Internal null controllability of the generalized Hirota-Satsuma system*

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Abstract

The generalized Hirota-Satsuma system consists of three coupled nonlinear Korteweg-de Vries (KdV) equations. By using two distributed controls it is proven in this paper that the local null controllability property holds when the system is posed on a bounded interval. First, the system is linearized around the origin obtaining two decoupled subsystems of third order dispersive equations. This linear system is controlled with two inputs, which is optimal. This is done with a duality approach and some appropriate Carleman estimates. Then, by means of an inverse function theorem, the local null controllability of the nonlinear system is proven.

Key words. Korteweg-de Vries equation, null controllability, Carleman estimates.

1 Introduction

In the eighties, Hirota and Satsuma introduced in [14] the set of two coupled Korteweg-de Vries (KdV) equations,

$$\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x, \\ v_t + \frac{1}{2}v_{xxx} = -3uv_x, \end{cases} \tag{1.1}$$

describing the interaction of two long waves with different dispersion relations. They studied the existence of soliton solutions and conserved quantities. Later, in [21] the same authors introduced a new system, coupling now three KdV equations,

$$\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x + 3w_x, \\ v_t + \frac{1}{2}v_{xxx} = -3uv_x, \\ w_t + \frac{1}{2}w_{xxx} = -3uw_x. \end{cases} \tag{1.2}$$

This set of equations was called in the literature the generalized Hirota-Satsuma (HS) system and has attracted the attention of many researchers mainly interested in soliton or explicit solutions. See for instance [12, 22] and the references therein.

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As far as we know, there is no studies of the control properties of this kind of coupled systems. Thus, in this article the goal is to fill this gap focusing on the null controllability with distributed controls. An important point is that we obtain our results on the control of this three-equation system using only two control inputs.

Let us precise which system we will control. We can see that the first equation in (1.2) is of KdV type with a negative dispersive term whereas the two others have positive dispersive term. Considering these facts, we propose to study equations (1.2) on a spatial domain $[0, L]$ with the usual boundary conditions for KdV equations, as for instance in [18],

$$\begin{cases} u(t,0) = u(t,L) = 0,\ u_x(t,0) = 0, \\ v(t,0) = v(t,L) = 0,\ v_x(t,L) = 0, \\ w(t,0) = w(t,L) = 0,\ w_x(t,L) = 0, \end{cases} \tag{1.3}$$

and the initial conditions

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x). \tag{1.4}$$

As mentioned previously, we consider here the internal control case. Thus, we study the following system, with $T > 0$ and $Q = (0,T) \times (0,L)$,

$$\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x + 3w_x, & (t,x) \in Q, \\ v_t + \frac{1}{2}v_{xxx} = -3uv_x + p\mathbb{1}_\gamma, & (t,x) \in Q, \\ w_t + \frac{1}{2}w_{xxx} = -3uw_x + q\mathbb{1}_\omega, & (t,x) \in Q, \\ u(t,0) = u(t,L) = 0,\ u_x(t,0) = 0, & t \in (0,T), \\ v(t,0) = v(t,L) = 0,\ v_x(t,L) = 0, & t \in (0,T), \\ w(t,0) = w(t,L) = 0,\ w_x(t,L) = 0, & t \in (0,T), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x), & x \in (0,L), \end{cases} \tag{1.5}$$

where $p = p(t,x)$ and $q = q(t,x)$ are the distributed controls acting on two subdomains γ and ω with $\gamma \subset (0,L)$ and either $\omega = (a,L)$ or $\omega = (0,a)$ for some $0 < a < L$. From now on, we only consider $\omega = (a,L)$ but everything can be done in similar ways for the other case.

The control of dispersive equations is an active research field. The first results for single KdV equations with internal controls were presented in [19, 20] where periodic domains were considered. Also in this framework we found the paper [15]. More related to this paper we can cite [6] where the authors study the internal control of a KdV equation on a bounded domain with the same kind of boundary conditions than here. They use duality arguments and a Carleman estimate to prove an observability inequality.

Regarding dispersive systems, we find papers dealing with the boundary controls of either KdV systems on a bounded domain [8, 16, 4, 5] or KdV equations posed on a network [1, 7]. Concerning the internal control of dispersive systems, the closest works are [17] where Ingham theorems are used to prove some observability inequalities for Boussinesq systems and [2] where a Carleman estimates approach is used to get the null controllability of a linear system coupling a KdV equation with a Schrödinger equation.

Summarizing the links with the existent literature, in this paper we follow the same methods than in [6] and [2] to study the null controllability property of a dispersive system with less controls than equations.

Let us go back to the control of system (1.5). The first step in our strategy is to linearize the system (1.5) around the origin, getting the linear system

$$\begin{cases} u_t - \frac{1}{4}u_{xxx} = f_1 + 3w_x, & (t,x) \in Q, \\ v_t + \frac{1}{2}v_{xxx} = f_2 + p\mathbb{1}_\gamma, & (t,x) \in Q, \\ w_t + \frac{1}{2}w_{xxx} = f_3 + q\mathbb{1}_\omega, & (t,x) \in Q, \\ u(t,0) = u(t,L) = 0,\ u_x(t,0) = 0, & t \in (0,T), \\ v(t,0) = v(t,L) = 0,\ v_x(t,L) = 0, & t \in (0,T), \\ w(t,0) = w(t,L) = 0,\ w_x(t,L) = 0, & t \in (0,T), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x), & x \in (0,L), \end{cases} \tag{1.6}$$

where f_1, f_2 and f_3 will play later the role of the nonlinearities. In order to study the null controllability of (1.6) we apply a duality approach that leads us to prove that the solutions of the adjoint system

$$\begin{cases} -\phi_t + \frac{1}{4}\phi_{xxx} = g_1, & (t,x) \in Q, \\ -\psi_t - \frac{1}{2}\psi_{xxx} = g_2, & (t,x) \in Q, \\ -\eta_t - \frac{1}{2}\eta_{xxx} = g_3 - 3\phi_x, & (t,x) \in Q, \\ \phi(t,0) = \phi(t,L) = 0,\ \phi_x(t,L) = 0, & t \in (0,T), \\ \psi(t,0) = \psi(t,L) = 0,\ \psi_x(t,0) = 0, & t \in (0,T), \\ \eta(t,0) = \eta(t,L) = 0,\ \eta_x(t,0) = 0, & t \in (0,T), \\ \phi(T,x) = \phi_T(x),\ \psi(T,x) = \psi_T(x),\ \eta(T,x) = \eta_T(x), & x \in (0,L), \end{cases} \tag{1.7}$$

satisfy an appropriate observability inequality. This is realized proving a Carleman estimate for system (1.7) where functions g_1, g_2 and g_3 are useful to get information on the solutions of (1.6) when using duality arguments.

Finally, the last step in our strategy is to go back to the original nonlinear system by using an inverse function theorem. In this way we will get our main result, stating the local null controllability of (1.5).

Theorem 1 *Let $\gamma \subset (0,L)$ and $\omega = (a,L)$, with $a \in (0,L)$. Assume that $(u_0,v_0,w_0) \in L^2(0,L)$. Then, for every $T > 0$ there exists $\delta > 0$ such that if $\|(u_0,v_0,w_0)\|_{L^2(0,L)^3} < \delta$, there are controls $p \in L^2(0,T;L^2(\gamma))$ and $q \in L^2(0,T;L^2(\omega))$ such that the solution (u,v,w) of (1.5) satisfies*

$$u(T,x) = v(T,x) = w(T,x) = 0 \quad in \quad (0,L).$$

The organization of this paper is the following. We start giving in Section 2 the well-posedness framework in which we work along this paper. Then, Section 3 is devoted to the proof of a Carleman estimate that is used to prove an appropriate observability inequality. Section 4 contains the control results for both the linear and nonlinear systems. Finally, we end this paper with some comments and related open problems.

2 Well-posedness results

In this section, we give the functional framework and some well-possedness results for the KdV equation, and the linear and nonlinear systems.

2.1 Functional spaces

We introduce the following functional spaces:

$$\begin{aligned} &X_0 := L^2(0,T;H^{-2}(0,L)), \quad X_1 := L^2(0,T;H_0^2(0,L)), \\ &\tilde{X}_0 := L^1(0,T;H^{-1}(0,L)), \quad \tilde{X}_1 := L^1(0,T;H^3(0,L) \cap H_0^2(0,L)), \end{aligned} \tag{2.1}$$

and

$$Y_0 := L^2(0,T;L^2(0,L)) \cap C([0,T];H^{-1}(0,L)),$$
$$Y_1 := L^2(0,T;H^4(0,L)) \cap C([0,T];H^3(0,L)). \tag{2.2}$$

These spaces are equipped with their usual norms. Moreover, we define for each $\theta \in [0,1]$ the interpolation spaces (see [3]):

$$X_\theta := (X_0, X_1)_{[\theta]}, \quad \tilde{X}_\theta := (\tilde{X}_0, \tilde{X}_1)_{[\theta]} \quad \text{and} \quad Y_\theta := (Y_0, Y_1)_{[\theta]}.$$

A sample of spaces that will be often used in the following is

$$X_{1/4} = L^2(0,T;H^{-1}(0,L)),\ \tilde{X}_{1/4} = L^1(0,T;L^2(0,L)),$$
$$Y_{1/4} = L^2(0,T;H^1(0,L)) \cap C([0,T];L^2(0,L)).$$

2.2 Regularity results for a single equation

We first consider a single KdV equation with a source term:

$$\begin{cases} \chi_t + \chi_{xxx} = g, & \text{in } Q, \\ \chi(t,0) = \chi(t,L) = \chi_x(t,L) = 0, & \text{in } (0,T), \\ \chi(0,x) = \chi_0(x), & \text{in } (0,L). \end{cases} \tag{2.3}$$

For this equation we have the following known results.

Proposition 2 *[13, Section 2.2.2] If $\chi_0 \in L^2(0,L)$ and $g \in G$ with $G = X_{1/4}$ or $G = \tilde{X}_{1/4}$, then system* (2.3) *admits a unique solution $\chi \in Y_{1/4}$. Moreover, there exists a constant $C > 0$ such that*

$$\|\chi\|_{Y_{1/4}} \leq C(\|g\|_G + \|\chi_0\|_{L^2(0,L)}). \tag{2.4}$$

Proposition 3 *[13, Section 2.3.1] If $\chi_0 \in H^3(0,L)$ is such that $\chi_0(0) = \chi_0(L) = \chi_0'(0) = 0$ and $g \in G$ with $G = X_1$ or $G = \tilde{X}_1$, then system* (2.3) *admits a unique solution $\chi \in Y_1$. Moreover, there exists a constant $C > 0$ such that*

$$\|\chi\|_{Y_1} \leq C(\|g\|_G + \|\chi_0\|_{H^3(0,L)}). \tag{2.5}$$

Proposition 4 *[13, Section 2.3.2] Let $\theta \in [1/4, 1]$ and $\chi_0 = 0$. If $g \in G$ with $G = X_\theta$ or $G = \tilde{X}_\theta$, then system* (2.3) *admits a unique solution $\chi \in Y_\theta$. Moreover, there exists a constant $C > 0$ such that*

$$\|\chi\|_{Y_\theta} \leq C\|g\|_G. \tag{2.6}$$

Notice that the same results are valid for the (backward-in-time) adjoint equation

$$\begin{cases} -\chi_t - d\chi_{xxx} = g, & \text{in } Q, \\ \chi(t,0) = \chi(t,L) = \chi_x(t,0) = 0, & \text{in } (0,T), \\ \chi(T,x) = \chi_0(x), & \text{in } (0,L), \end{cases} \tag{2.7}$$

and the reverse-in-space equation,

$$\begin{cases} \chi_t - d\chi_{xxx} = g, & \text{in } Q, \\ \chi(t,0) = \chi(t,L) = \chi_x(t,0) = 0, & \text{in } (0,T), \\ \chi(0,x) = \chi_0(x), & \text{in } (0,L), \end{cases} \tag{2.8}$$

for any dispersive coefficient $d > 0$.

2.3 Regularity results for the linear system

We first consider the linear system (1.6). Taking advantage of its cascade structure, notice that we can apply the results for a single equation in order to get the solutions v and w (Proposition 2, for instance). Then, we can see the term $3w_x$ as a source term in the equation satisfied by u. Therefore, we can easily obtain the following result.

Proposition 5 *Let $u_0, v_0, w_0 \in L^2(0,L)$, $p \in L^2(0,T;L^2(\gamma))$, $q \in L^2(0,T;L^2(\omega))$, and $f_1, f_2, f_3 \in G$ with $G = X_{1/4}$ or $G = \tilde{X}_{1/4}$. Then, system* (1.6) *admits a unique solution $(u,v,w) \in (Y_{1/4})^3$. Moreover, there exists a constant $C > 0$ such that*

$$\|(u,v,w)\|_{(Y_{1/4})^3} \leq C\Big(\|(u_0,v_0,w_0)\|_{L^2(0,L)^3} + \|(f_1,f_2,f_3)\|_{G^3} + \|p\|_{L^2(0,T;L^2(\gamma))} + \|q\|_{L^2(0,T;L^2(\omega))}\Big). \tag{2.9}$$

The regularity $p \in L^2(0,T;L^2(\gamma))$ and $q \in L^2(0,T;L^2(\omega))$ is enough to be sure that $p\mathbb{1}_\gamma$ and $q\mathbb{1}_\omega$ belong to both $L^2(0,T;H^{-1}(0,L))$ and $L^1(0,T;L^2(0,L))$. Consequently they can be seen as appropriate source terms in Proposition 2.

This result can be applied to the adjoint system (1.7) with appropriate functions g_1, g_2, and g_3. To do that we only need to perform a change of variable in space $x \approx L - x$ and time $t \approx T - t$.

2.4 Regularity results for the nonlinear system

In this section we apply a fixed point argument in order to establish the well-posedness of the nonlinear system (1.5). First of all, we prove the following lemma inspired from [18].

Lemma 6 *Let $y, z \in L^2(0,T;H^1(0,L))$. Then $yz_x \in L^1(0,T;L^2(0,L))$ and the map $(y,z) \in (L^2(0,T;H^1(0,L)))^2 \mapsto yz_x \in L^1(0,T;L^2(0,L))$ is continuous.*

Proof.

Let (y,z) and $(\tilde{y},\tilde{z})$ in $(L^2(0,T;H^1(0,L)))^2$, and let us denote by K the norm of the embedding $H^1(0,L) \hookrightarrow L^\infty(0,L)$. We then have

$$\begin{aligned}
\|yz_x - \tilde{y}\tilde{z}_x\|_{L^1(0,T;L^2(0,L))} &\leq \int_0^T \|(y-\tilde{y})z_x\|_{L^2(0,L)}dt + \int_0^T \|\tilde{y}(z-\tilde{z})_x\|_{L^2(0,L)}dt \\
&\leq \int_0^T \|y-\tilde{y}\|_{L^\infty(0,L)}\|z_x\|_{L^2(0,L)}dt + \int_0^T \|\tilde{y}\|_{L^\infty(0,L)}\|(z-\tilde{z})_x\|_{L^2(0,L)}dt \\
&\leq K\left(\int_0^T \|y-\tilde{y}\|_{H^1(0,L)}\|z\|_{H^1(0,L)}dt + \int_0^T \|\tilde{y}\|_{H^1(0,L)}\|z-\tilde{z}\|_{H^1(0,L)}dt\right) \\
&\leq K\|(y,z)\|_{(L^2(0,T;H^1(0,L)))^2}\|(y-\tilde{y}, z-\tilde{z})\|_{(L^2(0,T;H^1(0,L)))^2},
\end{aligned}$$

which proves Lemma 6. ∎

We can now prove the following well-posedness result.

Proposition 7 *Let $L > 0$ and $T > 0$. There exist $\varepsilon > 0$ and $C > 0$ such that for every $(u_0,v_0,w_0) \in L^2(0,L)^3$, $p \in L^2(0,T;L^2(\gamma))$, $q \in L^2(0,T;L^2(\omega))$, such that*

$$\|(u_0,v_0,w_0)\|_{L^2(0,L)^3} + \|p\|_{L^2(0,T;L^2(\gamma))} + \|q\|_{L^2(0,T;L^2(\omega))} \leq \varepsilon$$

there exists a unique solution $(u,v,w) \in (Y_{1/4})^3$ of the non linear equation (1.5) *that satisfies*

$$\|(u,v,w)\|_{(Y_{1/4})^3} \leq C\Big(\|(u_0,v_0,w_0)\|_{L^2(0,L)^3} + \|p\|_{L^2(0,T;L^2(\gamma))} + \|q\|_{L^2(0,T;L^2(\omega))}\Big).$$

Proof.

Let $(u_0, v_0, w_0) \in L^2(0,L)^3$, $p \in L^2(0,T;L^2(\gamma))$, $q \in L^2(0,T;L^2(\omega))$, such that

$$\|(u_0, v_0, w_0)\|_{L^2(0,L)^3} + \|p\|_{L^2(0,T;L^2(\gamma))} + \|q\|_{L^2(0,T;L^2(\omega))} \le \varepsilon$$

where ε will be chosen small enough later. Let $(u, v, w) \in (Y_{1/4})^3$ and consider the map $\Phi : (Y_{1/4})^3 \to (Y_{1/4})^3$ defined by $\Phi(u, v, w) = (\tilde{u}, \tilde{v}, \tilde{w})$ where $(\tilde{u}, \tilde{v}, \tilde{w})$ is the solution of the linear problem,

$$\begin{cases} \tilde{u}_t - \frac{1}{4}\tilde{u}_{xxx} = 3uu_x - 6vv_x + 3\tilde{w}_x, & \text{in } Q, \\ \tilde{v}_t + \frac{1}{2}\tilde{v}_{xxx} = -3uv_x + p\mathbb{1}_\gamma, & \text{in } Q, \\ \tilde{w}_t + \frac{1}{2}\tilde{w}_{xxx} = -3uw_x + q\mathbb{1}_\omega, & \text{in } Q, \\ \tilde{u}(t,0) = \tilde{u}(t,L) = 0,\ \tilde{u}_x(t,0) = 0, & \text{in } (0,T), \\ \tilde{v}(t,0) = \tilde{v}(t,L) = 0,\ \tilde{v}_x(t,L) = 0, & \text{in } (0,T), \\ \tilde{w}(t,0) = \tilde{w}(t,L) = 0,\ \tilde{w}_x(t,L) = 0, & \text{in } (0,T), \\ \tilde{u}(0,x) = u_0(x),\ \tilde{v}(0,x) = v_0(x),\ \tilde{w}(0,x) = w_0(x), & \text{in } (0,L). \end{cases}$$

By Proposition 5 we have

(2.10)

$$\|\Phi(u,v,w)\|_{Y^3_{1/4}} = \|(\tilde{u}, \tilde{v}, \tilde{w})\|_{Y^3_{1/4}} \le C\Big(\|(u_0, v_0, w_0)\|_{L^2(0,L)^3} + \|(3uu_x - 6vv_x, -3uv_x, -3uw_x)\|_{\tilde{X}^3_{1/4}} + \|p\|_{L^2(0,T;L^2(\gamma))} + \|q\|_{L^2(0,T;L^2(\omega))}\Big).$$

By Lemma 6, we obtain,

$$\|\Phi(u,v,w)\|_{Y^3_{1/4}} = \|(\tilde{u}, \tilde{v}, \tilde{w})\|_{Y^3_{1/4}} \le C\Big(\|(u_0, v_0, w_0)\|_{L^2(0,L)^3} + \|(u,v,w)\|^2_{Y^3_{1/4}} + \|p\|_{L^2(0,T;L^2(\gamma))} + \|q\|_{L^2(0,T;L^2(\omega))}\Big). \tag{2.11}$$

We also have, for any $(u_1, v_1, w_1) \in (Y_{1/4})^3$ and $(u_2, v_2, w_2) \in (Y_{1/4})^3$,

(2.12)

$$\|\Phi(u_1,v_1,w_1) - \Phi(u_2,v_2,w_2)\|_{Y^3_{1/4}} \le C\Big(\|(u_1,v_1,w_1)\|_{Y^3_{1/4}} + \|(u_2,v_2,w_2)\|_{Y^3_{1/4}}\Big)\|(u_1,v,w_1) - (u_2,v_2,w_2)\|_{Y^3_{1/4}}$$

Thus, if we restrict Φ to a closed ball $\overline{B(0,R)} = \{(u,v,w) \in (Y_{1/4})^3,\ \|(u,v,w)\|^2_{Y^3_{1/4}} \le R\}$ where $R > 0$ will be chosen later, we have the estimate,

$$\|\Phi(u,v,w)\|_{Y^3_{1/4}} \le C(\varepsilon + R^2) \text{ and } \|\Phi(u_1,v_1,w_1) - \Phi(u_2,v_2,w_2)\|_{Y^3_{1/4}} \le 2CR\|(u_1,v,w_1) - (u_2,v_2,w_2)\|_{Y^3_{1/4}}.$$

Then if we take R and ε such that $R < \frac{1}{2C}$ and $\varepsilon < \frac{R}{2C}$, we can apply the Banach fixed point theorem and Φ admits a unique fixed point, which ends the proof of Proposition 7.

■

3 Carleman inequalities

This section is dedicated to Carleman estimates. First, we present a general estimate for a KdV equation with observation in an interior domain. Then, we will prove a new Carleman estimate for the whole adjoint system (1.7).

3.1 Carleman weights

Let $\omega_0 = (a_0, b_0) \subset (0, L)$, and set $c_0 = (a_0 + b_0)/2$. Consider the weight functions defined in [2], namely for $K_1, K_2 > 0$, let

$$\varphi_0(x) = K_1(1 - e^{-K_2(x-c_0)^2}) + 1, \quad \xi(t) = \frac{1}{t(T-t)} \tag{3.1}$$

and

$$\varphi(t, x) := \xi(t)\varphi_0(x). \tag{3.2}$$

Notice that, for any $K_1, K_2 > 0$, we have

$$\varphi > 0 \text{ in } (0, T) \times [0, L], \tag{3.3}$$

$$|\varphi_x| > 0 \text{ in } (0, T) \times ([0, L] \setminus \bar{\omega}_0), \tag{3.4}$$

$$\varphi_x(t, 0) < 0, \ \varphi_x(t, L) > 0 \text{ in } (0, T). \tag{3.5}$$

Furthermore, K_1 and K_2 can be chosen such that

$$\varphi_{xx} < 0 \text{ in } (0, T) \times ([0, L] \setminus \bar{\omega}_0), \tag{3.6}$$

and

$$56\check{\varphi}(t) > 55\hat{\varphi}(t) \text{ in } (0, T), \tag{3.7}$$

where $\check{\varphi}(t) := \min\limits_{x\in[0,L]} \varphi(t, x)$ and $\hat{\varphi}(t) := \max\limits_{x\in[0,L]} \varphi(t, x)$.
Indeed, property (3.6) holds for

$$K_2 = \frac{4}{(b_0 - a_0)^2}.$$

Now, let us notice that

$$\check{\varphi}(t) = \varphi(t, c_0) = \xi(t),$$

and

$$\hat{\varphi}(t) = \max\{\varphi(t, 0), \varphi(t, L)\} = \xi(t) \max\{\varphi_0(0), \varphi_0(L)\},$$

since the extremum of the interval where the maximum is achieved depends on the location of c_0. Thus, if we call

$$C(K_2, c_0) = \max\{1 - e^{-K_2 c_0^2}, 1 - e^{-K_2(L-c_0)^2}\},$$

then, it suffices to take $K_1 = (110\, C(K_2, c_0))^{-1}$ for (3.7) to hold.

3.2 Carleman estimate for a single KdV equation

In this section, we establish a Carleman estimate for the general backward in time KdV equation of the following type, for $\nu \in \mathbb{R}^*$:

$$
(3.8) \qquad \begin{cases} y_t + \nu y_{xxx} = g, & \text{in } Q, \\ y(t,0) = y(t,L) = 0, & \text{in } (0,T), \\ (\frac{\nu}{|\nu|}+1) y_x(t,0) + (\frac{\nu}{|\nu|}-1) y_x(t,L) = 0, & \text{in } (0,T), \\ y(T,x) = y_T(x), & \text{in } (0,L). \end{cases}
$$

To begin, we recall a Carleman estimate for the linear KdV equation (3.8) obtained in [2, Theorem 3.1] and [6, Proposition 3.1]. Their results are obtained in the case $\nu > 0$, but they can easily be converted in the case $\nu < 0$ by using the change of variables $x \mapsto L - x$. We can re-write that estimate as follows.

Proposition 8 *Let $T > 0$ and $\omega_0 \subset (0,L)$ as in Section 3.1. There exist $C_0 > 0$, and $s_0 > 0$ such that for any $g \in L^2(0,T;L^2(0,L))$, $y_T \in L^2(0,L)$, and $s \geq s_0$, the solution y of (3.8) satisfies*

$$
(3.9) \quad \int_Q \left[s\xi |y_{xx}|^2 + (s\xi)^3 |y_x|^2 + (s\xi)^5 |y|^2 \right] e^{-2s\varphi} dxdt \leq C_0 \left(\int_Q e^{-2s\varphi} |g|^2 dxdt + \int_0^T \int_{\omega_0} \left[s^5 \xi^5 |y|^2 + s\xi |y_{xx}|^2 \right] e^{-2s\varphi} dxdt \right).
$$

The idea is to set the path for the Carleman estimate for the adjoint system (1.7). To this end, we will prove from estimate (3.9) the following inequality with more regular right-hand side in (3.8).

Proposition 9 *Let $T > 0$ and $\omega_0 \subset (0,L)$ as in Section 3.1. There exist $C_0 > 0$, and $s_0 > 0$ such that for any $y_T \in L^2(0,L)$, and $s \geq s_0$:*

If $g \in L^2(0,T;H^{1/3}(0,L))$, then the solution y of (3.8) satisfies

$$
(3.10) \quad \int_Q [s\xi |y_{xx}|^2 + (s\xi)^3 |y_x|^2 + (s\xi)^5 |y|^2] e^{-2s\varphi} dxdt + \int_0^T s\xi^{-3} e^{-2s\hat{\varphi}} \|y\|^2_{H^{7/3}(0,L)} dt \leq C \int_Q s^3 \xi e^{-2s\varphi} |g|^2 dxdt + C \int_0^T s\xi^{-3} e^{-2s\hat{\varphi}} \|g\|^2_{H^{1/3}(0,L)} dt + C \int_0^T \int_{\omega_0} s\xi^{25} |y|^2 e^{-2s(7\check{\varphi} - 6\hat{\varphi})} dxdt.
$$

If $g \in L^2(0,T;H^{2/3}(0,L))$, then the solution y of (3.8) satisfies

$$
(3.11) \quad \int_Q [s\xi |y_{xx}|^2 + (s\xi)^3 |y_x|^2 + (s\xi)^5 |y|^2] e^{-2s\varphi} dxdt + \int_0^T s\xi^{-3} e^{-2s\hat{\varphi}} \|y\|^2_{H^{8/3}(0,L)} dt \leq C \int_Q s^3 \xi e^{-2s\varphi} |g|^2 dxdt + C \int_0^T s\xi^{-3} e^{-2s\hat{\varphi}} \|g\|^2_{H^{2/3}(0,L)} dt + C \int_0^T \int_{\omega_0} s\xi^{13} |y|^2 e^{-2s(4\check{\varphi} - 3\hat{\varphi})} dxdt.
$$

Proof.

To begin the proof, notice that, from the properties of the weight function φ, we can write from (3.9),

$$\begin{aligned}\int_Q [s\xi|y_{xx}|^2 + (s\xi)^3|y_x|^2 + (s\xi)^5|y|^2]e^{-2s\varphi}dxdt \\ \leq C_0 \left(\int_Q e^{-2s\varphi}|g|^2 dxdt + \int_0^T\int_{\omega_0} s^5\xi^5|y|^2 e^{-2s\varphi}dxdt + \int_0^T\int_{\omega_0} s\xi|y_{xx}|^2 e^{-2s\check{\varphi}}dxdt \right).\end{aligned} \tag{3.12}$$

We will now apply a bootstrap argument in order to eliminate the local term of y_{xx} appearing in the right hand-side of (3.12). Let

$$I := \int_0^T\int_{\omega_0} s\xi e^{-2s\check{\varphi}}|y_{xx}|^2 dxdt.$$

Since $\check{\varphi}$ does not depend on space, we have

$$I \leq s\int_0^T \xi e^{-2s\check{\varphi}}\|y\|^2_{H^2(\omega_0)}dt.$$

Let $\mu \in (0,1]$ and $\varepsilon > 0$. Using an interpolation argument between the spaces $H^{2+\mu}(\omega_0)$ and $L^2(\omega_0)$, together with Young's inequality, we have

$$\begin{aligned} I &\leq C\int_0^T s\xi e^{-2s\check{\varphi}}\|y\|^{4/(2+\mu)}_{H^{2+\mu}(\omega_0)}\|y\|^{2\mu/(2+\mu)}_{L^2(\omega_0)}dt \\ &\leq \varepsilon\int_0^T s\xi^{-3}e^{-2s\hat{\varphi}}\|y\|^2_{H^{2+\mu}(\omega_0)}dt + C_\varepsilon\int_0^T s\xi^{1+8/\mu}e^{-2s[(1+2/\mu)\check{\varphi}-2/\mu\hat{\varphi}]}\|y\|^2_{L^2(\omega_0)}dt.\end{aligned} \tag{3.13}$$

The idea is now to remove the first term in the right-hand side of (3.13). We follow the same arguments as in [13, 6, 2] and adapt a technique of bootstrap. We define $y_1(t,x) := \theta_1(t)y(t,x)$ with $\theta_1(t) = s^{1/2}\xi^{1/2}e^{-s\hat{\varphi}}$. Thus y_1 is solution of the system,

$$\begin{cases} y_{1t} + \nu y_{1xxx} = f_1 := \theta_1 g + \theta_{1t}y & \text{in } Q,\\ y_1(t,0) = y_1(t,L) = 0, & \text{in } (0,T),\\ (\frac{\nu}{|\nu|}+1)y_{1x}(t,0) + (\frac{\nu}{|\nu|}-1)y_{1x}(t,L) = 0, & \text{in } (0,T),\\ y_1(T,x) = 0 & \text{in } (0,L).\end{cases}$$

As $|\theta_{1t}| \leq Cs^{3/2}\xi^{5/2}e^{-s\hat{\varphi}}$, we have for $C>0$ and all $s \geq s_0$, that $f_1 \in L^2(Q) = X_{1/2}$ and

$$\|f_1\|^2_{L^2(Q)} \leq C\int_Q s\xi e^{-2s\hat{\varphi}}|g|^2dxdt + C\int_Q s^3\xi^5 e^{-2s\hat{\varphi}}|y|^2dxdt. \tag{3.14}$$

Then, from Proposition 4, we have that $y_1 \in Y_{1/2}$, and, in particular,

$$\|y_1\|^2_{L^2(0,T;H^2(0,L))} \leq C\|f_1\|^2_{L^2(Q)}. \tag{3.15}$$

Now we take $y_2(t,x) := \theta_2(t)y(t,x)$ with $\theta_2(t) = s^{1/2}\xi^{-3/2}e^{-s\hat{\varphi}}$. Then, y_2 satisfies the system

$$\begin{cases} y_{2t} + \nu y_{2xxx} = f_2 := \theta_2 g + \theta_{2t}\theta_1^{-1}y_1 & \text{in } (0,T)\times(0,L),\\ y_2(t,0) = y_2(t,L) = 0 & \text{in } (0,T),\\ (\frac{\nu}{|\nu|}+1)y_{2x}(t,0) + (\frac{\nu}{|\nu|}-1)y_{2x}(t,L) = 0, & \text{in } (0,T),\\ y_2(T,x) = 0 & \text{in } (0,L).\end{cases}$$

Notice that since $|\theta_{2t}\theta_1^{-1}| \leq Cs$, and if $g \in L^2(0,T;H^\mu(0,L))$, we have that $f_2 \in L^2(0,T;H^\mu(0,L))$. From Proposition 4 (with $= X_{1/2+\mu/4}$), we deduce that

$$y_2 \in Y_{1/2+\mu/4} = L^2(0,T;H^{2+\mu}(0,L)) \cap L^\infty(0,T;H^{1+\mu}(0,L)),$$

and,

$$\|y_2\|^2_{Y_{1/2+\mu/4}} \leq C\|f_2\|^2_{L^2(0,T;H^\mu(0,L))}.$$

In particular,

$$\|y_2\|^2_{L^2(0,T;H^{2+\mu}(0,L))} \leq Cs\int_0^T \xi^{-3}e^{-2s\hat{\varphi}}\|g\|^2_{H^\mu(0,L)}dt + Cs^2\|y_1\|^2_{L^2(0,T;H^2(0,L))}. \tag{3.16}$$

Then we get, from (3.14), (3.15) and (3.16)

$$\begin{aligned}\int_0^T s\xi^{-3}e^{-2s\hat{\varphi}}\|y(t,.)\|^2_{H^{2+\mu}(0,L)}dt \\ \leq C\int_0^T \left(s\xi^{-3}\|g\|^2_{H^\mu(0,L)} + s^3\xi\|g\|^2_{L^2(0,L)}\right)e^{-2s\hat{\varphi}}dt + C\int_Q s^5\xi^5|y|^2e^{-2s\hat{\varphi}}dxdt.\end{aligned} \tag{3.17}$$

By combining (3.17), (3.13) and (3.12), together with a good choice of ε, we get Carleman estimates (3.10) and (3.11) taking μ equal to 1/3 and 2/3, respectively.

■

3.3 Carleman estimate for the adjoint system

We now prove a Carleman estimate for the adjoint system (1.7). For this, we will use two weight functions. Given $\omega_1 = (a_1,b_1)$, and $\gamma_1 = (a_2,b_2)$ two proper subsets of $(0,L)$, we define φ_0^1 and φ_0^2 as in (3.1) associated to the subsets ω_1 and γ_1, respectively. Then, for $i = 1,2$, let

$$\varphi_i(t,x) := \xi(t)\varphi_0^i(x),$$

$$I^i_{7/3}(y) := \int_Q \left[s\xi|y_{xx}|^2 + (s\xi)^3|y_x|^2 + (s\xi)^5|y|^2\right]e^{-2s\varphi_i}dxdt + \int_0^T s\xi^{-3}e^{-2s\hat{\varphi}_i}\|y\|^2_{H^{7/3}(0,L)}dt$$

and

$$I^i_{8/3}(y) := \int_Q \left[s\xi|y_{xx}|^2 + (s\xi)^3|y_x|^2 + (s\xi)^5|y|^2\right]e^{-2s\varphi_i}dxdt + \int_0^T s\xi^{-3}e^{-2s\hat{\varphi}_i}\|y\|^2_{H^{8/3}(0,L)}dt.$$

The main result of this section is the following.

Theorem 10 *Let ω, and γ subsets of $(0,L)$ as in Theorem 1. Fix ω_1 and γ_1 proper subsets of ω and γ, respectively, such that $\bar{\omega}_1 \subset \omega$ and $\bar{\gamma}_1 \subset \gamma$. Then, there exist $C_0 > 0$, and $s_0 > 0$ such that for any $g_1 \in L^2(0,T;H^{2/3}(0,L))$, $g_2 \in L^2(0,T;H^{1/3}(0,L))$, and $g_3 \in L^2(0,T;H^{1/3}(0,L))$ and $s \geq s_0$, the solution (ϕ,ψ,η) of system (1.7) satisfies*

$$\begin{aligned}I^1_{8/3}(\phi) + I^2_{7/3}(\psi) + I^1_{7/3}(\eta) \\ \leq C\int_0^T\int_\gamma s\xi^{25}e^{-2s(7\check{\varphi}_2-6\hat{\varphi}_2)}|\psi|^2dxdt + C\int_0^T\int_\omega s\xi^{221}e^{-2s(56\check{\varphi}_1-55\hat{\varphi}_1)}|\eta|^2dxdt \\ + C\int_0^T s^3\xi e^{-2s\check{\varphi}_1}\|g_1\|^2_{H^{2/3}(0,L)}dt + C\int_0^T s^3\xi e^{-2s\check{\varphi}_2}\|g_2\|^2_{H^{1/3}(0,L)}dt \\ + C\int_0^T s^3\xi^{23}e^{-2s(8\check{\varphi}_1-7\hat{\varphi}_1)}\|g_3\|^2_{H^{1/3}(0,L)}dt,\end{aligned} \tag{3.18}$$

where φ_1 and φ_2 are the weight functions associated to ω_1 and γ_1, respectively.

Proof.

We begin applying Proposition 9 to the equation in (1.7) satisfied by ψ, taking $\omega_0 = \gamma_1$, $\varphi = \varphi_2$, $\nu = 1/2$, and $g = -g_2$. From (3.10), we obtain

$$I^2_{7/3}(\psi) \leq C \int_Q s^3 \xi e^{-2s\varphi_2} |g_2|^2 dxdt + C \int_0^T s\xi^{-3} e^{-2s\hat{\varphi}_2} \|g_2\|^2_{H^{1/3}(0,L)} dt + C \int_0^T \int_{\gamma_1} s\xi^{25} e^{-2s(7\check{\varphi}_2 - 6\hat{\varphi}_2)} |\psi|^2 dxdt.$$

Using the properties of the weight functions, we have

$$I^2_{7/3}(\psi) \leq C \int_0^T s^3 \xi e^{-2s\check{\varphi}_2} \|g_2\|^2_{H^{1/3}(0,L)} dt + C \int_0^T \int_{\gamma_1} s\xi^{25} e^{-2s(7\check{\varphi}_2 - 6\hat{\varphi}_2)} |\psi|^2 dxdt. \tag{3.19}$$

Now, for ϕ we apply the second inequality of Proposition 9 with $\omega_0 = \omega_1$, $\varphi = \varphi_1$, $\nu = -1/4$, and $g = -g_1$. In this way, from (3.11) we get, after using the properties of the weight functions, the estimate

$$I^1_{8/3}(\phi) \leq C \int_0^T s^3 \xi e^{-2s\check{\varphi}_1} \|g_1\|^2_{H^{2/3}(0,L)} dt + C \int_0^T \int_{\omega_1} s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} |\phi|^2 dxdt. \tag{3.20}$$

Lastly, we apply Proposition 9 to the equation in (1.7) satisfied by η, with $\omega_0 = \omega_1$, $\varphi = \varphi_1$, $\nu = 1/2$, and $g = -g_3 + 3\phi_x$. From (3.10), we obtain

$$I^1_{7/3}(\eta) \leq C \int_Q s^3 \xi e^{-2s\varphi_1} |g_3 - 3\phi_x|^2 dxdt + C \int_0^T s\xi^{-3} e^{-2s\hat{\varphi}_1} \|g_3 - 3\phi_x\|^2_{H^{1/3}(0,L)} dt + C \int_0^T \int_{\omega_1} s\xi^{25} e^{-2s(7\check{\varphi}_1 - 6\hat{\varphi}_1)} |\eta|^2 dxdt,$$

from where we deduce

$$I^1_{7/3}(\eta) \leq C \int_0^T s^3 \xi e^{-2s\check{\varphi}_1} \|g_3\|^2_{H^{1/3}(0,L)} dt + C \int_0^T \int_{\omega_1} s\xi^{25} e^{-2s(7\check{\varphi}_1 - 6\hat{\varphi}_1)} |\eta|^2 dxdt + C I^1_{8/3}(\phi). \tag{3.21}$$

Putting together inequalities (3.19)-(3.20) we have

$$\begin{aligned} I^1_{8/3}(\phi) + I^2_{7/3}(\psi) + I^1_{7/3}(\eta) \leq{}& C \int_0^T s^3 \xi e^{-2s\check{\varphi}_1} \|g_1\|^2_{H^{2/3}(0,L)} dt \\ &+ C \int_0^T s^3 \xi e^{-2s\check{\varphi}_2} \|g_2\|^2_{H^{1/3}(0,L)} dt + C \int_0^T s^3 \xi e^{-2s\check{\varphi}_1} \|g_3\|^2_{H^{1/3}(0,L)} dt + C \int_0^T \int_{\gamma_1} s\xi^{25} e^{-2s(7\check{\varphi}_2 - 6\hat{\varphi}_2)} |\psi|^2 dxdt \\ &+ C \int_0^T \int_{\omega_1} s\xi^{25} e^{-2s(7\check{\varphi}_1 - 6\hat{\varphi}_1)} |\eta|^2 dxdt + C \int_0^T \int_{\omega_1} s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} |\phi|^2 dxdt. \end{aligned} \tag{3.22}$$

To finish the proof of estimate (3.18), it remains to absorb the last term of this inequality. The idea is to use the coupling of the equation satisfied by η in system (1.7) to express ϕ in terms of η. However, since the coupling is of first order, this cannot be done directly. Here, we will need the fact that ω "touches" the boundary of $(0, L)$. Let us call

$$J := \int_0^T \int_{\omega_1} s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} |\phi|^2 dxdt,$$

and consider $\omega_2 := (\delta, L)$, with $\delta \in (0,L)$ such that $\omega_1 \subset \omega_2 \subset \omega$, where all the inclusions are strict. Since $\phi(t,L) = 0$, we have with Poincar's inequality that

$$J \leq C \int_0^T \int_{\omega_2} s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} |\phi_x|^2 dx dt.$$

We concentrate on this term. Let $\theta \in C^\infty([0,L])$ a non-negative function such that $\theta(x) = 0$ for $x \in [0,L] \setminus \omega$, and $\theta(x) = 1$ for $x \in \omega_2$. Then, using the equation satisfied by η in system (1.7), we have

$$\begin{aligned} J &\leq C \int_0^T \int_\omega \theta(x) s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} |\phi_x|^2 dx dt \\ &= \frac{C}{6} \int_0^T \int_\omega \theta(x) s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} \phi_x \big(2g_3 + \eta_{xxx} + 2\eta_t\big) dx dt \\ &=: J_1 + J_2 + J_3. \end{aligned} \tag{3.23}$$

Let $\varepsilon > 0$. We estimate each one of these terms. Using Young's inequality, we have

$$J_1 \leq C_\varepsilon \int_Q s^{-1} \xi^{23} e^{-2s(8\check{\varphi}_1 - 7\hat{\varphi}_1)} |g_3|^2 dx dt + \varepsilon I^1_{8/3}(\phi). \tag{3.24}$$

For J_2, taking into account that $\phi_x(t,L) = 0$, we integrate by parts in space:

$$J_2 = -\frac{C}{6} \int_0^T \int_\omega \theta'(x) s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} \phi_x \eta_{xx} dx dt - \frac{C}{6} \int_0^T \int_\omega \theta(x) s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} \phi_{xx} \eta_{xx} dx dt,$$

where we use Young's inequality to obtain

$$J_2 \leq C_\varepsilon \int_0^T \int_\omega s\xi^{25} e^{-2s(8\check{\varphi}_1 - 7\hat{\varphi}_1)} |\eta_{xx}|^2 dx dt + \varepsilon I^1_{8/3}(\phi). \tag{3.25}$$

The third and last term is the more delicate one. We integrate by parts once in time and space in the term J_3. We get

$$\begin{aligned} J_3 &= -\frac{C}{3} \int_0^T \int_\omega \theta(x) s \big(\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)}\big)_t \phi_x \eta dx dt + \frac{C}{3} \int_0^T \int_\omega s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} \phi_t \big(\theta(x)\eta\big)_x dx dt \\ &=: J_{31} + J_{32}. \end{aligned}$$

For the first term, since

$$\big|\big(\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)}\big)_t\big| \leq Cs\xi^{15} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)},$$

we have that

$$J_{31} \leq C_\varepsilon \int_0^T \int_\omega s\xi^{27} e^{-2s(8\check{\varphi}_1 - 7\hat{\varphi}_1)} |\eta|^2 dx dt + \varepsilon I^1_{8/3}(\phi).$$

For the second one, we use the fact that $\phi_t = \frac{1}{4}\phi_{xxx} - g_1$ and integrate by parts in space. This is:

$$\begin{aligned} J_{32} =& \frac{C}{12} \int_0^T \int_\omega s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} (\phi_{xxx} - 4g_1) \big(\theta(x)\eta\big)_x dx dt \\ =& -\frac{C}{3} \int_0^T \int_\omega s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} g_1 \big(\theta(x)\eta\big)_x dx dt \\ &- \frac{C}{12} \int_0^T \int_\omega s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} \phi_{xx} \big(\theta(x)\eta\big)_{xx} dx dt + \frac{C}{12} \int_0^T s\xi^{13} e^{-2s(4\check{\varphi}_1 - 3\hat{\varphi}_1)} \phi_{xx}(t,L) \eta_x(t,L) dt. \end{aligned}$$

We observe the following:

- $$-\frac{C}{3}\int_0^T\int_\omega s\xi^{13}e^{-2s(4\check{\varphi}_1-3\hat{\varphi}_1)}g_1\big(\theta(x)\eta\big)_x dxdt$$
$$\leq C\int_0^T\int_\omega s^3\xi e^{-2s\varphi_1}|g_1|^2dxdt + C\int_0^T\int_\omega s^{-1}\xi^{25}e^{-2s(8\check{\varphi}_1-7\hat{\varphi}_1)}\big(|\eta|^2+|\eta_x|^2\big)dxdt;$$

- $$-\frac{C}{12}\int_0^T\int_\omega s\xi^{13}e^{-2s(4\check{\varphi}_1-3\hat{\varphi}_1)}\phi_{xx}\big(\theta(x)\eta\big)_{xx}dxdt$$
$$\leq \varepsilon I^1_{8/3}(\phi) + C_\varepsilon\int_0^T\int_\omega s\xi^{25}e^{-2s(8\check{\varphi}_1-7\hat{\varphi}_1)}\big(|\eta|^2+|\eta_x|^2+|\eta_{xx}|^2\big)dxdt;\ \text{and}$$

- $$\frac{C}{12}\int_0^T s\xi^{13}e^{-2s(4\check{\varphi}_1-3\hat{\varphi}_1)}\phi_{xx}(t,L)\eta_x(t,L)dt \leq \varepsilon I^1_{8/3}(\phi) + C_\varepsilon\int_0^T s\xi^{29}e^{-2s(8\check{\varphi}_1-7\hat{\varphi}_1)}\|\eta\|^2_{H^2(\omega)}dt.$$

Going back to the expression of J_3, we obtain

$$J_3 \leq C\int_Q s^3\xi e^{-2s\varphi_1}|g_1|^2dxdt + C_\varepsilon\int_0^T s\xi^{29}e^{-2s(8\check{\varphi}_1-7\hat{\varphi}_1)}\|\eta\|^2_{H^2(\omega)}dt + 3\varepsilon I^1_{8/3}(\phi). \tag{3.26}$$

Let us gather what we have so far. Putting together estimates (3.24)-(3.26) in (3.23), we have

$$\begin{aligned} J \leq & C\int_Q s^3\xi e^{-2s\varphi_1}|g_1|^2dxdt + \int_Q s^{-1}\xi^{23}e^{-2s(8\check{\varphi}_1-7\hat{\varphi}_1)}|g_3|^2dxdt \\ & + C_\varepsilon\int_0^T s\xi^{29}e^{-2s(8\check{\varphi}_1-7\hat{\varphi}_1)}\|\eta\|^2_{H^2(\omega)}dt + 5\varepsilon I^1_{8/3}(\phi). \end{aligned}$$

We estimate now the local term of η. Regarding $H^2(\omega)$ as the interpolation of the spaces $H^{7/3}(\omega)$ and $L^2(\omega)$, and Young's inequality, we obtain

$$\begin{aligned} C_\varepsilon\int_0^T s\xi^{29}e^{-2s(8\check{\varphi}_1-7\hat{\varphi}_1)}\|\eta\|^2_{H^2(\omega)}dt &\leq C\int_0^T s\xi^{29}e^{-2s(8\check{\varphi}_1-7\hat{\varphi}_1)}\|\eta\|^{12/7}_{H^{7/3}(\omega)}\|\eta\|^{2/7}_{L^2(\omega)}dt \\ &\leq \varepsilon I^1_{7/3}(\eta) + C_\varepsilon\int_0^T s\xi^{221}e^{-2s(56\check{\varphi}_1-55\hat{\varphi}_1)}\|\eta\|^2_{L^2(\omega)}dt. \end{aligned}$$

Then, finally, we get

$$\begin{aligned} J \leq & C\int_Q s^3\xi e^{-2s\varphi_1}|g_1|^2dxdt + \int_Q s^{-1}\xi^{23}e^{-2s(8\check{\varphi}_1-7\hat{\varphi}_1)}|g_3|^2dxdt \\ & + C_\varepsilon\int_0^T\int_\omega s\xi^{221}e^{-2s(56\check{\varphi}_1-55\hat{\varphi}_1)}|\eta|^2dxdt + \varepsilon I^1_{7/3}(\eta) + 5\varepsilon I^1_{8/3}(\phi). \end{aligned}$$

Going back to (3.22), we deduce (3.18) by choosing the biggest weight functions and ε sufficiently small.

∎

4 Control results

In this section, we establish an observability inequality for the solutions of system (1.7) and deduce a null controllability result for the linear system (1.6). Moreover, we prove our main result getting the local null controllability of system (1.5).

4.1 Observability inequality

The observability inequality will be deduced from Carleman estimate (3.18), but first, to be able to deduce null controllability, we need to change the weight functions in such a way that they do not vanish at $t = 0$. Before that, let us deduce a somewhat simpler version of the Carleman estimate (3.18) which will be useful in what follows.

Let

$$\varphi_M := \max\Big\{ \max_{x\in[0,L]} \varphi_0^1(x), \max_{x\in[0,L]} \varphi_0^2(x) \Big\}$$

and

$$\varphi_m := \min\Big\{ \min_{x\in[0,L]} \varphi_0^1(x), \min_{x\in[0,L]} \varphi_0^2(x) \Big\}.$$

Notice that if we call $\hat{\varphi}(t) := \xi(t)\varphi_M$ and $\check{\varphi}(t) := \xi(t)\varphi_m$, under the assumptions of Theorem 10 we can deduce from (3.18) the following inequality:

$$\begin{aligned}(4.1)\quad &\int_Q s^3\xi^3 e^{-2s\hat{\varphi}}(|\phi_x|^2 + |\psi_x|^2 + |\eta_x|^2)dxdt \\ &\le C\int_0^T\!\!\int_\gamma s\xi^{25} e^{-2s(7\check{\varphi}-6\hat{\varphi})}|\psi|^2 dxdt + C\int_0^T\!\!\int_\omega s\xi^{221} e^{-2s(56\check{\varphi}-55\hat{\varphi})}|\eta|^2 dxdt \\ &\quad + C\int_0^T s^3\xi^{23} e^{-2s(8\check{\varphi}-7\hat{\varphi})}\big(\|g_1\|^2_{H^{2/3}(0,L)} + \|g_2\|^2_{H^{1/3}(0,L)} + \|g_3\|^2_{H^{1/3}(0,L)}\big)dt.\end{aligned}$$

Now, let $\beta \in C^1(0,T)$ be defined by

$$\beta(t) = \begin{cases} \dfrac{4}{T^2}, & \text{if } t \in (0, T/2), \\ \dfrac{1}{t(T-t)}, & \text{if } t \in [T/2, T), \end{cases}$$

and let us call

$$\hat{\alpha}(t) := \beta(t)\varphi_M \text{ and } \check{\alpha}(t) := \beta(t)\varphi_m.$$

Furthermore, we will assume also that g_1, g_2, and g_3 in system (1.7) belong to $L^2(0,T;H_0^1(0,L))$. This will make the analysis of the controllability of system (1.6) simpler later on.

Proposition 11 *Let s be fixed such that Carleman estimate* (3.18) *holds. Assume that $g_1, g_2, g_3 \in L^2(0,T;H_0^1(0,L))$. Then, every solution (ϕ,ψ,η) of system* (1.7) *satisfies*

$$\begin{aligned}(4.2)\quad &\int_0^L (|\phi(0,x)|^2 + |\psi(0,x)|^2 + |\eta(0,x)|^2)dx + \int_Q \beta^3 e^{-2s\hat{\alpha}}(|\phi_x|^2 + |\psi_x|^2 + |\eta_x|^2)dxdt \\ &+ \|\beta^{1/2}e^{-s\hat{\alpha}}\phi\|^2_{L^\infty(0,T;L^2(0,L))} + \|\beta^{1/2}e^{-s\hat{\alpha}}\psi\|^2_{L^\infty(0,T;L^2(0,L))} + \|\beta^{1/2}e^{-s\hat{\alpha}}\eta\|^2_{L^\infty(0,T;L^2(0,L))} \\ &\le C\int_0^T\!\!\int_\gamma \beta^{25} e^{-2s(7\check{\alpha}-6\hat{\alpha})}|\psi|^2 dxdt + C\int_0^T\!\!\int_\omega \beta^{221} e^{-2s(56\check{\alpha}-55\hat{\alpha})}|\eta|^2 dxdt \\ &\quad + C\int_Q \beta^{23} e^{-2s(8\check{\alpha}-7\hat{\alpha})}(|(g_1)_x|^2 + |(g_2)_x|^2 + |(g_3)_x|^2)dxdt.\end{aligned}$$

Proof.

Let $\lambda \in C^1([0,T])$ be a non-negative function such that $\lambda(t) = 1$ if $t \leq T/2$ and $\lambda(t) = 0$ if $t \geq 3T/4$. Then, from the system satisfied by $(\lambda\phi, \lambda\psi, \lambda\eta)$ and the estimate in Proposition 2, we deduce that

$$\begin{aligned}\|\lambda(t)(\phi,\psi,\eta)\|^2_{L^2(0,T;H^1_0(0,L))} &+ \|\lambda(t)(\phi,\psi,\eta)\|^2_{L^\infty(0,T;L^2(0,L))}\\ &\leq C\|\lambda(t)(g_1,g_2,g_3)\|^2_{L^2(0,T;L^2(0,L))} + C\|\lambda'(t)(\phi,\psi,\eta)\|^2_{L^2(0,T;L^2(0,L))},\end{aligned}$$

from where

$$\begin{aligned}\|(\phi,\psi,\eta)\|^2_{L^2(0,T/2;H^1_0(0,L))} &+ \|(\phi(0),\psi(0),\eta(0))\|^2_{L^2(0,L)}\\ &\leq C\|(g_1,g_2,g_3)\|^2_{L^2(0,3T/4;L^2(0,L))} + C\|(\phi,\psi,\eta)\|^2_{L^2(T/2,3T/4;L^2(0,L))}.\end{aligned}$$

Since $e^{-2s\hat{\varphi}} \geq C > 0$ in $(T/2, 3T/4)$, the last term of this estimate can be bounded from above by the left-hand side of (4.1). Thus, we get

$$\begin{aligned}\int_0^L (|\phi(0)|^2 + |\psi(0)|^2 + |\eta(0)|^2)dx &+ \int_0^{T/2}\int_0^L \beta^3 e^{-2s\hat{\alpha}}(|\phi_x|^2 + |\psi_x|^2 + |\eta_x|^2)dxdt\\ \leq C\int_0^T\int_\gamma \beta^{25} e^{-2s(7\check{\alpha}-6\hat{\alpha})}|\psi|^2 dxdt &+ C\int_0^T\int_\omega \beta^{221} e^{-2s(56\check{\alpha}-55\hat{\alpha})}|\eta|^2 dxdt\\ &+ C\int_Q \beta^{23} e^{-2s(8\check{\alpha}-7\hat{\alpha})}(|(g_1)_x|^2 + |(g_2)_x|^2 + |(g_3)_x|^2)dxdt,\end{aligned}$$

where we have also used the fact that $\xi \equiv \beta$ in $(T/2, T)$. Actually, using this last property again, we see that

$$\int_{T/2}^T\int_0^L \beta^3 e^{-2s\hat{\alpha}}(|\phi_x|^2 + |\psi_x|^2 + |\eta_x|^2)dxdt$$

is bounded from above by the left-hand side of (4.1).

To conclude, it suffices to apply the estimate of Proposition 2 to the equations satisfied by $(\beta^{1/2}e^{-s\hat{\alpha}}\phi, \beta^{1/2}e^{-s\hat{\alpha}}\psi, \beta^{1/2}e^{-s\hat{\alpha}}\eta)$.

$\blacksquare$

4.2 Null controllability of the linear system

Now, we are in position to prove the null controllability of the linear system (1.6). In the following, consider the notation

$$L^r(\rho(t)(0,T);H) := \{y \in L^r(0,T;H) : \rho(t)y \in L^r(0,T;H)\}, \quad r \in [1,+\infty].$$

Let E be the space of quintuples (u,v,w,p,q) such that

- $(u,v,w) \in L^2(\beta^{23/2}e^{s(8\check{\alpha}-7\hat{\alpha})}(0,T);H^{-1}(0,L))^3$,
- $p\mathbb{1}_\gamma \in L^2(\beta^{-25/2}e^{s(7\check{\alpha}-6\hat{\alpha})}(0,T);L^2(0,L))$,
- $q\mathbb{1}_\omega \in L^2(\beta^{-221/2}e^{s(56\check{\alpha}-55\hat{\alpha})}(0,T);L^2(0,L))$,
- $(u,v,w) \in (L^2(\beta^{-3/4}e^{s/2\hat{\alpha}}(0,T);H^1_0(0,L)) \cap L^\infty(\beta^{-3/4}e^{s/2\hat{\alpha}}(0,T);L^2(0,L)))^3$,

- $(u_t - \frac{1}{4}u_{xxx} - 3w_x,\, v_t - \frac{1}{2}v_{xxx} - p\mathbb{1}_\gamma,\, w_t + \frac{1}{2}w_{xxx} - q\mathbb{1}_\omega) \in L^2(\beta^{-3/2}e^{s\hat{\alpha}}(0,T);H^{-1}(0,L))^3$.

Actually, the space E becomes a Banach space endowed with its natural norm.

The following result establishes the null controllability of the linearized system (1.6).

Proposition 12 *Let $u_0, v_0, w_0 \in L^2(0,L)$ and assume that*

$$(f_1, f_2, f_3) \in L^2(\beta^{-3/2}e^{s\hat{\alpha}}(0,T);H^{-1}(0,L))^3. \tag{4.3}$$

Then, there exist two controls p and q, such that the associated solution (u,v,w) to (1.6) satisfies $(u,v,w,p,q) \in E$. In particular,

$$u(T,x) = v(T,x) = w(T,x) = 0 \quad in \quad (0,L).$$

Proof.

We follow an approach introduced in [11]. Let P_0 be the space of triplets $(\phi,\psi,\eta) \in C^4([0,T]\times[0,L])$ such that:

- $\phi(t,0) = \phi(t,0) = \phi_x(t,L) = \phi_{xxx}(t,0) = \phi_{xxx}(t,L) = 0$,
- $\psi(t,0) = \psi(t,0) = \psi_x(t,0) = \psi_{xxx}(t,0) = \psi_{xxx}(t,L) = 0$,
- $\eta(t,0) = \eta(t,0) = \eta_x(t,0) = \eta_{xxx}(t,L) = 0$,
- $-\frac{1}{2}\eta_{xxx}(t,0) + 3\phi_x(t,0) = 0$.

Notice that the observability inequality (4.2) holds for every $(\phi,\psi,\eta) \in P_0$ taking $g_1 = -\phi_t + \frac{1}{4}\phi_{xxx}$, $g_2 = -\psi_t - \frac{1}{2}\psi_{xxx}$, and $g_3 = -\eta_t - \frac{1}{2}\eta_{xxx} + 3\phi_x$.

Let $a : P_0 \times P_0 \to \mathbb{R}$ be the bilinear form

$$\begin{aligned} a((\hat{\phi},\hat{\psi},\hat{\eta}),(\phi,\psi,\eta)) = &\int_Q \beta^{23}e^{-2s(8\check{\alpha}-7\hat{\alpha})}(-\hat{\phi}_t + \tfrac{1}{4}\hat{\phi}_{xxx})_x(-\phi_t + \tfrac{1}{4}\phi_{xxx})_x dxdt \\ &+ \int_Q \beta^{23}e^{-2s(8\check{\alpha}-7\hat{\alpha})}(-\hat{\psi}_t - \tfrac{1}{2}\hat{\psi}_{xxx})_x(-\psi_t - \tfrac{1}{2}\psi_{xxx})_x dxdt \\ &+ \int_Q \beta^{23}e^{-2s(8\check{\alpha}-7\hat{\alpha})}(-\hat{\eta}_t - \tfrac{1}{2}\hat{\eta}_{xxx} + 3\hat{\phi}_x)_x(-\eta_t - \tfrac{1}{2}\eta_{xxx} + 3\phi_x)_x dxdt \\ &+ \int_0^T\int_\gamma \beta^{25}e^{-2s(7\check{\alpha}-6\hat{\alpha})}\hat{\psi}\psi dxdt + \int_0^T\int_\omega \beta^{221}e^{-2s(56\check{\alpha}-55\hat{\alpha})}\hat{\eta}\eta dxdt, \end{aligned}$$

and $\ell : P_0 \to \mathbb{R}$ the linear form

$$\begin{aligned} \ell(\phi,\psi,\eta) = &\int_0^L (u_0\phi(0,x) + v_0\psi(0,x) + w_0\eta(0,x))dx \\ &+ \int_0^T (\langle f_1,\phi\rangle + \langle f_2,\psi\rangle + \langle f_3,\eta\rangle)dt, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $H^{-1}(0,L)$ and $H_0^1(0,L)$.

Thanks to Proposition 11, the bilinear form above induces a norm $\|\cdot\|_a := a(\cdot,\cdot)^{1/2}$ in P_0. Call P the completion of P_0 with respect to $\|\cdot\|_a$, which is a Hilbert space for the scalar product $a(\cdot,\cdot)$. From assumption (4.3) and using Cauchy-Schwarz inequality, we readily check that

$$\ell(\phi,\psi,\eta) \le \Big(\|\beta^{-3/2}e^{s\hat{\alpha}}(f_1,f_2,f_3)\|_{L^2(0,T;H^{-1}(0,L))} + \|(u_0,v_0,w_0)\|_{L^2(0,L)}\Big)\|(\phi,\psi,\eta)\|_a,$$

for every $(\phi,\psi,\eta)\in P$, from where we see that ℓ is bounded in P. Therefore, we deduce that there exists a unique triplet $(\hat\phi,\hat\psi,\hat\eta)\in P$ such that

$$a((\hat\phi,\hat\psi,\hat\eta),(\phi,\psi,\eta))=\ell(\phi,\psi,\eta),\quad \text{for all } (\phi,\psi,\eta)\in P. \tag{4.4}$$

We define $(\hat u,\hat v,\hat w,\hat p,\hat q)$ by

- $\hat u := -\beta^{23}e^{-2s(8\check\alpha-7\hat\alpha)}(-\hat\phi_t+\frac14\hat\phi_{xxx})_{xx}$,
- $\hat v := -\beta^{23}e^{-2s(8\check\alpha-7\hat\alpha)}(-\hat\psi_t-\frac12\hat\psi_{xxx})_{xx}$,
- $\hat w := -\beta^{23}e^{-2s(8\check\alpha-7\hat\alpha)}(-\hat\eta_t-\frac12\hat\eta_{xxx}+3\hat\phi_x)_{xx}$,
- $\hat p := -\beta^{25}e^{-2s(7\check\alpha-6\hat\alpha)}\hat\psi\mathbb{1}_\gamma$,
- $\hat q := -\beta^{221}e^{-2s(56\check\alpha-55\hat\alpha)}\hat\eta\mathbb{1}_\omega$.

Let us show now that $(\hat u,\hat v,\hat w,\hat p,\hat q)$ is the quintuple that we are looking for. First, let us prove that $(\hat u,\hat v,\hat w)$ is actually the solution of (1.6) with $p=\hat p$ and $q=\hat q$. Let $(\tilde u,\tilde v,\tilde w)$ be the (unique) weak solution of (1.6) associated to $p=\hat p$ and $q=\hat q$. This triplet is also the unique solution by transposition of (1.6), that is, it satisfies

$$\begin{aligned}\int_Q\big(\tilde u g_1+\tilde v g_2+\tilde w g_3\big)dxdt&=\int_0^L(u_0\phi(0,x)+v_0\psi(0,x)+w_0\eta(0,x))dx\\&+\int_0^T(\langle f_1,\phi\rangle+\langle f_2,\psi\rangle+\langle f_3,\eta\rangle)dt\\&+\int_0^T\int_\gamma\hat p\psi dxdt+\int_0^T\int_\omega\hat q\eta dxdt,\end{aligned}\tag{4.5}$$

for all $g_1,g_2,g_3\in L^2(0,T;H_0^1(0,L))$, where (ϕ,ψ,η) is the solution of

$$\begin{cases}-\phi_t+\frac14\phi_{xxx}=g_1, & \text{in } Q,\\ -\psi_t-\frac12\psi_{xxx}=g_2, & \text{in } Q,\\ -\eta_t-\frac12\eta_{xxx}=g_3-3\phi_x, & \text{in } Q,\\ +\text{ b.c.} & \\ \phi(T,x)=\psi(T,x)=\eta(T,x)=0, & \text{in } (0,L).\end{cases}\tag{4.6}$$

Actually, one usually takes $g_1,g_2,g_3\in L^2(0,T;L^2(0,L))$ in (4.5), but given the density of $H_0^1(0,L)$ in $L^2(0,L)$ (together with energy estimates for system (4.6)), these two ways of taking the g_i functions are equivalent. On the other hand, from (4.4), we see that

$$a((\hat\phi,\hat\psi,\hat\eta),(\phi,\psi,\eta))=\ell(\phi,\psi,\eta),\quad \text{for all } (\phi,\psi,\eta)\in P_0, \tag{4.7}$$

where $(\hat\phi,\hat\psi,\hat\eta)\in P$ is the unique solution of (4.4). Integrating by parts in space once, we find that

$$\begin{aligned}&\int_Q\big(\hat u(-\phi_t+\tfrac14\phi_{xxx})+\hat v(-\psi_t-\tfrac12\psi_{xxx})+\hat w(-\eta_t-\tfrac12\eta_{xxx}+3\phi_x)\big)dxdt\\&=\int_0^L(u_0\phi(0,x)+v_0\psi(0,x)+w_0\eta(0,x))dx\\&+\int_0^T(\langle f_1,\phi\rangle+\langle f_2,\psi\rangle+\langle f_3,\eta\rangle)dt\\&+\int_0^T\int_\gamma\hat p\psi dxdt+\int_0^T\int_\omega\hat q\eta dxdt,\end{aligned}\tag{4.8}$$

for all $(\phi, \psi, \eta) \in P_0$. Using the density of P_0 in P with respect to the norm $\|\cdot\|_a$, we show that (4.8) holds for all $(\phi, \psi, \eta) \in P$. Therefore, the triplets $(\tilde{u}, \tilde{v}, \tilde{w})$ and $(\hat{u}, \hat{v}, \hat{w})$ must coincide, and $(\hat{u}, \hat{v}, \hat{w})$ is the solution of (1.6) associated to $\hat{p}$ and $\hat{q}$.

Now, notice that

$$
\begin{aligned}
\int_0^T \beta^{-23} e^{2s(8\check{\alpha}-7\hat{\alpha})} \|\hat{u}\|^2_{H^{-1}(0,L)} dt &= \int_0^T \beta^{-23} e^{2s(8\check{\alpha}-7\hat{\alpha})} \sup_{\|y\|_{H_0^1(0,L)}=1} \langle \hat{u}, y\rangle^2 dt \\
&= \int_0^T \beta^{23} e^{-2s(8\check{\alpha}-7\hat{\alpha})} \sup_{\|y\|_{H_0^1(0,L)}=1} \langle -(-\hat{\phi}_t + \tfrac{1}{4}\hat{\phi}_{xxx})_{xx}, y\rangle^2 dt \\
&= \int_0^T \beta^{23} e^{-2s(8\check{\alpha}-7\hat{\alpha})} \sup_{\|y\|_{H_0^1(0,L)}=1} \left(\int_0^L (-\hat{\phi}_t + \tfrac{1}{4}\hat{\phi}_{xxx})_x y_x \, dx \right)^2 dt \\
&\leq \int_Q \beta^{23} e^{-2s(8\check{\alpha}-7\hat{\alpha})} |(-\hat{\phi}_t + \tfrac{1}{4}\hat{\phi}_{xxx})_x|^2 dxdt \\
&\leq \|(\hat{\phi}, \hat{\psi}, \hat{\eta})\|_a^2 < +\infty.
\end{aligned}
$$

Proceeding in the same way for $\hat{v}$ and $\hat{w}$, we can prove that

$$
\int_0^T \beta^{-23} e^{2s(8\check{\alpha}-7\hat{\alpha})} \big(\|\hat{u}\|^2_{H^{-1}(0,L)} + \|\hat{v}\|^2_{H^{-1}(0,L)} + \|\hat{w}\|^2_{H^{-1}(0,L)} \big) dt < +\infty, \tag{4.9}
$$

and, directly from the definition,

$$
\int_0^T \int_\gamma \beta^{-25} e^{2s(7\check{\alpha}-6\hat{\alpha})} |\hat{p}|^2 dxdt + \int_0^T \int_\omega \beta^{-221} e^{2s(56\check{\alpha}-55\hat{\alpha})} |\hat{q}|^2 dxdt < +\infty. \tag{4.10}
$$

It only remains to check that

$$
(u, v, w) \in (L^2(\beta^{-3/4} e^{s/2\hat{\alpha}}(0,T); H_0^1(0,L)) \cap L^\infty(\beta^{-3/4} e^{s/2\hat{\alpha}}(0,T); L^2(0,L)))^3.
$$

To do this, let $(\bar{u}, \bar{v}, \bar{w}) := \beta^{-3/4} e^{s/2\hat{\alpha}} (\hat{u}, \hat{v}, \hat{w})$. From (1.6), the triplet $(\bar{u}, \bar{v}, \bar{w})$ satisfies the system

$$
\begin{cases}
\bar{u}_t - \frac{1}{4}\bar{u}_{xxx} - 3\bar{w}_x = \beta^{-3/4} e^{s/2\hat{\alpha}} f_1 + (\beta^{-3/4} e^{s/2\hat{\alpha}})_t \hat{u}, & \text{in } Q, \\
\bar{v}_t + \frac{1}{2}\bar{v}_{xxx} = \beta^{-3/4} e^{s/2\hat{\alpha}} (f_2 + \hat{p}\mathbb{1}_\gamma) + (\beta^{-3/4} e^{s/2\hat{\alpha}})_t \hat{v}, & \text{in } Q, \\
\bar{w}_t + \frac{1}{2}\bar{w}_{xxx} = \beta^{-3/4} e^{s/2\hat{\alpha}} (f_3 + \hat{q}\mathbb{1}_\omega) + (\beta^{-3/4} e^{s/2\hat{\alpha}})_t \hat{w}, & \text{in } Q, \\
+ \text{ b.c.} \\
(\bar{u}(0,x), \bar{v}(0,x), \bar{w}(0,x)) = \beta^{-3/4}(0) e^{s/2\hat{\alpha}(0)} (u_0(x), v_0(x), w_0(x)) & \text{in } (0,L).
\end{cases} \tag{4.11}
$$

Since

$$
|(\beta^{-3/4} e^{s/2\hat{\alpha}})_t| \leq C\beta^{5/4} e^{s/2\hat{\alpha}} \leq C\beta^{23/2} e^{s(8\check{\alpha}-7\hat{\alpha})},
$$

we have from (4.3), (4.9) and (4.10) that the right-hand sides of the previous systems belong to $L^2(0,T, H^{-1}(0,L))$. Then, from Proposition 2, we conclude that

$$
(\bar{u}, \bar{v}, \bar{w}) \in \big(L^2(0,T; H_0^1(0,L)) \cap L^\infty(0,T; L^2(0,L))\big)^3,
$$

which concludes the proof of Proposition 12.

$\blacksquare$

A similar controllability result holds if, instead of (4.3), we assume that

$$
(f_1, f_2, f_3) \in L^1(\beta^{-1/2} e^{s\hat{\alpha}}(0,T); L^2(0,L))^3. \tag{4.12}
$$

Indeed, the proof is analogous to the one of Proposition 12 with a few changes:

1. Take
$$\ell(\phi,\psi,\eta)=\int_0^L (u_0\phi(0,x)+v_0\psi(0,x)+w_0\eta(0,x))dx+\langle f_1,\phi\rangle+\langle f_2,\psi\rangle+\langle f_3,\eta\rangle$$
where $\langle\cdot,\cdot\rangle$ denotes the duality product between $L^\infty(0,T;L^2(0,L))$ and $L^1(0,T;L^2(0,L))$. From (4.2), we check that ℓ is a linear bounded operator in P.

2. Call $\tilde{E}$ the space of quintuples (u,v,w,p,q) that satisfy the first four points of the space E above, and replacing the last condition by
$$(u_t-\tfrac{1}{4}u_{xxx}-3w_x,\ v_t-\tfrac{1}{2}v_{xxx}-p\mathbb{1}_\gamma,\ w_t+\tfrac{1}{2}w_{xxx}-q\mathbb{1}_\omega)\in L^1(\beta^{-1/2}e^{s\hat{\alpha}}(0,T);L^2(0,L))^3.$$

Then, we can establish the following controllability result for system (1.6).

Proposition 13 *Let $u_0,v_0,w_0\in L^2(0,L)$ and assume that* (4.12) *holds. Then, there exist two controls p and q, such that the associated solution (u,v,w) to* (1.6) *satisfies $(u,v,w,p,q)\in\tilde{E}$. In particular,*
$$u(T,x)=v(T,x)=w(T,x)=0\quad in\quad (0,L).$$

4.3 Local null controllability of the nonlinear system

In this section, we prove the local null controllability of the Hirota-Satsuma system (1.5), that means Theorem 1, using a local inversion argument.

Proof.
Let $\mathcal{F}:E\to L^2(\beta^{-3/2}e^{s\hat{\alpha}}(0,T);H^{-1}(0,L)\times L^2(0,L))^3$ be an operator defined by
$$\begin{aligned}\mathcal{F}(u,v,w,p,q):=\big((&u_t-\tfrac{1}{4}u_{xxx}-3uu_x+6vv_x-3w_x,u(0,\cdot),\\ &v_t+\tfrac{1}{2}v_{xxx}+3uv_x-p\mathbb{1}_\gamma,v(0,\cdot),\\ &w_t+\tfrac{1}{2}w_{xxx}+3uw_x-q\mathbb{1}_\omega,w(0,\cdot)\big).\end{aligned}$$
Recall that the space E is the Banach space defined at the beginning of Section 4.2.

We will check that the following two points are verified:

- $\mathcal{F}'$ is an operator of class C^1 from E to $L^2(\beta^{-3/2}e^{s\hat{\alpha}}(0,T);H^{-1}(0,L))^3$.
- $\mathcal{F}'(0):E\to L^2(\beta^{-3/2}e^{s\hat{\alpha}}(0,T);H^{-1}(0,L))^3$ is surjective.

Then, since $\mathcal{F}(0)=0$, there exists $\delta>0$ such that if $\|(u_0,v_0,w_0)\|_{L^2(0,L)}<\delta$, there exists $(u,v,w,p,q)\in E$ such that
$$\mathcal{F}(u,v,w,p,q)=(0,u_0,0,v_0,0,w_0).$$

Let us check the two points above.

• $\underline{\mathcal{F}' \text{ is an operator of class } C^1 \text{ from } E \text{ to } L^2(\beta^{-3/2}e^{s\hat{\alpha}}(0,T);H^{-1}(0,L))^3}$.

It is fairly clear to see that it suffices to prove that the bilinear terms in $\mathcal{F}$ are bounded. Indeed, let y and z two functions in E. We have
$$\begin{aligned}\|yz_x\|_{L^2(\beta^{-3/2}e^{s\hat{\alpha}}(0,T);H^{-1}(0,L))}&\le C\|y\|_{L^2(\beta^{-3/4}e^{s/2\hat{\alpha}}(0,T);H_0^1(0,L))}\|z\|_{L^\infty(\beta^{-3/4}e^{s/2\hat{\alpha}}(0,T);L^2(0,L))}\\ &\le C\|y\|_E\|z\|_E.\end{aligned}$$

• $\underline{\mathcal{F}'(0):E\to L^2(\beta^{-3/2}e^{s\hat{\alpha}}(0,T);H^{-1}(0,L))^3 \text{ is surjective}}$.

Notice that
$$\mathcal{F}'(0):=\big((u_t-\tfrac{1}{4}u_{xxx}-3w_x,u(0,\cdot),v_t+\tfrac{1}{2}v_{xxx}-p\mathbb{1}_\gamma,v(0,\cdot),w_t+\tfrac{1}{2}w_{xxx}-q\mathbb{1}_\omega,w(0,\cdot)\big),$$
which is surjective thanks to Proposition 12. This completes the proof of Theorem 1. ■

5 Final comments

We finish our paper with some comments and open problems.

- We have proven in Theorem 1 the local null controllability of the generalized HS system (1.5). Given the strategy followed in this paper, we have done the best possible: to control the three-equation system with two internal controls. This optimality is clear from the fact that when we linearize we obtain two decoupled subsystems and consequently we need two controls to achieve our results.

- A very nice open problem is to get the control of the generalized HS system (1.5) using only one control input. To do that, the strategy used here is not good enough as explained in the previous point. A possible strategy is the use of nonlinear arguments as the return method as done for instance in [9, 10] for parabolic systems and in [23] for hyperbolic systems. This strategy should be also useful to control the HS system (1.1) with only one control, for instance:
$$\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x, & (t,x) \in Q, \\ v_t + \frac{1}{2}v_{xxx} = -3uv_x + p\mathbb{1}_\gamma, & (t,x) \in Q, \\ u(t,0) = u(t,L) = 0,\ u_x(t,0) = 0, & t \in (0,T), \\ v(t,0) = v(t,L) = 0,\ v_x(t,L) = 0, & t \in (0,T), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in (0,L). \end{cases}$$

- Other interesting open problem it is to study the boundary controllability of the generalized HS system, trying to get some results when some equations are not directly controlled.

References

[1] K. Ammari and E. Crépeau, *Feedback stabilization and boundary controllability of the Korteweg–de Vries equation on a star-shaped network*, SIAM J. Control Optim., 56 (2018), pp. 1620–1639.

[2] F. D. Araruna, E. Cerpa, A. Mercado, and M. C. Santos, *Internal null controllability of a linear Schrödinger-KdV system on a bounded interval*, J. Differential Equations, 260 (2016), pp. 653–687.

[3] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[4] R. A. Capistrano-Filho, F. A. Gallego, and A. F. Pazoto, *Neumann boundary controllability of the Gear-Grimshaw system with critical size restrictions on the spatial domain*, Z. Angew. Math. Phys., 67 (2016), pp. Art. 109, 36.

[5] ——, *Boundary controllability of a nonlinear coupled system of two Korteweg–de Vries equations with critical size restrictions on the spatial domain*, Math. Control Signals Systems, 29 (2017), pp. Art. 6, 37.

[6] R. A. Capistrano-Filho, A. F. Pazoto, and L. Rosier, *Internal controllability of the Korteweg–de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var., 21 (2015), pp. 1076–1107.

[7] E. Cerpa, E. Crépeau, and C. Moreno, *On the boundary controllability of the Korteweg-de Vries equation on a star-shaped network*, IMA J. Math. Control and Inform., (in press).

[8] E. Cerpa and A. F. Pazoto, *A note on the paper "On the controllability of a coupled system of two Korteweg-de Vries equations" [mr2561938]*, Commun. Contemp. Math., 13 (2011), pp. 183–189.

[9] J.-M. Coron, S. Guerrero, and L. Rosier, *Null controllability of a parabolic system with a cubic coupling term*, SIAM J. Control Optim., 48 (2010), pp. 5629–5653.

[10] J.-M. Coron and J.-P. Guilleron, *Control of three heat equations coupled with two cubic nonlinearities*, SIAM J. Control Optim., 55 (2017), pp. 989–1019.

[11] A. V. Fursikov and O. Y. Imanuvilov, *Controllability of evolution equations*, vol. 34 of Lecture Notes Series, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[12] X. Geng, H. Ren, and G. He, *Darboux transformation for a generalized Hirota-Satsuma coupled Korteweg-de Vries equation*, Phys. Rev. E (3), 79 (2009), pp. 056602, 9.

[13] O. Glass and S. Guerrero, *Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit*, Asymptot. Anal., 60 (2008), pp. 61–100.

[14] R. Hirota and J. Satsuma, *Soliton solutions of a coupled Korteweg-de Vries equation*, Phys. Lett. A, 85 (1981), pp. 407–408.

[15] C. Laurent, L. Rosier, and B.-Y. Zhang, *Control and stabilization of the Korteweg-de Vries equation on a periodic domain*, Comm. Partial Differential Equations, 35 (2010), pp. 707–744.

[16] S. Micu, J. H. Ortega, and A. F. Pazoto, *On the controllability of a coupled system of two Korteweg-de Vries equations*, Commun. Contemp. Math., 11 (2009), pp. 799–827.

[17] S. Micu, J. H. Ortega, L. Rosier, and B.-Y. Zhang, *Control and stabilization of a family of Boussinesq systems*, Discrete Contin. Dyn. Syst., 24 (2009), pp. 273–313.

[18] L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var., 2 (1997), pp. 33–55.

[19] D. L. Russell and B. Y. Zhang, *Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain*, SIAM J. Control Optim., 31 (1993), pp. 659–676.

[20] D. L. Russell and B. Y. Zhang, *Exact controllability and stabilizability of the Korteweg-de Vries equation*, Trans. Amer. Math. Soc., 348 (1996), pp. 3643–3672.

[21] J. Satsuma and R. Hirota, *A coupled KdV equation is one case of the four-reduction of the KP hierarchy*, J. Phys. Soc. Japan, 51 (1982), pp. 3390–3397.

[22] L. Xue, Q. P. Liu, and D. Wang, *A generalized Hirota-Satsuma coupled KdV system: Darboux transformations and reductions*, J. Math. Phys., 57 (2016), pp. 083506, 13.

[23] C. Zhang, *Internal controllability of systems of semilinear coupled one-dimensional wave equations with one control*, SIAM J. Control Optim., 56 (2018), pp. 3092–3127.