L,t) = \varphi_x(L,t) = 0 & \text{in } (0,T), \\ \psi(0,t) = 0, \quad \psi(L,t) = 0 & \text{in } (0,T), \\ \varphi(x,T) = \varphi_T(x), \quad \psi(x,T) = \psi_T(x) & \text{in } (0,L), \end{cases} \tag{1.4}$$

with $g_0, g_1 \in L^2(Q)$. This Carleman estimate looks like

$$\iint_Q \rho_1 \big(|\varphi|^2 + |\psi|^2\big)\, \mathrm{d}x\, \mathrm{d}t \le C \iint_Q \rho_2 \big(|g_0|^2 + |g_1|^2\big)\, \mathrm{d}x\, \mathrm{d}t + C \iint_{\omega\times(0,T)} \rho_3 |\psi|^2\, \mathrm{d}x\, \mathrm{d}t$$

with $\rho_i = \rho_i(x,t)$ some positive weight functions. The idea to prove this type of inequalities is to combine Carleman estimates for φ and ψ, and then use the coupling between the equations to eliminate the local term of φ (see, for instance, [21,9]). Since the coupling occurs at the first order, we need a local term like

$$\iint_{\omega\times(0,T)} \rho_3 |\varphi_x|^2\, \mathrm{d}x\, \mathrm{d}t.$$

However, one can never obtain an estimate as

$$\iint_{\omega^*\times(0,T)} \rho_3 |\varphi|^2\, \mathrm{d}x\, \mathrm{d}t \le C \iint_{\omega\times(0,T)} \rho_3 |\varphi_x|^2\, \mathrm{d}x\, \mathrm{d}t, \quad \omega^* \subset \omega,$$

due to the lack of homogeneous boundary conditions in ω^*. This obstruction motivates the use of a Carleman estimate for the fourth order parabolic equation with nonhomogeneous boundary data (see Theorem 3.5). To our knowledge, such an estimate has not been proven for a fourth-order equation and is the second main result and novelty of this article.

To end this introductory section, let us mention how this work is organized. First, in Section 2, we recall some well-posedness results concerning systems (1.1), (1.3) and (1.4). Next, Section 3 is devoted to Carleman inequalities. In particular, we give the proof of our new Carleman estimate. In Sections 4 and 5, we deal with the null controllability of the linear and nonlinear systems, respectively. Finally, in Section 6, we make final comments about this work and discuss some open problems.

2. Some well-posedness results

Let us begin with an existence and uniqueness result concerning the adjoint system (1.4). From the proofs of [8, Theorem 2.1] and [9, Proposition 2.1], one can readily deduce the following proposition.

Proposition 2.1. *Let $g_0 \in L^2(Q)$, $g_1 \in L^2(Q)$, $\varphi_T \in H_0^2(0,L)$ and $\psi_T \in H_0^1(0,L)$. There exists a unique solution (φ, ψ) to system (1.4) that belongs to the space*

$$L^2(0,T; H^4(0,L) \times H^2(0,L)) \cap \mathcal{C}([0,T]; H_0^2(0,L) \times H_0^1(0,L))$$

and, there exists a constant $C > 0$ (independent of T, φ and ψ) such that

$$\begin{aligned} &\|\varphi\|^2_{L^2(0,T;H^4(0,L))\cap L^\infty(0,T;H_0^2(0,L))} + \|\psi\|^2_{L^2(0,T;H^2(0,L))\cap L^\infty(0,T;H_0^1(0,L))} \\ &\quad \le C^* \big(\|g_0\|^2_{L^2(Q)} + \|g_1\|^2_{L^2(Q)} + \|\varphi_T\|^2_{H^2(0,L)} + \|\psi_T\|^2_{H^1(0,L)}\big), \end{aligned} \tag{2.1}$$

where

$$C^* = C(1 + \|\bar{y}\|_\infty^2) \exp\big(C(1 + \|\bar{y}\|_\infty^2)T\big).$$

Furthermore, if $g_0 \in L^1(0,T;H_0^2(0,L))$ *and* $g_1 \in L^1(0,T;H_0^1(0,L))$*, then*

$$\begin{aligned}
&\|\varphi\|^2_{L^2(0,T;H^4(0,L))\cap L^\infty(0,T;H_0^2(0,L))} + \|\psi\|^2_{L^2(0,T;H^2(0,L))\cap L^\infty(0,T;H_0^1(0,L))} \\
&\quad \le C^*\big(\|g_0\|^2_{L^1(0,T;H_0^2(0,L))} + \|g_1\|^2_{L^1(0,T;H_0^1(0,L))} + \|\varphi_T\|^2_{H^2(0,L)} + \|\psi_T\|^2_{H^1(0,L)}\big).
\end{aligned} \tag{2.2}$$

The solutions of the linear system (1.3) will be understood in the sense of *transposition.*

Definition 2.2. Given $y_0 \in H^{-2}(0,L)$, $z_0 \in H^{-1}(0,L)$, $f_0 \in L^1(0,T;W^{-1,1}(0,L))$, $f_1 \in L^2(0,T;H^{-1}(0,L))$ and $v \in L^2(\omega\times(0,T))$. We say that $(y,z) \in L^2(Q)^2$ is a solution (in the sense of transposition) of system (1.3) if, for any $(g_0,g_1) \in L^2(Q)^2$, it satisfies

$$\begin{aligned}
\iint\limits_Q \big(yg_0 + zg_1\big)\,\mathrm{d}x\,\mathrm{d}t &= \iint\limits_{\omega\times(0,T)} v\psi\,\mathrm{d}x\,\mathrm{d}t + \langle f_0,\varphi\rangle_{L^1(0,T;W^{-1,1}(0,L)),L^\infty(0,T;W^{1,\infty}(0,L))} \\
&\quad + \langle f_1,\psi\rangle_{L^2(0,T;H^{-1}(0,L)),L^2(0,T;H_0^1(0,L))} + \langle y_0,\varphi(\cdot,0)\rangle_{H^{-2}(0,L),H_0^2(0,L)} + \langle z_0,\psi(\cdot,0)\rangle_{H^{-1}(0,L),H_0^1(0,L)},
\end{aligned}$$

where (φ,ψ) is the unique solution of (1.4) with $(\varphi_T,\psi_T) \equiv (0,0)$.

The following result states the existence and uniqueness of solutions of system (1.3). It is a direct consequence of Riesz's representation theorem and Proposition 2.1.

Proposition 2.3. *(See [9, Theorem 2.4].) Let* $y_0 \in H^{-2}(0,L)$*,* $z_0 \in H^{-1}(0,L)$*,* $f_0 \in L^1(0,T;W^{-1,1}(0,L))$*,* $f_1 \in L^2(0,T;H^{-1}(0,L))$ *and* $v \in L^2(\omega\times(0,T))$*. There exists a unique solution* $(y,z) \in L^2(Q)^2$ *of system (1.3). Furthermore, this solution belongs to the space* $\mathcal{C}([0,T];H^{-2}(0,L)\times H^{-1}(0,L))$ *and there exists a constant* $C > 0$ *such that*

$$\begin{aligned}
&\|y\|^2_{L^2(Q)\cap L^\infty(0,T;H^{-2}(0,L))} + \|z\|^2_{L^2(Q)\cap L^\infty(0,T;H^{-1}(0,L))} \\
&\quad \le C\big(\|f_0\|^2_{L^1(0,T;W^{-1,1}(0,L))} + \|f_1\|^2_{L^2(0,T;H^{-1}(0,L))} + \|y_0\|^2_{H^{-2}(0,L)} + \|z_0\|^2_{H^{-1}(0,L)}\big).
\end{aligned}$$

Finally, let us recall a result concerning the well posedness of system (1.1). This result is proved in [9] (see also [8]).

Proposition 2.4. *(See [9, Theorem 2.5].) There exists* $\delta > 0$ *such that for any* $y_0 \in H^{-2}(0,L)$*,* $z_0 \in H^{-1}(0,L)$ *and* $v \in L^2(\omega\times(0,T))$ *such that*

$$\|y_0\|_{H^{-2}(0,L)} + \|z_0\|_{H^{-1}(0,L)} + \|v\|_{L^2(\omega\times(0,T))} \le \delta,$$

the system (1.1) has a unique solution $(y,z) \in L^2(Q)^2 \cap \mathcal{C}([0,T];H^{-2}(0,L)\times H^{-1}(0,L))$*.*

3. Carleman estimates

The objective of this section is to prove an appropriate Carleman inequality for the solutions of the adjoint system (1.4). We need first to introduce some weight functions. Let $\eta(x) \in \mathcal{C}^4([0,L])$ be a function that satisfies

$$\eta(x) > 0 \quad \forall x \in (0,L), \quad \eta(0) = \eta(L) = 0 \tag{3.1}$$

and

$$|\eta'(x)| \geq \delta > 0 \quad \forall x \in [0, L] \setminus \omega_0, \tag{3.2}$$

for some $\omega_0 \Subset \omega \subset (0, L)$. Notice that (3.1) and (3.2) imply that

$$\eta'(0) \geq \delta \text{ and } -\eta'(L) \geq \delta. \tag{3.3}$$

The existence of such a function η (in higher dimension) is given in [19] (see also [12]). Now, let us define

$$\begin{aligned}
&\alpha(x,t) := \frac{e^{k\frac{m+1}{m}\lambda\|\eta\|_\infty} - e^{\lambda(k\|\eta\|_\infty + \eta(x))}}{t^m(T-t)^m}, \quad \xi(x,t) := \frac{e^{\lambda(k\|\eta\|_\infty+\eta(x))}}{t^m(T-t)^m},\\
&\alpha^*(t) := \max_{x\in[0,L]} \alpha(x,t) = \alpha(0,t) = \alpha(L,t), \quad \xi^*(t) := \min_{x\in[0,L]} \xi(x,t) = \xi(0,t) = \xi(L,t),\\
&\widehat{\alpha}(t) := \min_{x\in[0,L]} \alpha(x,t), \quad \widehat{\xi}(t) := \max_{x\in[0,L]} \xi(x,t),
\end{aligned} \tag{3.4}$$

where $\lambda > 1$, $k > m > 0$. Weights functions like (3.4) have been used extensively over the last 20 years. Notice that we have

$$|\alpha_t| \leq CT\xi^{1+1/m}, \quad 1 \leq CT^{2m}\xi, \quad |\partial_x^n \xi| \leq C\lambda^n \xi \tag{3.5}$$

for any $n \in \mathbb{N}$ and some positive constant C independent of λ.

We recall now two Carleman estimates that will be used in the following. The first one concerns the solutions of the heat equation that has been proved in [19].

Lemma 3.1. *Let $f \in L^2(Q)$, $m \geq 1$ and $\omega \subset (0, L)$. Then, there exist $\lambda_0 > 0$ and $C > 0$ (depending only on ω) such that any solution u of*

$$\begin{cases} u_t - u_{xx} = f & \text{in } Q,\\ u(0,t) = 0, \quad u(L,t) = 0 & \text{in } (0,T),\\ u(x,0) = u_0(x) & \text{in } (0,L), \end{cases} \tag{3.6}$$

satisfies

$$\begin{aligned}
&\iint_Q e^{-2s\alpha}\Big(s^5\lambda^6\xi^5(|u_t|^2 + |u_{xx}|^2) + s^7\lambda^8\xi^7|u_x|^2 + s^9\lambda^{10}\xi^9|u|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad \leq Cs^6\lambda^6 \iint_Q e^{-2s\alpha}\xi^6|f|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^9\lambda^{10} \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^9|u|^2\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.7}$$

for every $\lambda \geq \lambda_0$ and $s \geq C(T^{2m} + T^{2m-1})$.

Remark 3.2. The powers of the weights ξ present in (3.7) are different from the ones in [19]. However, this version of the inequality can be easily achieved with the change of variables $\tilde{u} := \xi^3 u$.

The next result is a Carleman inequality proved in [9] for the Kuramoto–Sivashinsky equation with homogeneous Dirichlet boundary conditions.

Lemma 3.3. *Let $f \in L^2(Q)$, $m \geq 2/5$ and $\omega \subset (0, L)$. Then, there exist $\lambda_0 > 0$ and $C > 0$ (depending only on ω) such that any solution u of*

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} = f & \text{in } Q, \\ u(0,t) = u_x(0,t) = 0, \quad u(L,t) = u_x(L,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,L) \end{cases} \tag{3.8}$$

satisfies

$$\begin{aligned} & s^{-1} \iint\limits_Q e^{-2s\alpha} \xi^{-1} \left(|u_t|^2 + |u_{xxxx}|^2 \right) \mathrm{d}x \, \mathrm{d}t \\ & \quad + \iint\limits_Q e^{-2s\alpha} \left(s\lambda^2 \xi |u_{xxx}|^2 + s^3 \lambda^4 \xi^3 |u_{xx}|^2 + s^5 \lambda^6 \xi^5 |u_x|^2 + s^7 \lambda^8 \xi^7 |u|^2 \right) \mathrm{d}x \, \mathrm{d}t \\ & \leq C \iint\limits_Q e^{-2s\alpha} |f|^2 \, \mathrm{d}x \, \mathrm{d}t + C s^7 \lambda^8 \iint\limits_{\omega \times (0,T)} e^{-2s\alpha} \xi^7 |u|^2 \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{3.9}$$

for every $\lambda \geq \lambda_0$ and $s \geq C(T^{2m} + T^{2m-2/5})$.

Remark 3.4. In [9], inequality (3.9) is proved using the same weight functions (3.4) with $m \geq 3$ and in [7], the boundary observation version of (3.9) is proved with $m = 1$. Now, a closer look to the proof in these works shows that these inequalities are still valid for any $m \geq 2/5$ and $s \geq C(T^{2m} + T^{2m-2/5})$, taking into consideration the properties (3.5). However, a slight change in the proofs present in [9] and [7] will show that the optimal power is actually $m = 1/3$, which improves the estimation of the *cost of null controllability* of the linear KS equation. Further applications of this fact are beyond the scope of this paper (see Section 6). Additionally, for the sake of completeness, we have added the term u_t in the inequality which is not present in [7] nor in [9]. Again, this is readily seen from the proofs in these references.

3.1. Carleman estimate with nonhomogeneous boundary conditions

Let us state one of the main results of this paper. It concerns a new Carleman estimate for the KS equation with nonhomogeneous boundary conditions and right-hand side in $L^2(0,T; H^{-2}(0,L))$.

Theorem 3.5. *Let $B_0, B_1, B_2 \in L^2(Q)$, $b_1, b_2, b_3, b_4 \in L^2(0,T)$, $m \geq 2/5$ and $\omega \subset (0, L)$. There exist $\lambda_0 > 0$ and $C > 0$ (depending only on ω, γ and a) such that any solution u of*

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} = B_0 + \partial_x B_1 + \partial_x^2 B_2 & \text{in } Q, \\ u(0,t) = b_1(t), \quad u(L,t) = b_2(t) & \text{in } (0,T), \\ u_x(0,t) = b_3(t), \quad u_x(L,t) = b_4(t) & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,L) \end{cases} \tag{3.10}$$

satisfies, for every $\lambda \geq \lambda_0$ and $s \geq C(T^{2m} + T^{2m-2/5})$,

$$\begin{aligned} s^7 \lambda^8 \iint\limits_Q e^{-2s\alpha} \xi^7 |u|^2 \, \mathrm{d}x \, \mathrm{d}t & \leq C \iint\limits_Q e^{-2s\alpha} \Big(|B_0|^2 + s^2 \lambda^2 \xi^2 |B_1|^2 + s^4 \lambda^4 \xi^4 |B_2|^2 \Big) \mathrm{d}x \, \mathrm{d}t \\ & \quad + C s^7 \lambda^7 \int_0^T e^{-2s\alpha^*} (\xi^*)^7 (|b_1|^2 + |b_2|^2) \, \mathrm{d}t + C s^5 \lambda^5 \int_0^T e^{-2s\alpha^*} (\xi^*)^5 (|b_3|^2 + |b_4|^2) \, \mathrm{d}t \end{aligned}$$

$$+ Cs^7\lambda^8 \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^7|u|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{3.11}$$

The proof of Theorem 3.5 is based on a duality argument introduced in [22] (see also [17]). It is clear that given the regularity of the boundary conditions, we cannot expect to have stronger norms in the left-hand side of (3.11), even with homogeneous right-hand sides. Nevertheless, this will be enough to our purposes. It would be interesting to prove that, given more regular data on the boundary, norms of derivatives of the solution can be obtained in the left-hand side of (3.11).

Proof. We start by viewing u as a solution by transposition of (3.10), that is, u is the unique function in $L^2(Q)$ that satisfies

$$\begin{aligned}\iint\limits_Q u\,g\,\mathrm{d}x\,\mathrm{d}t &= \iint\limits_Q B_0\,w\,\mathrm{d}x\,\mathrm{d}t - \iint\limits_Q B_1\,w_x\,\mathrm{d}x\,\mathrm{d}t + \iint\limits_Q B_2\,w_{xx}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \int_0^T b_1\left(\gamma w_{xxx}(0,t) - w_{xx}(0,t)\right)\mathrm{d}t + \int_0^T b_2\left(\gamma w_{xxx}(L,t) - w_{xx}(L,t)\right)\mathrm{d}t\\ &\quad + \gamma\int_0^T b_3\,w_{xx}(0,t)\,\mathrm{d}t - \gamma\int_0^T b_4\,w_{xx}(L,t)\,\mathrm{d}t + \int_0^L u_0\,w(x,0)\,\mathrm{d}x,\end{aligned} \tag{3.12}$$

for every $g \in L^2(Q)$, where w is the solution of

$$\begin{cases} -w_t + \gamma w_{xxxx} - w_{xxx} + a w_{xx} = g & \text{in } Q,\\ w(0,t) = w_x(0,t) = 0, \quad w(L,t) = w_x(L,t) = 0 & \text{in } (0,T),\\ w(x,T) = 0 & \text{in } (0,L). \end{cases} \tag{3.13}$$

Let us consider the following null controllability problem: find (w,h) such that

$$\begin{cases} -w_t + \gamma w_{xxxx} - w_{xxx} + a w_{xx} = s^7\lambda^8\xi^7 e^{-2s\alpha}u + h\mathbb{1}_\omega & \text{in } Q,\\ w(0,t) = w_x(0,t) = 0, \quad w(L,t) = w_x(L,t) = 0 & \text{in } (0,T),\\ w(x,0) = 0, \quad w(x,T) = 0 & \text{in } (0,L). \end{cases} \tag{3.14}$$

We define the space

$$E_0 = \{q \in \mathcal{C}^\infty(\overline{Q}) : q(0,t) = q(L,t) = q_x(0,t) = q_x(L,t) = 0\}.$$

We denote $P(q) = q_t + \gamma q_{xxxx} + q_{xxx} + a q_{xx}$. Let $\kappa : E_0 \times E_0 \to \mathbb{R}$ be the bilinear form

$$\kappa(q_1,q_2) = \iint\limits_Q e^{-2s\alpha}P(q_1)\,P(q_2)\,\mathrm{d}x\,\mathrm{d}t + s^7\lambda^8 \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^7 q_1\,q_2\,\mathrm{d}x\,\mathrm{d}t$$

and $\ell : E_0 \to \mathbb{R}$ the linear form

$$\ell(q) = s^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7 u\,q\,\mathrm{d}x\,\mathrm{d}t.$$

In this part of the proof, λ and s can be thought to be fixed and large enough such that Carleman estimate (3.9) holds. From (3.9), $\kappa(\cdot,\cdot)^{1/2}$ defines a norm in the space E_0. Let us denote by E the closure of the space E_0 with respect to the norm induced by $\kappa(\cdot,\cdot)$, which is a Hilbert space with the inner product $\kappa(\cdot,\cdot)$. Moreover, ℓ is a bounded operator. Indeed, by Cauchy–Schwarz inequality and (3.9) we have

$$|\ell(q)| \le C\Big(s^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7|u|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\kappa(q,q)^{1/2} \tag{3.15}$$

Therefore, by Lax–Milgram's lemma, there exists a unique solution $\widehat{q}\in E$ such that

$$\kappa(\widehat{q},q)=\ell(q)\quad \forall q\in E. \tag{3.16}$$

We define

$$\widehat{w}:=e^{-2s\alpha}P(\widehat{q}),\quad \widehat{h}:=-s^7\lambda^8\xi^7e^{-2s\alpha}\widehat{q}\mathbb{1}_\omega \tag{3.17}$$

and take $q=\widehat{q}$ in (3.16). Using (3.15) we get

$$\iint\limits_Q e^{2s\alpha}|\widehat{w}|^2\,\mathrm{d}x\,\mathrm{d}t + s^{-7}\lambda^{-8}\iint\limits_{\omega\times(0,T)} e^{2s\alpha}\xi^{-7}|\widehat{h}|\,\mathrm{d}x\,\mathrm{d}t \le Cs^7\lambda^8\iint\limits_Q e^{-2s\alpha}\xi^7|u|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{3.18}$$

From (3.16)–(3.18) one can easily check that $(\widehat{w},\widehat{h})$ is a solution to the control problem (3.14).

Now, we take $g=s^7\lambda^8\xi^7e^{-2s\alpha}u+\widehat{h}\mathbb{1}_\omega$ in (3.12)–(3.13). We obtain

$$\begin{aligned} s^7\lambda^8\iint\limits_Q e^{-2s\alpha}\xi^7|u|^2\,\mathrm{d}x\,\mathrm{d}t &= \iint\limits_Q B_0\,\widehat{w}\,\mathrm{d}x\,\mathrm{d}t - \iint\limits_Q B_1\,\widehat{w}_x\,\mathrm{d}x\,\mathrm{d}t + \iint\limits_Q B_2\,\widehat{w}_{xx}\,\mathrm{d}x\,\mathrm{d}t \\ &- \iint\limits_{\omega\times(0,T)} \widehat{h}\,u\,\mathrm{d}x - \int_0^T b_1\left(\gamma\widehat{w}_{xxx}(0,t)-\widehat{w}_{xx}(0,t)\right)\mathrm{d}t + \int_0^T b_2\left(\gamma\widehat{w}_{xxx}(L,t)-\widehat{w}_{xx}(L,t)\right)\mathrm{d}t \\ &+\gamma\int_0^T b_3\,\widehat{w}_{xx}(0,t)\,\mathrm{d}t - \gamma\int_0^T b_4\,\widehat{w}_{xx}(L,t)\,\mathrm{d}t. \end{aligned} \tag{3.19}$$

In the following, we will estimate some weighted norms of $\widehat{w}$ $\widehat{w}_x$, $\widehat{w}_{xx}$, $\widehat{h}\mathbb{1}_\omega$, $\widehat{w}_{xx}(0,\cdot)$, $\widehat{w}_{xx}(L,\cdot)$, $\widehat{w}_{xxx}(0,\cdot)$ and $\widehat{w}_{xxx}(L,\cdot)$ by the left-hand side of (3.19). In fact, from (3.18), this is already done for $\widehat{w}$ and $\widehat{h}\mathbb{1}_\omega$.

We multiply equation (3.14) by $s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}\widehat{w}$. After several integrations by parts and using Young's inequality we obtain

$$\begin{aligned} s^{-4}\lambda^{-4}\iint\limits_Q e^{2s\alpha}\xi^{-4}|\widehat{w}_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t &\le C\iint\limits_Q e^{2s\alpha}|\widehat{w}|^2\,\mathrm{d}x\,\mathrm{d}t \\ &+ Cs^6\lambda^8\iint\limits_Q e^{2s\alpha}\xi^6|u|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{-8}\lambda^{-8}\iint\limits_{\omega\times(0,T)} e^{2s\alpha}\xi^{-8}|\widehat{h}|^2\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{3.20}$$

for every $\lambda\ge C$ and every $s\ge C(T^{2m}+T^{2m-1/3})$. We can also obtain the estimate

$$s^{-2}\lambda^{-2}\iint\limits_Q e^{2s\alpha}\xi^{-2}|\widehat{w}_x|^2\,\mathrm{d}x\,\mathrm{d}t \le C\iint\limits_Q e^{2s\alpha}|\widehat{w}|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{-4}\lambda^{-4}\iint\limits_Q e^{2s\alpha}\xi^{-4}|\widehat{w}_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.21}$$

just by integrating by parts and Cauchy–Schwarz inequality, for every $\lambda \geq C$ and every $s \geq CT^{2m}$. Therefore, from (3.18):

$$s^{-4}\lambda^{-4}\iint\limits_{Q} e^{2s\alpha}\xi^{-4}|\widehat{w}_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + s^{-2}\lambda^{-2}\iint\limits_{Q} e^{2s\alpha}\xi^{-2}|\widehat{w}_{x}|^2\,\mathrm{d}x\,\mathrm{d}t \leq Cs^7\lambda^8\iint\limits_{Q} e^{2s\alpha}\xi^{7}|u|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{3.22}$$

for every $\lambda \geq C$ and every $s \geq C(T^{2m} + T^{2m-1/3})$.

We now estimate the boundary terms. Integration by parts shows that

$$\begin{aligned} &s^{-6}\lambda^{-6}\int_0^T \left[\left(e^{2s\alpha}\xi^{-6}\right)_x |\widehat{w}_{xx}|^2\right]_0^L \mathrm{d}t \\ &\quad = 2s^{-6}\lambda^{-6}\iint\limits_{Q}\left(e^{2s\alpha}\xi^{-6}\right)_x \widehat{w}_{xx}\widehat{w}_{xxx}\,\mathrm{d}x\,\mathrm{d}t + 2s^{-6}\lambda^{-6}\iint\limits_{Q}\left(e^{2s\alpha}\xi^{-6}\right)_{xx}|\widehat{w}_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.23}$$

From (3.4) we have

$$\begin{aligned} &s^{-6}\lambda^{-6}\int_0^T \left[\left(e^{2s\alpha}\xi^{-6}\right)_x |\widehat{w}_{xx}|^2\right]_0^L \mathrm{d}t = s^{-6}\lambda^{-5}\int_0^T e^{2s\alpha^*}(\xi^*)^{-5}(2s+6)\left[\eta'|\widehat{w}_{xx}|^2\right]_L^0 \mathrm{d}t \\ &\quad = s^{-6}\lambda^{-5}\int_0^T e^{2s\alpha^*}(\xi^*)^{-5}(2s+6)\left(\eta'(0)|\widehat{w}_{xx}(0,t)|^2 - \eta'(L)|\widehat{w}_{xx}(L,t)|^2\right)\mathrm{d}t. \end{aligned} \tag{3.24}$$

Using property (3.3) in (3.24), and then Young's inequality in (3.23) we obtain

$$\begin{aligned} &\delta\, s^{-5}\lambda^{-5}\int_0^T e^{2s\alpha^*}(\xi^*)^{-5}\left(|\widehat{w}_{xx}(0,t)|^2 + |\widehat{w}_{xx}(L,t)|^2\right)\mathrm{d}t \\ &\quad \leq \varepsilon s^{-6}\lambda^{-6}\iint\limits_{Q} e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{-4}\lambda^{-4}\iint\limits_{Q} e^{2s\alpha}\xi^{-4}|\widehat{w}_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{3.25}$$

for every $\lambda \geq C$ and every $s \geq CT^{2m}$, where ε is a positive constant to be chosen small enough later on.

Similarly, we can get

$$\begin{aligned} &\delta\, s^{-7}\lambda^{-7}\int_0^T e^{2s\alpha^*}(\xi^*)^{-7}\left(|\widehat{w}_{xxx}(0,t)|^2 + |\widehat{w}_{xxx}(L,t)|^2\right)\mathrm{d}t \\ &\quad \leq Cs^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}|\widehat{w}_{xxxx}|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{-6}\lambda^{-6}\iint\limits_{Q} e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.26}$$

for every $\lambda \geq C$ and every $s \geq CT^{2m}$.

In view of (3.25) and (3.26), we need to estimate the weighted norms of $\widehat{w}_{xxx}$ and $\widehat{w}_{xxxx}$. To do this, we first multiply (3.14) by $-s^{-8}\lambda^{-8}\xi^{-8}e^{2s\alpha}\widehat{w}_t$. After some integrations by parts we have

$$s^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}|\widehat{w}_t|^2\,\mathrm{d}x\,\mathrm{d}t + \gamma s^{-8}\lambda^{-8}\iint\limits_{Q}\left(e^{2s\alpha}\xi^{-8}\right)_t|\widehat{w}_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&+ 2\gamma s^{-8}\lambda^{-8}\iint\limits_{Q}\left(e^{2s\alpha}\xi^{-8}\right)_x \widehat{w}_{xxx}\widehat{w}_t\,\mathrm{d}x\,\mathrm{d}t + \gamma s^{-8}\lambda^{-8}\iint\limits_{Q}\left(e^{2s\alpha}\xi^{-8}\right)_{xx}\widehat{w}_{xx}\widehat{w}_t\,\mathrm{d}x\,\mathrm{d}t\\
&+ s^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}\widehat{w}_{xxx}\widehat{w}_t\,\mathrm{d}x\,\mathrm{d}t - as^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}\widehat{w}_{xx}\widehat{w}_t\,\mathrm{d}x\,\mathrm{d}t\\
&= -s^{-1}\iint\limits_{Q}\xi^{-1}u\,\widehat{w}_t\,\mathrm{d}x\,\mathrm{d}t - s^{-8}\lambda^{-8}\iint\limits_{\omega\times(0,T)} e^{2s\alpha}\xi^{-8}\widehat{h}\,\widehat{w}_t\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Using Young's inequality, we get

$$
\begin{aligned}
s^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}|\widehat{w}_t|^2\,\mathrm{d}x\,\mathrm{d}t &\le Cs^{-4}\lambda^{-4}\iint\limits_{Q} e^{2s\alpha}\xi^{-4}|\widehat{w}_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+ Cs^{-6}\lambda^{-6}\iint\limits_{Q} e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{6}\lambda^{8}\iint\limits_{Q} e^{-2s\alpha}\xi^{6}|u|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+ Cs^{-8}\lambda^{-8}\iint\limits_{\omega\times(0,T)} e^{2s\alpha}\xi^{-8}|\widehat{h}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}s^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}|\widehat{w}_t|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for every $\lambda \ge C$ and $s \ge C(T^{2m} + T^{2m-1/3})$. From (3.18) and (3.22), we obtain

$$
\begin{aligned}
&s^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}|\widehat{w}_t|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le Cs^{-6}\lambda^{-6}\iint\limits_{Q} e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{7}\lambda^{8}\iint\limits_{Q} e^{-2s\alpha}\xi^{7}|u|^2\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{3.27}
$$

for every $\lambda \ge C$ and $s \ge C(T^{2m} + T^{2m-1/3})$.

Using equation (3.14), we see that

$$
\begin{aligned}
s^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}|\widehat{w}_{xxxx}|^2\,\mathrm{d}x\,\mathrm{d}t &\le Cs^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}|\widehat{w}_t|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+ Cs^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}|\widehat{w}_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+ Cs^{6}\lambda^{8}\iint\limits_{Q} e^{-2s\alpha}\xi^{6}|u|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{-8}\lambda^{-8}\iint\limits_{\omega\times(0,T)} e^{2s\alpha}\xi^{-8}|\widehat{h}|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Again from (3.18), (3.22) and (3.27) we get

$$
\begin{aligned}
&s^{-8}\lambda^{-8}\iint\limits_{Q} e^{2s\alpha}\xi^{-8}\left(|\widehat{w}_t|^2 + |\widehat{w}_{xxxx}|^2\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad\le Cs^{-6}\lambda^{-6}\iint\limits_{Q} e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{7}\lambda^{8}\iint\limits_{Q} e^{-2s\alpha}\xi^{7}|u|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{3.28}
$$

for every $\lambda \ge C$ and $s \ge C(T^{2m} + T^{2m-1/3})$.

Now, we estimate $\widehat{w}_{xxx}$ using integration by parts. We have

$$s^{-6}\lambda^{-6}\iint\limits_Q e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t = -s^{-6}\lambda^{-6}\iint\limits_Q e^{2s\alpha}\xi^{-6}\widehat{w}_{xxxx}\widehat{w}_{xx}\,\mathrm{d}x\,\mathrm{d}t$$

$$- s^{-6}\lambda^{-6}\iint\limits_Q \left(e^{2s\alpha}\xi^{-6}\right)_x \widehat{w}_{xxx}\widehat{w}_{xx}\,\mathrm{d}x\,\mathrm{d}t + s^{-6}\lambda^{-6}\int\limits_0^T e^{2s\alpha^*}(\xi^*)^{-6}\widehat{w}_{xxx}\widehat{w}_{xx}\Big|_0^L\,\mathrm{d}t.$$

Here, we use Young's inequality to get

$$s^{-6}\lambda^{-6}\iint\limits_Q e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon s^{-8}\lambda^{-8}\iint\limits_Q e^{2s\alpha}\xi^{-8}|\widehat{w}_{xxxx}|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \varepsilon s^{-6}\lambda^{-6}\iint\limits_Q e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{-4}\lambda^{-4}\iint\limits_Q e^{2s\alpha}\xi^{-4}|\widehat{w}_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \varepsilon s^{-7}\lambda^{-7}\int\limits_0^T e^{2s\alpha^*}(\xi^*)^{-7}\left(|\widehat{w}_{xxx}(0,t)|^2 + |\widehat{w}_{xxx}(L,t)|^2\right)\mathrm{d}t$$

$$+ Cs^{-5}\lambda^{-5}\int\limits_0^T e^{2s\alpha^*}(\xi^*)^{-5}\left(|\widehat{w}_{xx}(0,t)|^2 + |\widehat{w}_{xx}(L,t)|^2\right)\mathrm{d}t$$

for every $\lambda \ge C$ and $s \ge CT^{2m}$. Notice that using (3.25) and (3.26) in this estimate, we obtain

$$s^{-6}\lambda^{-6}\iint\limits_Q e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t \le 2\varepsilon s^{-8}\lambda^{-8}\iint\limits_Q e^{2s\alpha}\xi^{-8}|\widehat{w}_{xxxx}|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$+ 3\varepsilon s^{-6}\lambda^{-6}\iint\limits_Q e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{-4}\lambda^{-4}\iint\limits_Q e^{2s\alpha}\xi^{-4}|\widehat{w}_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t$$

for every $\lambda \ge C$ and $s \ge CT^{2m}$.

Using (3.28) and (3.22), choosing ε small enough, we obtain from this last inequality

$$s^{-6}\lambda^{-6}\iint\limits_Q e^{2s\alpha}\xi^{-6}|\widehat{w}_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t \le Cs^7\lambda^8\iint\limits_Q e^{-2s\alpha}\xi^7|u|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.29}$$

for every $\lambda \ge C$ and $s \ge C(T^{2m} + T^{2m-1/3})$.

Finally, from (3.25), (3.26), (3.22), (3.28) and (3.29) we readily obtain

$$s^{-7}\lambda^{-7}\int\limits_0^T e^{2s\alpha^*}(\xi^*)^{-7}\left(|\widehat{w}_{xxx}(0,t)|^2 + |\widehat{w}_{xxx}(L,t)|^2\right)\mathrm{d}t$$

$$+ s^{-5}\lambda^{-5}\int\limits_0^T e^{2s\alpha^*}(\xi^*)^{-5}\left(|\widehat{w}_{xx}(0,t)|^2 + |\widehat{w}_{xx}(L,t)|^2\right)\mathrm{d}t \le Cs^7\lambda^8\iint\limits_Q e^{-2s\alpha}\xi^7|u|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.30}$$

for every $\lambda \ge C$ and $s \ge C(T^{2m} + T^{2m-1/3})$. Using (3.18), (3.22) and (3.30) in (3.19) along with Cauchy–Schwarz's inequality, we obtain (3.11). Notice that the dependence of the parameter s in (3.11) is justified by

$$T^{2m-1/3} \leq \frac{T^{2m}}{6} + \frac{5T^{2m-2/5}}{6}$$

which is obtained using Young's inequality. This ends the proof of Theorem 3.5. □

3.2. Carleman estimate for the adjoint system

We prove now a Carleman inequality for the solutions of the adjoint system (1.4) with observation term only on ψ.

Proposition 3.6. *Let $g_0, g_1 \in L^2(Q)$, $m \geq 2$ and $\omega \subset (0, L)$. There exist $\lambda_0 > 0$ and $C > 0$ (depending only on ω, γ, a, Γ and c) such that any solution (φ, ψ) of (1.4) satisfies*

$$\begin{aligned} &s^7\lambda^8 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^7|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + s^9\lambda^{10} \iint\limits_Q e^{-2s\alpha}\xi^9|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq Cs^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7\left(|g_0|^2 + |g_1|^2\right)\mathrm{d}x\,\mathrm{d}t + Cs^{23}\lambda^{16} \iint\limits_{\omega\times(0,T)} e^{-6s\alpha+4s\alpha^*}\xi^{23}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.31}$$

for every $\lambda \geq C(1 + \|\bar{y}\|_\infty + e^{C(1+\|\bar{y}\|_\infty^2)T})$ and $s \geq C(T^{2m} + T^m)$.

Proof. The proof follows a by now classical strategy to eliminate one of the observation terms (see for instance [21,4,9]). In a first step, we use inequality (3.11) for the equation satisfied by φ_x and estimate the remaining boundary terms. Then, we use the usual Carleman estimate of the heat equation for ψ. This allows to estimate the global terms coming from the previous step. Finally, using the coupling in the second equation of (1.4), we eliminate the local term of φ.

Step 1. Carleman for φ_x
The goal of this step is to prove

$$\begin{aligned} &s^7\lambda^8 \iint\limits_Q \left(e^{-2s\alpha}\xi^7|\varphi_x|^2 + e^{-2s\alpha^*}(\xi^*)^7|\varphi|^2\right)\mathrm{d}x\,\mathrm{d}t + \|s^{5/2-1/m}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2-1/m}\varphi\|^2_{L^2(0,T;H^4(0,L))} \\ &\leq C\iint\limits_Q e^{-2s\alpha}\left(T^{4m+2}s^{5-2/m}\lambda^8\xi^7|\psi_x|^2 + T^{10m}\xi^5|\psi_{xx}|^2 + s^7\lambda^8\xi^7|g_0|^2\right)\mathrm{d}x\,\mathrm{d}t \\ &\quad + Cs^7\lambda^8 \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.32}$$

for every $\lambda \geq C(1 + \|\bar{y}\|_\infty + e^{C(1+\|\bar{y}\|_\infty^2)T})$ and $s \geq C(T^{2m} + T^m)$.

We start by deriving with respect to x the equation of φ. We denote $\widetilde{\varphi} := \varphi_x$. It is easily checked that $\widetilde{\varphi}$ satisfies the equation

$$\begin{cases} -\widetilde{\varphi}_t + \gamma\widetilde{\varphi}_{xxxxx} - \widetilde{\varphi}_{xxx} + a\widetilde{\varphi}_{xx} = -\psi_{xx} + (\bar{y}\varphi_x)_x + (g_0)_x & \text{in } Q, \\ \widetilde{\varphi}(0,t) = 0, \quad \widetilde{\varphi}(L,t) = 0 & \text{in } (0,T), \\ \widetilde{\varphi}_x(0,t) = \varphi_{xx}(0,t), \quad \widetilde{\varphi}_x(L,t) = \varphi_{xx}(L,t) & \text{in } (0,T), \\ \widetilde{\varphi}(x,T) = \varphi'_T(x) & \text{in } (0,L). \end{cases}$$

Theorem 3.5 with $B_0 = -\psi_{xx}$, $B_1 = \bar{y}\varphi_x + g_0$, $B_2 \equiv 0$, $b_1 \equiv 0$, $b_2 \equiv 0$, $b_3 = \varphi_{xx}(0,t)$, and $b_4 = \varphi_{xx}(L,t)$ gives

$$s^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t \le C \iint\limits_Q e^{-2s\alpha}\Big(|\psi_{xx}|^2 + s^2\lambda^2\xi^2|\bar{y}\varphi_x|^2 + s^2\lambda^2\xi^2|g_0|^2\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ Cs^5\lambda^5\int_0^T e^{-2s\alpha^*}(\xi^*)^5(|\varphi_{xx}(0,t)|^2 + |\varphi_{xx}(L,t)|^2)\,\mathrm{d}t$$
$$+ Cs^7\lambda^8 \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.33}$$

for every $\lambda \ge C$ and $s \ge C(T^{2m} + T^{2m-2/5})$. Here, the subdomain $\omega_0 \Subset \omega$ is, without loss of generality, the one in (3.2).

We now deal with the boundary terms in (3.33). First, notice that since $\varphi(0,t) = \varphi(L,t) = 0$, we have

$$s^7\lambda^8 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^7|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t \le Cs^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{3.34}$$

where we have also used (3.4).

Next, let $\rho(t) := s^{5/2-1/m}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2-1/m}$ and $\varphi^* := \rho\,\varphi$. From (1.4) we verify that φ^* satisfies the equation

$$\begin{cases} -\varphi^*_t + \gamma\varphi^*_{xxxx} - \varphi^*_{xxx} + a\varphi^*_{xx} - \bar{y}\varphi^*_x = -\rho\,\psi_x + \rho\,g_0 - \rho'\varphi & \text{in } Q, \\ \varphi^*(0,t) = \varphi^*_x(0,t) = 0, \quad \varphi^*(L,t) = \varphi^*_x(L,t) = 0 & \text{in } (0,T), \\ \varphi^*(x,T) = 0 & \text{in } (0,L). \end{cases}$$

From regularity estimates for the KS equation (see [7, Proposition 2.1] or Lemma 2.1), we have

$$\|\varphi^*\|^2_{L^2(0,T;H^4(0,L))} \le C^*\big(\|\rho\,\psi_x\|^2_{L^2(Q)} + \|\rho\,g_0\|^2_{L^2(Q)} + \|s^{7/2}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{7/2}\varphi\|^2_{L^2(Q)}\big) \tag{3.35}$$

for every $s \ge C(T^{2m} + T^m)$. The constant C^* is given by Lemma 2.1. Observe that we have used that

$$|\rho'(t)| \le CTs^{5/2-1/m}\big(s + T^{2m}\big)\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{7/2} \le Cs^{7/2}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{7/2}$$

for every $s \ge C(T^{2m} + T^m)$. See (3.5).

Now, let us estimate the boundary terms. From interpolation between the spaces $L^2(0,T;H^1(0,L))$ and $L^2(0,T;H^4(0,L))$ we have

$$\begin{aligned} &\|s^{5/2}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2}\varphi\|_{L^2(0,T;H^3(0,L))} \\ &\quad\le \|s^{7/2}\lambda^{7/2}e^{-s\alpha^*}(\xi^*)^{7/2}\varphi\|^{1/3}_{L^2(0,T;H^1(0,L))}\|s^2\lambda^2e^{-s\alpha^*}(\xi^*)^2\varphi\|^{2/3}_{L^2(0,T;H^4(0,L))}. \end{aligned} \tag{3.36}$$

Since

$$\begin{aligned} &\|s^{5/2}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2}\varphi_{xx}(0,\cdot)\|^2_{L^2(0,T)} + \|s^{5/2}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2}\varphi_{xx}(L,\cdot)\|^2_{L^2(0,T)} \\ &\quad\le C\|s^{5/2}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2}\varphi\|_{L^2(0,T;H^3(0,L))}, \end{aligned}$$

we have, combining this estimate with (3.36) and Young's inequality,

$$\begin{aligned} &\|s^{5/2}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2}\varphi_{xx}(0,\cdot)\|^2_{L^2(0,T)} + \|s^{5/2}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2}\varphi_{xx}(L,\cdot)\|^2_{L^2(0,T)} \\ &\quad\le C\|s^{7/2}\lambda^{7/2}e^{-s\alpha^*}(\xi^*)^{7/2}\varphi\|^2_{L^2(0,T;H^1(0,L))} + C\|s^2\lambda^2e^{-s\alpha^*}(\xi^*)^2\varphi\|^2_{L^2(0,T;H^4(0,L))}. \end{aligned} \tag{3.37}$$

Using (3.34)–(3.35)–(3.37) in (3.33) we obtain at this point

$$
\begin{aligned}
& s^7\lambda^8 \iint\limits_Q \left(e^{-2s\alpha}\xi^7|\varphi_x|^2 + e^{-2s\alpha^*}(\xi^*)^7|\varphi|^2\right) \mathrm{d}x\,\mathrm{d}t + \|s^{5/2-1/m}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2-1/m}\varphi\|^2_{L^2(0,T;H^4(0,L))} \\
&\quad \le C \iint\limits_Q e^{-2s\alpha}\Big(s^{5-2/m}\lambda^8\xi^{5-2/m}|\psi_x|^2 + |\psi_{xx}|^2 + (s^2\lambda^2\xi^2 + s^{5-2/m}\lambda^8\xi^{5-2/m})|g_0|^2\Big) \mathrm{d}x\,\mathrm{d}t \\
&\qquad + C \iint\limits_Q e^{-2s\alpha}\big(s^7\lambda^7\xi^7 + s^2\lambda^2\xi^2\|\bar{y}\|^2_\infty\big)|\varphi_x|^2 \,\mathrm{d}x\,\mathrm{d}t + C\|s^2\lambda^2 e^{-s\alpha^*}(\xi^*)^2\varphi\|^2_{L^2(0,T;H^4(0,L))} \\
&\qquad + Cs^7\lambda^8 \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^7|\varphi_x|^2 \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for every $\lambda \ge C(1 + \|\bar{y}\|_\infty + e^{C(1+\|\bar{y}\|^2_\infty)T})$ and $s \ge C(T^{2m} + T^m)$.

Since $m \ge 2$, the global terms concerning φ_x in the right-hand side can be absorbed by the left-hand side by taking $\lambda \ge C(1 + \|\bar{y}\|_\infty)$ and $s \ge CT^{2m}$. The remaining global term of φ is absorbed in the same way with $\lambda \ge C$ and $s \ge CT^{2m}$. Then, from here it is easy to deduce (3.32) taking into account (3.5).

Step 2. Carleman for ψ

For ψ we apply Lemma 3.1 to obtain

$$
\begin{aligned}
& \iint\limits_Q e^{-2s\alpha}\big(s^5\lambda^6\xi^5(|\psi_t|^2 + |\psi_{xx}|^2) + s^7\lambda^8\xi^7|\psi_x|^2 + s^9\lambda^{10}\xi^9|\psi|^2\big) \mathrm{d}x\,\mathrm{d}t \\
&\quad \le Cs^6\lambda^6 \iint\limits_Q e^{-2s\alpha}\xi^6(|\psi_x|^2 + |\varphi_x|^2 + |g_1|^2) \,\mathrm{d}x\,\mathrm{d}t + Cs^9\lambda^{10} \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^9|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.38}
$$

for every $\lambda \ge \lambda_0$ and $s \ge C(T^{2m} + T^{2m-1})$.

Step 3. Combining both Carleman estimates

Adding (3.32) and (3.38), we can absorb the global terms related to φ_x, ψ_x and ψ_{xx} present in the right-hand sides as long as $\lambda \ge C$ and $s \ge CT^{2m}$. Thus, we get

$$
\begin{aligned}
& s^7\lambda^8 \iint\limits_Q \left(e^{-2s\alpha}\xi^7|\varphi_x|^2 + e^{-2s\alpha^*}(\xi^*)^7|\varphi|^2\right) \mathrm{d}x\,\mathrm{d}t + \|s^{5/2-1/m}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2-1/m}\varphi\|^2_{L^2(0,T;H^4(0,L))} \\
&\qquad + \iint\limits_Q e^{-2s\alpha}\big(s^5\lambda^6\xi^5(|\psi_t|^2 + |\psi_{xx}|^2) + s^7\lambda^8\xi^7|\psi_x|^2 + s^9\lambda^{10}\xi^9|\psi|^2\big) \mathrm{d}x\,\mathrm{d}t \\
&\quad \le C \iint\limits_Q e^{-2s\alpha}(s^7\lambda^8\xi^7|g_0|^2 + s^6\lambda^6\xi^6|g_1|^2) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + Cs^9\lambda^{10} \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^9|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + Cs^7\lambda^8 \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^7|\varphi_x|^2 \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.39}
$$

for every $\lambda \ge C(1 + \|\bar{y}\|_\infty + e^{C(1+\|\bar{y}\|^2_\infty)T})$ and $s \ge C(T^{2m} + T^m)$.

Step 4. Eliminating the local term of φ_x

This is the longest part of the proof. What is left to do is to estimate the last term in (3.39). Let $\theta(x) \in \mathcal{C}^2_0(\omega)$ such that $0 \le \theta \le 1$ and $\theta_{|\omega_0} \equiv 1$. From the equation satisfied by ψ in (1.4), we have

$$
\begin{aligned}
Cs^7\lambda^8 \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t &\le Cs^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t \\
&= Cs^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7\varphi_x(g_1+(\Gamma-a)\psi_{xx}+c\psi_x+\psi_t+a\psi_{xx})\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.40)
\end{aligned}
$$

Now we deal with the terms at the right-hand side in (3.40). First, by Young's inequality we get

$$
\begin{aligned}
&s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7\varphi_x\,g_1\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \varepsilon s^7\lambda^8 \iint\limits_{Q} e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^7\lambda^8 \iint\limits_{Q} e^{-2s\alpha}\xi^7|g_1|^2\,\mathrm{d}x\,\mathrm{d}t, \qquad (3.41)
\end{aligned}
$$

where ε is a positive number to be chosen small enough later on. Next, integration by parts yields

$$
\begin{aligned}
&s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7\varphi_x\left((\Gamma-a)\psi_{xx}+c\psi_x\right)\mathrm{d}x\,\mathrm{d}t \\
&= (\Gamma-a)s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \left(\theta(x)e^{-2s\alpha}\xi^7\right)_{xx}\varphi_x\,\psi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - 2(\Gamma-a)s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \left(\theta(x)e^{-2s\alpha}\xi^7\right)_{x}\varphi_{xx}\,\psi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + (\Gamma-a)s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7\varphi_{xxx}\,\psi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - cs^7\lambda^8 \iint\limits_{\omega\times(0,T)} \left(\theta(x)e^{-2s\alpha}\xi^7\right)_{x}\varphi_x\,\psi\,\mathrm{d}x\,\mathrm{d}t - cs^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7\varphi_{xx}\,\psi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

From this, Young's inequality and (3.5) we get

$$
\begin{aligned}
&Cs^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7\varphi_x\left((\Gamma-a)\psi_{xx}+c\psi_x\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \varepsilon\|s^{5/2-1/m}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2-1/m}\varphi\|^2_{L^2(0,T;H^4(0,L))} \\
&\qquad + Cs^{18}\lambda^{15} \iint\limits_{\omega\times(0,T)} e^{-4s\alpha+2s\alpha^*}\xi^{18}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \qquad (3.42)
\end{aligned}
$$

for every $s \ge CT^{2m}$.

For the remaining terms in (3.40), we obtain by integration by parts

$$
\begin{aligned}
&s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7\varphi_x\left(\psi_t+a\psi_{xx}\right)\mathrm{d}x\,\mathrm{d}t \\
&= s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7(-\varphi_{xt}+a\varphi_{xxx})\,\psi\,\mathrm{d}x\,\mathrm{d}t - s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)\left(e^{-2s\alpha}\xi^7\right)_t\varphi_x\,\psi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$+ as^7\lambda^8 \iint\limits_{\omega\times(0,T)} \Big(\theta(x)e^{-2s\alpha}\xi^7\Big)_{xx} \varphi_x\,\psi\,\mathrm{d}x\,\mathrm{d}t - 2as^7\lambda^8 \iint\limits_{\omega\times(0,T)} \Big(\theta(x)e^{-2s\alpha}\xi^7\Big)_x \varphi_{xx}\,\psi\,\mathrm{d}x\,\mathrm{d}t. \quad (3.43)$$

The last two integrals can be bounded as in (3.42). The second integral in the right-hand side can be estimated using Young's inequality and (3.5)

$$\begin{aligned} & s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)\Big(e^{-2s\alpha}\xi^7\Big)_t \varphi_x\,\psi\,\mathrm{d}x\,\mathrm{d}t \\ & \quad \le \varepsilon s^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{9+2/m}\lambda^8 \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^{9+2/m}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \quad (3.44)$$

for every $s \ge C(T^{2m} + T^m)$. For the first integral in the right-hand side of (3.43), we use the equation satisfied by φ in (1.4):

$$\begin{aligned} & s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7(-\varphi_{xt} + a\varphi_{xxx})\,\psi\,\mathrm{d}x\,\mathrm{d}t = s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7(\varphi_{xxx} - \psi_{xx})\,\psi\,\mathrm{d}x\,\mathrm{d}t \\ & \quad + s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7(-\gamma\varphi_{xxxxx} + (\bar{y}\varphi_x)_x + (g_0)_x)\,\psi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \quad (3.45)$$

The first integral in the right-hand side of (3.45) can be estimated as (3.42). For the second one, after one integration by parts we obtain

$$\begin{aligned} & s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7(-\gamma\varphi_{xxxxx} + (\bar{y}\varphi_x)_x + (g_0)_x)\,\psi\,\mathrm{d}x\,\mathrm{d}t \\ & = \gamma s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \Big(\theta(x)e^{-2s\alpha}\xi^7\Big)_x \varphi_{xxxx}\,\psi\,\mathrm{d}x\,\mathrm{d}t + \gamma s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7\varphi_{xxxx}\,\psi_x\,\mathrm{d}x\,\mathrm{d}t \\ & \quad - s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \Big(\theta(x)e^{-2s\alpha}\xi^7\Big)_x g_0\,\psi\,\mathrm{d}x\,\mathrm{d}t - s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7 g_0\,\psi_x\,\mathrm{d}x\,\mathrm{d}t \\ & \quad - s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \Big(\theta(x)e^{-2s\alpha}\xi^7\Big)_x \bar{y}\varphi_x\,\psi\,\mathrm{d}x\,\mathrm{d}t - s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7\bar{y}\varphi_x\,\psi_x\,\mathrm{d}x\,\mathrm{d}t \\ & =: \sum_{i=1}^6 I_i \end{aligned}$$

where we have denoted by I_i the i-th integral in this equation. The term I_1 can be bounded as in (3.42) and I_2 by

$$\varepsilon\|s^{5/2-1/m}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2-1/m}\varphi\|^2_{L^2(0,T;H^4(0,L))} + Cs^{14}\lambda^{11} \iint\limits_{\omega\times(0,T)} \theta(x)e^{-4s\alpha+2s\alpha^*}\xi^{14}\,|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t \quad (3.46)$$

for every $s \ge CT^{2m}$. Integrals I_3 and I_4 are bounded by

$$\varepsilon \iint\limits_Q e^{-2s\alpha}\big(s^7\lambda^8\xi^7|\psi_x|^2 + s^9\lambda^{10}\xi^9|\psi|^2\big)\,\mathrm{d}x\,\mathrm{d}t + Cs^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7|g_0|^2\,\mathrm{d}x\,\mathrm{d}t \quad (3.47)$$

for every $\lambda \geq C$ and $s \geq CT^{2m}$. Finally, I_5 and I_6 can be estimated as follows

$$I_5 \leq C\|\bar{y}\|_\infty^2 s^9\lambda^{10} \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^9|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon s^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.48}$$

and

$$I_6 \leq C\|\bar{y}\|_\infty^2 s^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon s^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.49}$$

for every $s \geq CT^{2m}$.

Thus, it only remains to estimate the second term in (3.46), since it bounds the first term in (3.49) if $\lambda \geq C(1+\|\bar{y}\|_\infty)$ and $s \geq CT^{2m}$ (recall (3.4)). We integrate by parts:

$$\begin{aligned}
s^{14}\lambda^{11} \iint\limits_{\omega\times(0,T)} \theta(x)e^{-4s\alpha+2s\alpha^*}\xi^{14}\,|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t &= -s^{14}\lambda^{11} \iint\limits_{\omega\times(0,T)} \theta(x)e^{-4s\alpha+2s\alpha^*}\xi^{14}\,\psi_{xx}\,\psi\,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{2}s^{14}\lambda^{11} \iint\limits_{\omega\times(0,T)} \left(\theta(x)e^{-4s\alpha+2s\alpha^*}\xi^{14}\right)_{xx}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Young's inequality yields

$$\begin{aligned}
Cs^{14}\lambda^{11} \iint\limits_{\omega\times(0,T)} \theta(x)e^{-4s\alpha+2s\alpha^*}\xi^{14}\,|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t &\leq \varepsilon s^5\lambda^6 \iint\limits_Q e^{-2s\alpha}\xi^5\,|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&+ Cs^{23}\lambda^{16} \iint\limits_{\omega\times(0,T)} e^{-6s\alpha+4s\alpha^*}\xi^{23}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{16}\lambda^{13} \iint\limits_{\omega\times(0,T)} e^{-4s\alpha+2s\alpha^*}\xi^{16}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.50}$$

for every $\lambda \geq C$ and $s \geq CT^{2m}$.

Gathering (3.44)–(3.50) in (3.43) we obtain

$$\begin{aligned}
&Cs^7\lambda^8 \iint\limits_{\omega\times(0,T)} \theta(x)e^{-2s\alpha}\xi^7\varphi_x\left(\psi_t + a\psi_{xx}\right)\mathrm{d}x\,\mathrm{d}t \\
&\leq 4\varepsilon\|s^{5/2-1/m}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2-1/m}\varphi\|_{L^2(0,T;H^4(0,L))}^2 + Cs^{18}\lambda^{15} \iint\limits_{\omega\times(0,T)} e^{-4s\alpha+2s\alpha^*}\xi^{18}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 3\varepsilon s^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{9+2/m}\lambda^8 \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^{9+2/m}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \varepsilon \iint\limits_Q e^{-2s\alpha}\left(s^5\lambda^6\xi^5|\psi_{xx}|^2 + s^7\lambda^8\xi^7|\psi_x|^2 + s^9\lambda^{10}\xi^9|\psi|^2\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + Cs^7\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^7|g_0|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{23}\lambda^{16} \iint\limits_{\omega\times(0,T)} e^{-6s\alpha+4s\alpha^*}\xi^{23}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\|\bar{y}\|_\infty^2 s^9\lambda^{10} \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^9|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{16}\lambda^{13} \iint\limits_{\omega\times(0,T)} e^{-4s\alpha+2s\alpha^*}\xi^{16}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.51}$$

for every $\lambda \geq C(1+\|\bar{y}\|_\infty)$ and $s \geq C(+T^{2m}+T^m)$.

Finally, using (3.41), (3.42) and (3.51) in (3.40), and choosing ε small enough, we obtain from (3.39)

$$\begin{aligned}
& s^7\lambda^8 \iint\limits_Q \left(e^{-2s\alpha}\xi^7|\varphi_x|^2 + e^{-2s\alpha^*}(\xi^*)^7|\varphi|^2\right) \mathrm{d}x\,\mathrm{d}t + \|s^{5/2-1/m}\lambda^{5/2}e^{-s\alpha^*}(\xi^*)^{5/2-1/m}\varphi\|^2_{L^2(0,T;H^4(0,L))} \\
& \quad + \iint\limits_Q e^{-2s\alpha}\left(s^5\lambda^6\xi^5(|\psi_t|^2+|\psi_{xx}|^2) + s^7\lambda^8\xi^7|\psi_x|^2 + s^9\lambda^{10}\xi^9|\psi|^2\right) \mathrm{d}x\,\mathrm{d}t \\
& \leq C \iint\limits_Q e^{-2s\alpha}(s^7\lambda^8\xi^7|g_0|^2 + \left(s^6\lambda^6\xi^6 + s^7\lambda^8\xi^7\right)|g_1|^2)\,\mathrm{d}x\,\mathrm{d}t \\
& \quad + C \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\left(s^{9+2/m}\lambda^8\xi^{9+2/m} + \|\bar{y}\|^2_\infty s^9\lambda^{10}\xi^9\right)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\
& \quad + C \iint\limits_{\omega\times(0,T)} e^{-4s\alpha+2s\alpha^*}\left(s^{18}\lambda^{15}\xi^{18} + s^{16}\lambda^{13}\xi^{16}\right)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\
& \quad + Cs^{23}\lambda^{16} \iint\limits_{\omega\times(0,T)} e^{-6s\alpha+4s\alpha^*}\xi^{23}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.52}$$

for every $\lambda \geq C(1+\|\bar{y}\|_\infty + e^{C(1+\|\bar{y}\|^2_\infty)T})$ and $s \geq C(T^{2m}+T^m)$. From (3.52), (3.4) and (3.5), we can deduce (3.31). $\square$

4. Controllability of the linearized system

In this section we establish the null controllability of the linear system (1.3). First, we deduce an observability inequality from the Carleman estimate (3.31). Then, we construct a control and a solution of system (1.3) in appropriate weighted spaces.

4.1. Observability inequality

Let us introduce a function $\tau \in \mathcal{C}([0,T])$ such that $\tau(t) := (T^{2m}/4^m)\mathbb{1}_{[0,T/2)} + t^m(T-t)^m\mathbb{1}_{[T/2,T]}$. Then, we define

$$\begin{aligned}
& \beta(x,t) := \frac{e^{k\frac{m+1}{m}\lambda\|\eta\|_\infty} - e^{\lambda(k\|\eta\|_\infty+\eta(x))}}{\tau(t)}, \quad \mu(x,t) := \frac{e^{\lambda(k\|\eta\|_\infty+\eta(x))}}{\tau(t)}, \\
& \beta^*(t) := \max_{x\in[0,L]}\beta(x,t) = \beta(0,t) = \beta(L,t), \quad \mu^*(t) := \min_{x\in[0,L]}\mu(x,t) = \mu(0,t) = \mu(L,t), \\
& \widehat{\beta}(t) := \min_{x\in[0,L]}\beta(x,t), \quad \widehat{\mu}(t) := \max_{x\in[0,L]}\mu(x,t).
\end{aligned} \tag{4.1}$$

Notice that with these definitions we have that

$$\alpha \equiv \beta,\ \xi \equiv \mu \text{ in } [T/2,T]\times(0,L) \tag{4.2}$$

where α and ξ are defined in (3.4). With this new weight functions, we have the following result.

Proposition 4.1. *Let m, s and λ be fixed such that Proposition 3.6 holds. Then, there exists a constant $C > 0$ such that any solution of (1.4) satisfies*

$$\int_0^L |\varphi_{xx}(x,0)|^2\,\mathrm{d}x + \int_0^L |\psi_x(x,0)|^2\,\mathrm{d}x$$
$$+ \|e^{-s\beta^*}(\mu^*)^{5/2-1/m}\varphi\|^2_{L^\infty(0,T;W^{1,\infty}(0,L))} + \|e^{-s\beta^*}(\mu^*)^{5/2-1/m}\psi\|^2_{L^2(0,T;H_0^1(0,L))}$$
$$\le C\iint_Q e^{-2s\widehat{\beta}}\widehat{\mu}^7\big(|g_0|^2+|g_1|^2\big)\,\mathrm{d}x\,\mathrm{d}t + C\iint_{\omega\times(0,T)} e^{-6s\widehat{\beta}+4s\beta^*}\,\widehat{\mu}^{23}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.3}$$

Proof. We divide the proof of (4.3) in two steps:

Step 1. Let us show the estimate

$$\iint_Q e^{-2s\beta^*}(\mu^*)^7|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_Q e^{-2s\beta^*}(\mu^*)^9|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^L (|\varphi_{xx}(x,0)|^2+|\psi_x(x,0)|^2)\,\mathrm{d}x$$
$$\le C\iint_Q e^{-2s\widehat{\beta}}\widehat{\mu}^7\big(|g_0|^2+|g_1|^2\big)\,\mathrm{d}x\,\mathrm{d}t + C\iint_{\omega\times(0,T)} e^{-6s\widehat{\beta}+4s\beta^*}\,\widehat{\mu}^{23}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.4}$$

Let $\nu \in \mathcal{C}^1([0,T])$ be a non-negative and non-increasing function such that $\nu \equiv 1$ in $[0,T/2]$ and $\nu \equiv 0$ in $[3T/4,T]$. From Lemma 2.1 applied to $(\nu\varphi,\nu\psi)$ we have,

$$\|\nu\varphi\|^2_{L^2(0,T;H^4(0,L))\cap L^\infty(0,T;H_0^2(0,L))} + \|\nu\psi\|^2_{L^2(0,T;H^2(0,L))\cap L^\infty(0,T;H_0^1(0,L))}$$
$$\le C\Big(\|\nu g_0\|^2_{L^2(0,3T/4;L^2(0,L))} + \|\nu g_1\|^2_{L^2(0,3T/4;L^2(0,L))}$$
$$+ \|\nu'\varphi\|^2_{L^2(T/2,3T/4;L^2(0,L))} + \|\nu'\psi\|^2_{L^2(T/2,3T/4;L^2(0,L))}\Big). \tag{4.5}$$

The fact that we can truncate the integrals away from T allows us to add the exponential weight functions at the right-hand side of (4.5). Indeed, from (3.4) we get

$$\int_{T/2}^{3T/4}\int_0^L (|\varphi|^2+|\psi|^2)\,\mathrm{d}x\,\mathrm{d}t \le C\int_{T/2}^{3T/4}\int_0^L (e^{-2s\alpha^*}(\xi^*)^7|\varphi|^2 + e^{-2s\alpha}\xi^9|\psi|^2)\,\mathrm{d}x\,\mathrm{d}t.$$

Combining this with (3.31), (4.5) and (4.2) we obtain

$$\int_0^{T/2}\int_0^L (e^{-2s\beta^*}(\mu^*)^7|\varphi|^2 + e^{-2s\beta^*}(\mu^*)^9|\psi|^2)\,\mathrm{d}x\,\mathrm{d}t + \int_0^L (|\varphi_{xx}(x,0)|^2+|\psi_x(x,0)|^2)\,\mathrm{d}x$$
$$\le C\iint_Q e^{-2s\widehat{\beta}}\widehat{\mu}^7\big(|g_0|^2+|g_1|^2\big)\,\mathrm{d}x\,\mathrm{d}t + C\iint_{\omega\times(0,T)} e^{-6s\widehat{\beta}+4s\beta^*}\,\widehat{\mu}^{23}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.6}$$

Now, it is not difficult to see that from (3.31) and (4.2) we can get the estimate

$$\int_{T/2}^{T}\int_0^L e^{-2s\beta^*}(\mu^*)^7|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \int_{T/2}^{T}\int_0^L e^{-2s\beta^*}(\mu^*)^9|\psi|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C\iint_Q e^{-2s\widehat{\beta}}\widehat{\mu}^7\big(|g_0|^2+|g_1|^2\big)\,\mathrm{d}x\,\mathrm{d}t + C\iint_{\omega\times(0,T)} e^{-6s\widehat{\beta}+4s\beta^*}\,\widehat{\mu}^{23}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t,$$

which combined with (4.6) yields (4.4).

Step 2. We prove that

$$\begin{aligned}&\|e^{-s\beta^*}(\mu^*)^{5/2-1/m}\varphi\|^2_{L^\infty(0,T;H_0^2(0,L))}+\|e^{-s\beta^*}(\mu^*)^{5/2-1/m}\psi\|^2_{L^\infty(0,T;H_0^1(0,L))}\\&\quad\le C\iint_Q e^{-2s\widehat{\beta}}\widehat{\mu}^7\big(|g_0|^2+|g_1|^2\big)\,\mathrm{d}x\,\mathrm{d}t+C\iint_{\omega\times(0,T)} e^{-6s\widehat{\beta}+4s\beta^*}\widehat{\mu}^{23}\,|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{4.7}$$

which, together with (4.4), implies (4.3).

As in the proof of Proposition 3.6 (Step 1), we define $\rho(t):=e^{-s\beta^*}(\mu^*)^{5/2-1/m}$ and $(\varphi^*,\psi^*):=(\rho\varphi,\rho\psi)$. From (1.4), this couple satisfies the system

$$\begin{cases}-\varphi^*_t+\gamma\varphi^*_{xxxx}-\varphi^*_{xxx}+a\varphi^*_{xx}-\bar{y}\varphi^*_x=-\psi^*_x+\rho g_0-\rho'\varphi & \text{in } Q,\\ -\psi^*_t-\Gamma\psi^*_{xx}-c\psi^*_x=-\varphi^*_x+\rho g_1-\rho'\psi & \text{in } Q,\\ \ \varphi^*(0,t)=\varphi^*_x(0,t)=0,\quad \varphi^*(L,t)=\varphi^*_x(L,t)=0 & \text{in } (0,T),\\ \ \psi^*(0,t)=0,\quad \psi^*(L,t)=0 & \text{in } (0,T),\\ \ \varphi^*(x,T)=0,\quad \psi(x,T)=0 & \text{in } (0,L).\end{cases}$$

From Lemma 2.1 we obtain

$$\begin{aligned}&\|\varphi^*\|^2_{L^\infty(0,T;H_0^2(0,L))}+\|\psi^*\|^2_{L^\infty(0,T;H_0^1(0,L))}\\&\quad\le C\big(\|\rho g_0\|^2_{L^2(Q)}+\|\rho g_1\|^2_{L^2(Q)}+\|\rho'\varphi\|^2_{L^2(Q)}+\|\rho'\psi\|^2_{L^2(Q)}\big).\end{aligned}\tag{4.8}$$

It is readily checked that the weight functions defined in (4.1) satisfy (3.5). Thus, we have

$$|\rho(t)|\le Ce^{-s\widehat{\beta}}\widehat{\mu}^{7/2}\text{ and }|\rho'(t)|\le Ce^{-s\beta^*}(\mu^*)^{7/2}.$$

Hence, the last two terms in (4.8) are estimated by the left-hand side of (4.4) and we obtain (4.7). This finishes the proof of Proposition 4.1. □

4.2. Null controllability result

Let us start by defining the linear operators

$$\mathcal{L}_0 q:=q_t+\gamma q_{xxxx}+q_{xxx}+aq_{xx}+\bar{y}q_x+\bar{y}_x q,$$
$$\mathcal{L}_1 q:=q_t-\Gamma q_{xx}+cq_x,$$

their respective (formal) adjoints

$$\mathcal{L}_0^* q:=-q_t+\gamma q_{xxxx}-q_{xxx}+aq_{xx}-\bar{y}q_x,$$
$$\mathcal{L}_1^* q:=-q_t-\Gamma q_{xx}-cq_x$$

and the following Banach space (endowed with its natural norm)

$$\begin{aligned}\mathcal{E} := \big\{(y,z,v) :\ & e^{s\widehat{\beta}}\widehat{\mu}^{-7/2}(y,z) \in L^2(Q)^2,\\ & e^{3s\widehat{\beta}-2s\beta^*}(\widehat{\mu})^{-21/2-1/m}(y,z) \in \mathcal{C}([0,T];H^{-2}(0,L)\times H^{-1}(0,L)),\\ & e^{3s\widehat{\beta}-2s\beta^*}(\widehat{\mu})^{-23/2} v\mathbb{1}_\omega \in L^2(Q),\\ & e^{s\beta^*}(\mu^*)^{-5/2+1/m}(\mathcal{L}_0 y - z_x) \in L^1(0,T;W^{-1,1}(0,L)),\\ & e^{s\beta^*}(\mu^*)^{-5/2+1/m}(\mathcal{L}_1 z - y_x - v\mathbb{1}_\omega) \in L^2(0,T;H^{-1}(0,L))\big\}.\end{aligned}$$

The null controllability result is given by the following proposition.

Proposition 4.2. *Let m, s and λ be fixed as in Proposition 4.1, $y_0 \in H^{-2}(0,L)$ and $z_0 \in H^{-1}(0,L)$. Let f_0 and f_1 two functions verifying that*

$$e^{s\beta^*}(\mu^*)^{-5/2+1/m} f_0 \in L^1(0,T;W^{-1,1}(0,L))$$

and

$$e^{s\beta^*}(\mu^*)^{-5/2+1/m} f_1 \in L^2(0,T;H^{-1}(0,L)).$$

Then, there exist a control v and a solution (y,z) to system (1.3) such that $(y,z,v) \in \mathcal{E}$. In particular, $(y(\cdot,T), z(\cdot,T)) = 0$ in $(0,L)$.

Proof. We define the space

$$E_0 = \{(q,r) \in \mathcal{C}^\infty(\overline{Q})^2 : q(0,t) = q(L,t) = q_x(0,t) = q_x(L,t) = r(0,t) = r(L,t) = 0 \text{ in } (0,T)\},$$

the bilinear operator $\kappa : E_0 \times E_0 \to \mathbb{R}$ given by

$$\begin{aligned}\kappa((q_1,r_1),(q_2,r_2)) := &\iint_Q e^{-2s\widehat{\beta}}\widehat{\mu}^7(\mathcal{L}_0^* q_1 + \partial_x r_1)(\mathcal{L}_0^* q_2 + \partial_x r_2)\,\mathrm{d}x\,\mathrm{d}t\\ &+ \iint_Q e^{-2s\widehat{\beta}}\widehat{\mu}^7(\mathcal{L}_1^* r_1 + \partial_x q_1)(\mathcal{L}_1^* r_2 + \partial_x q_2)\,\mathrm{d}x\,\mathrm{d}t + \iint_{\omega\times(0,T)} e^{-6s\widehat{\beta}+4s\beta^*}\widehat{\mu}^{23} r_1 r_2\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and the linear operator $\ell : E_0 \to \mathbb{R}$ defined by

$$\begin{aligned}\ell(q,r) := &\langle f_0, q\rangle_{L^1(0,T;W^{-1,1}(0,L)),L^\infty(0,T;W^{1,\infty}(0,L))} + \langle f_1, r\rangle_{L^2(0,T;H^{-1}(0,L)),L^2(0,T;H_0^1(0,L))}\\ &+ \langle y_0, q(\cdot,0)\rangle_{H^{-2}(0,L),H_0^2(0,L)} + \langle z_0, r(\cdot,0)\rangle_{H^{-1}(0,L),H_0^1(0,L)}.\end{aligned}$$

As usual, the observability inequality (4.3) allows us to deduce that $\kappa(\cdot,\cdot)^{1/2}$ is a norm in the space E_0 and we call $E := \overline{E_0}^{\sqrt{\kappa(\cdot,\cdot)}}$. It follows that E is a Hilbert space with this inner product and ℓ is bounded thanks to (4.3). Therefore, Lax–Milgram's lemma gives the existence of a unique pair $(\widehat{q},\widehat{r}) \in E$ such that

$$\kappa((\hat{q},\hat{r}),(q,r)) = \ell(q,r) \quad \forall (q,r) \in E. \tag{4.9}$$

Let

$$\widehat{y} := e^{-2s\widehat{\beta}}\widehat{\mu}^7(\mathcal{L}_0^*\widehat{q} + \widehat{r}_x), \quad \widehat{z} := e^{-2s\widehat{\beta}}\widehat{\mu}^7(\mathcal{L}_1^*\widehat{r} + \widehat{q}_x), \quad \widehat{v} := -e^{-6s\widehat{\beta}+4s\beta^*}\widehat{\mu}^{23}\widehat{r}\mathbb{1}_\omega \tag{4.10}$$

which, from (4.9)–(4.10), verify

$$\iint\limits_{Q} e^{2s\widehat{\beta}}\widehat{\mu}^{-7}(|\widehat{y}|^2+|\widehat{z}|^2)\,\mathrm{d}x\,\mathrm{d}t+\iint\limits_{\omega\times(0,T)} e^{6s\widehat{\beta}-4s\beta^*}\widehat{\mu}^{-23}|\widehat{v}|^2\,\mathrm{d}x\,\mathrm{d}t<\infty \tag{4.11}$$

and that $(\widehat{y},\widehat{z})$ is the (unique) solution by transposition (see Definition 2.2) of system (1.3).

To conclude the proof, we set $(y^*,z^*):=e^{3s\widehat{\beta}-2s\beta^*}(\widehat{\mu})^{-21/2-1/m}(\widehat{y},\widehat{z})$. Then it satisfies the system

$$\begin{cases}\mathcal{L}_0y^*-z^*_x=f^*_0+(e^{3s\widehat{\beta}-2s\beta^*}(\widehat{\mu})^{-21/2-1/m})_ty & \text{in } Q,\\ \mathcal{L}_1z^*-y^*_x=f^*_1+v^*\mathbb{1}_\omega+(e^{3s\widehat{\beta}-2s\beta^*}(\widehat{\mu})^{-21/2-1/m})_tz & \text{in } Q,\\ y^*(0,t)=y_x(0,t)=0,\quad y^*(L,t)=y_x(L,t)=0 & \text{in } (0,T),\\ z^*(0,t)=0,\quad z^*(L,t)=0 & \text{in } (0,T),\\ (y^*(x,0),z(x,0)^*)=e^{3s\widehat{\beta}(0)-2s\beta(0)^*}\widehat{\mu}(0)^{-21/2-1/m}(y_0(x),z_0(x)) & \text{in } (0,L),\end{cases}$$

where $(f^*_0,f^*_1,v^*):=e^{3s\widehat{\beta}-2s\beta^*}(\widehat{\mu})^{-21/2-1/m}(f_0,f_1,\widehat{v})$. From (3.5), (4.1), (4.11) and the assumptions on (f_0,f_1), we have that the right-hands sides of the equations of this system belong to $L^1(0,T;W^{-1,1}(0,L))$ and $L^2(0,T;H^{-1}(0,L))$, respectively. In addition, $(y^*(x,0),z(x,0)^*)\in H^{-2}(0,L)\times H^{-1}(0,L)$. Then, Lemma 2.3 shows that

$$(y^*,z^*)\in\mathcal{C}([0,T];H^{-2}(0,L)\times H^{-1}(0,L)).$$

Therefore, $(\widehat{y},\widehat{z},\widehat{v})\in\mathcal{E}$ is the triplet we were looking for in Proposition 4.2. $\square$

5. Controllability of the nonlinear system

In this section we prove Theorem 1.1. Actually, this result is equivalent to show the local null controllability of the system

$$\begin{cases}y_t+\gamma y_{xxxx}+y_{xxx}+ay_{xx}+yy_x+\bar{y}y_x+\bar{y}_xy=z_x & \text{in } Q,\\ z_t-\Gamma z_{xx}+cz_x=y_x+v\mathbb{1}_\omega & \text{in } Q,\\ y(0,t)=y_x(0,t)=0,\quad y(L,t)=y_x(L,t)=0 & \text{in } (0,T),\\ z(0,t)=0,\quad z(L,t)=0 & \text{in } (0,T),\\ y(x,0)=y_0(x),\quad z(x,0)=z_0(x) & \text{in } (0,L).\end{cases} \tag{5.1}$$

Indeed, it suffices to take the change of unknowns $(\widetilde{y},\widetilde{z})=(y-\bar{y},z-\bar{z})$ and subtract systems (1.1) and (1.2). Therefore, we concentrate in showing that there exists $\delta>0$ such that if

$$\|y_0\|_{H^{-2}(0,L)}+\|z_0\|_{H^{-1}(0,L)}\le\delta,$$

then there exists a control v such that $y(\cdot,T)=z(\cdot,T)=0$ in $(0,L)$.

We use the following inverse mapping theorem (see [1]).

Theorem 5.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{F}:\mathcal{B}_1\to\mathcal{B}_2$ satisfy $\mathcal{F}\in\mathcal{C}^1(\mathcal{B}_1;\mathcal{B}_2)$. Assume that $b_1\in\mathcal{B}_1$, $\mathcal{F}(b_1)=b_2$ and that $\mathcal{F}'(b_1):\mathcal{B}_1\to\mathcal{B}_2$ is surjective. Then, there exists $\delta>0$ such that, for every $b'\in\mathcal{B}_2$ satisfying $\|b'-b_2\|_{\mathcal{B}_2}<\delta$, there exists a solution of the equation*

$$\mathcal{F}(b)=b',\quad b\in\mathcal{B}_1.$$

Let us prove the local null controllability of system (5.1). The idea is to apply Theorem 5.1 with the following elements:

$$\mathcal{B}_1 := \mathcal{E},$$
$$\mathcal{B}_2 := Y \times H^{-2}(0,L) \times Z \times H^{-1}(0,L),$$

and

$$\mathcal{F}(y,z,v) := \big(\mathcal{L}_0 y + yy_x - z_x, y(\cdot,0), \mathcal{L}_1 z - y_x - v\mathbb{1}_\omega, z(\cdot,0)\big),$$

where we have denoted

$$Y := \{u : e^{s\beta^*}(\mu^*)^{-5/2+1/m} u \in L^1(0,T;W^{-1,1}(0,L))\}$$

and

$$Z := \{u : e^{s\beta^*}(\mu^*)^{-5/2+1/m} u \in L^2(0,T;H^{-1}(0,L))\}.$$

Let us check that $\mathcal{F}$ is of class $\mathcal{C}^1(\mathcal{B}_1;\mathcal{B}_2)$. In fact, all the terms appearing in $\mathcal{F}$ are well defined (by the definition of $\mathcal{B}_1$ and $\mathcal{B}_2$) and linear, and therefore of class $\mathcal{C}^1$, except for yy_x. Then, it suffices to show that the map

$$\big((y_1,z_1,v_1),(y_2,z_2,v_2)\big) \to \frac{1}{2}(y_1\, y_2)_x$$

is continuous from $\mathcal{B}_1 \times \mathcal{B}_2$ to Y. Indeed,

$$\begin{aligned}
\|e^{s\beta^*}(\mu^*)^{-5/2+1/m}(y_1y_2)_x\|_{L^1(0,T;W^{-1,1}(0,L))} &\le C\|e^{2s\widehat{\beta}}\widehat{\mu}^{-7}(y_1y_2)_x\|_{L^1(0,T;W^{-1,1}(0,L))} \\
&\le C\|e^{2s\widehat{\beta}}\widehat{\mu}^{-7}y_1y_2\|_{L^1(0,T;L^1(0,L))} \\
&\le C\|e^{s\widehat{\beta}}\widehat{\mu}^{-7/2}y_1\|_{L^2(Q)}\|e^{s\widehat{\beta}}\widehat{\mu}^{-7/2}y_2\|_{L^2(Q)} \\
&\le C\|y_1\|_{\mathcal{B}_1}\|y_2\|_{\mathcal{B}_1}.
\end{aligned}$$

Furthermore, notice that

$$\mathcal{F}'(0,0,0)(y,z,v) = \big(\mathcal{L}_0 y - z_x, y(\cdot,0), \mathcal{L}_1 z - y_x - v\mathbb{1}_\omega, z(\cdot,0)\big),$$

is surjective from $\mathcal{B}_1$ to $\mathcal{B}_2$ thanks to Proposition 4.2. From Theorem 5.1 with $b_1 = (0,0,0)$, $b_2 = (0,0,0,0)$ and $b' = (0,y_0,0,z_0)$ we obtain the existence of a positive number δ such that if

$$\|y_0\|_{H^{-2}(0,L)} + \|z_0\|_{H^{-1}(0,L)} \le \delta,$$

there exists (y,z,v) solution to system (5.1) belonging to $\mathcal{E}$. In particular, $y(\cdot,T) = z(\cdot,T) = 0$ in $(0,L)$. Therefore, the proof of Theorem 1.1 is complete.

6. Final comments

In this final section, let us briefly discuss some natural questions that arise from this work.

- *Other alternatives to deal with the nonlinearity.* One could think of other alternatives to obtain controllability properties of system (1.1). For instance, we could have tried to use a fixed-point argument as in [15]. However, the nonlinearity in (1.1) is stronger than the one considered usually for the heat equation. The inverse mapping theorem approach used in this work (Section 5) seems to be the appropriate tool to treat this kind of nonlinearities (see [18]).

Another alternative to prove controllability of system (1.1) is Coron's return method (see [12, Chapter 6] for a complete presentation of the method). It was used for the first time for controllability of a partial differential equation in [11]. An example of how the return method, together with an algebraic method to solve PDE's, can be used in the context of controllability with a reduced number of controls, is [13], where the authors prove the local null controllability of the three dimensional Navier–Stokes system with only one scalar control. Although the return method is a very useful tool to prove local null controllability of nonlinear equations (or systems), it is usually applied to overcome the noncontrollability of the linearized equation. As it is shown in Section 4, this is not the case here.

- *Global null controllability in large time.* A global null controllability result can be obtained, for instance, if the system is asymptotically stable and locally null controllable. Indeed, one just needs to let the system evolve freely for a time large enough to reach the vicinity of the origin from where there exists a control driving the solution to zero. It is known from [24] that the Kuramoto–Sivashinsky equation

$$y_t + y_{xxxx} + \lambda y_{xx} + y y_x = 0$$

is exponentially stable if $\lambda < 4\pi^2/L$ and is unstable otherwise. Later, in [6], exponential stability was proved if $\lambda \geq 4\pi^2/L$ by means of a suitable feedback law.
For system (1.1), a simple computation shows that, if $a < 0$, the total energy of system (1.1) decreases exponentially by taking the feedback control $v := -z$. It is an open problem to know if there exists a feedback law such that the system (1.1) is asymptotically stable if $a > 0$.

- *Boundary controllability with one single control.* In [8], a local null controllability result for system (1.1) is proved using three boundary controls at $x = 0$. It is then natural to wonder if a similar controllability result would hold if less number of controls are present on the boundary. We can consider the system with one boundary control

$$\begin{cases} y_t + \gamma y_{xxxx} + y_{xxx} + a y_{xx} + y y_x = z_x & \text{in } Q, \\ z_t - \Gamma z_{xx} + c z_x = y_x & \text{in } Q, \\ y(0,t) = y_x(0,t) = 0, \quad y(L,t) = y_x(L,t) = 0 & \text{in } (0,T), \\ z(0,t) = v(t), \quad z(L,t) = 0 & \text{in } (0,T), \\ y(x,0) = y_0(x), \quad z(x,0) = z_0(x) & \text{in } (0,L). \end{cases} \tag{6.1}$$

Proving controllability properties for (6.1) via Carleman estimates does not seem to be a good strategy, since there is no way to relate the states of the adjoint equation on the boundary. It would be interesting to know if other techniques used for systems of heat equations (like the moment method in [16]) work in this setting.

- *The limit case $\gamma \to 0^+$.* An interesting problem would be to study the behavior of the control in (1.1) when $\gamma \to 0^+$ (and/or $\Gamma \to 0^+$). Some related works are [14] and [20], where it is proven that the boundary controls, for the Burgers and heat equations, respectively, remain bounded uniformly with respect to the diffusion parameter as it tends to zero for large control times. As for controllability of systems of parabolic equations degenerating in parabolic–elliptic systems, we find [10] and [3].
It is hard to say what to expect of (1.1) in the limit case, because of the structure of the system. Nevertheless, let us mention the recent work [5], where the authors study the cost of null controllability of the equation

$$\begin{cases} y_t + \gamma y_{xxxx} + y_x = 0 & \text{in } Q, \\ y(0,t) = v_1(t), \quad y(L,t) = 0 & \text{in } (0,T), \\ y_{xx}(0,t) = v_2(t), \quad y_{xx}(L,t) = 0 & \text{in } (0,T), \\ y(x,0) = y_0(x) & \text{in } (0,L), \end{cases}$$

in the limit $\gamma \to 0^+$. They show that for large times, the controls can be bounded uniformly with respect to γ small and, furthermore, decrease exponentially to zero as $\exp(-C(T)\gamma^{-1/3})$. The proof is based on the combination of an exponential dissipation result and a Carleman estimate with optimal weight functions (see Remark 3.4).

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