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Insensitizing controls for a phase field system

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ABSTRACT

In this paper, a nonlinear parabolic system modeling phase field phenomena is considered. This system consists of two coupled parabolic equations, the first one describes the temperature of the material and the second one describes a phase field function. Under small perturbations of the initial data, we study the existence of controls insensitizing the phase field function and acting only on the temperature equation. This problem is equivalent to the null controllability of a parabolic system, which is studied by means of duality arguments, Carleman estimates, and fixed point theorems.

1. Introduction

Motivated by the difficulties in determining data in applications of distributed systems, Lions introduced in [29] the topic of insensitizing controls. This notion deals with the existence of controls making some functional of the solution insensible to small perturbations of the initial data. For some particular functionals, it has been proven that this problem is equivalent to control properties of cascade systems [7,23]. The insensitivity can be defined in an approximate or exact way. Approximate insensitivity is equivalent to approximate controllability of the cascade system, while exact insensitivity is equivalent to its null controllability. The first mathematical results concerned the insensitivity of the L^2-norm of the solution restricted to a subdomain, called the observatory. In [7], approximate results were proven for the heat equation by getting unique continuation properties for parabolic systems. In [14], de Teresa used a global Carleman estimate

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approach to get the existence of exact insensitizing controls for a semilinear heat equation. Other papers on the heat equation are [8], where superlinear nonlinearities were considered, and [23], where the L^2-norm of the gradient of the solution is chosen as the functional to deal with. The same kind of problems have been studied for other systems, as the wave equation [13,2] and fluids equations [22,24,12,11].

In this paper, we address the insensitizing problem for a phase transition model introduced in [9]. We consider a material that may be in two phases, liquid and solid. Instead of studying a free boundary problem, a phase field function is introduced to indicate if the material is in solid or liquid state, varying in a continuous way from one phase to the other. The interface between these two states is supposed to be of finite thickness and defined where the phase field function is not constant. These hypothesis lead to the following nonlinear model (see [9])

$$\begin{cases} y_t - \Delta y = -\ell z_t, \quad \ell > 0, \\ z_t - \Delta z - a(z - z^3) = y, \quad a > 0, \end{cases}$$

describing the evolution of the temperature of the material ($y = y(x,t)$) and the phase field function ($z = z(x,t)$). In [9], well-posedness and asymptotic results (with respect to the thickness of the interface) were proved. From a control point of view, some optimal controls are obtained in [25]. Null controllability of this system is proven in [6] by means of two distributed controls, and in [3,21,4] with only one.

Let us describe in detail the system we are concerned with. Let $T > 0$ and $\omega, \Omega \subset \mathbb{R}^N$, with $N = 2$ or 3, two nonempty bounded open sets such that $\omega \subset \Omega$ with $\partial\Omega$ of class C^2. We define $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$ and $\mathbb{1}_\omega$ the characteristic function of ω. Our phase field system, for a material which occupies the region Ω, is described for the temperature $y = y(x,t)$, and the phase field function $z = z(x,t)$ in the coupled system given by

$$\begin{cases} y_t - d_1\Delta y = -\ell z_t + f_1 + h\mathbb{1}_\omega & \text{in } Q, \\ z_t - d_2\Delta z - (az + bz^2 - z^3) = y + f_2 & \text{in } Q, \\ \dfrac{\partial y}{\partial n} = 0, \qquad \dfrac{\partial z}{\partial n} = 0 & \text{on } \Sigma, \\ y(0) = y_0 + \tau_1\widehat{y}_0, \qquad z(0) = z_0 + \tau_2\widehat{z}_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $d_1, d_2, \ell, a > 0$, $b \in \mathbb{R}$, and the source term h is viewed as a control function supported in ω. Moreover, $f_1, f_2 \in L^2(Q)$ are given external forces. The initial data is composed by a fixed known part given by $y_0, z_0 \in L^2(\Omega)$, and some unknown perturbations given by $\tau_1\widehat{y}_0, \tau_2\widehat{z}_0 \in L^2(\Omega)$. Here, τ_1 and τ_2 are small unknown real numbers and, $\widehat{y}_0, \widehat{z}_0 \in L^2(\Omega)$ are such that $\|\widehat{y}_0\|_{L^2(\Omega)} = \|\widehat{z}_0\|_{L^2(\Omega)} = 1$.

We aim at insensitizing the functional

$$J_{\tau_1,\tau_2}(y,z) = \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |z|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{1.2}$$

This means that we look for a control $h \in L^2(0,T;L^2(\omega))$ such that

$$\left.\frac{\partial J_{\tau_1,\tau_2}(y,z)}{\partial \tau_i}\right|_{(\tau_1,\tau_2)=(0,0)} = 0 \quad \forall \widehat{y}_0, \widehat{z}_0 \in L^2(\Omega) \text{ with } \|\widehat{y}_0\|_{L^2(\Omega)} = \|\widehat{z}_0\|_{L^2(\Omega)} = 1,\ i = 1,2. \tag{1.3}$$

The existence of a control satisfying (1.3) is equivalent to the (partial) control problem of finding a function h such that the solution of system

$$\begin{cases} w_t - d_1\Delta w = -\ell u_t + f_1 + h\mathbb{1}_\omega & \text{in } Q, \\ -\theta_t - d_1\Delta\theta = -\ell\theta + q & \text{in } Q, \\ u_t - d_2\Delta u = (a + bu - u^2)u + w + f_2 & \text{in } Q, \\ -q_t - d_2\Delta q = -\ell d_2\Delta\theta - \ell(a + 2bu - 3u^2)\theta + (a + 2bu - 3u^2)q + u\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial w}{\partial n} = 0, \qquad \dfrac{\partial \theta}{\partial n} = 0, \qquad \dfrac{\partial u}{\partial n} = 0, \qquad \dfrac{\partial q}{\partial n} = 0 & \text{on } \Sigma, \\ w(0) = y_0, \qquad \theta(T) = 0, \qquad u(0) = z_0, \qquad q(T) = 0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

satisfies $\theta(0) = 0$ and $q(0) = 0$ in Ω. Indeed, let $i \in \{1, 2\}$. For every $\widehat{y}_0 \in L^2(\Omega)$ and every $\widehat{z}_0 \in L^2(\Omega)$ we have

$$\left.\frac{\partial J_{\tau_1,\tau_2}(y,z)}{\partial \tau_i}\right|_{(\tau_1,\tau_2)=(0,0)} = \iint_{\mathcal{O}\times(0,T)} u z^{\tau_i} \,\mathrm{d}x\,\mathrm{d}t, \tag{1.5}$$

where $w := y|_{(\tau_1,\tau_2)=(0,0)}$, $u := z|_{(\tau_1,\tau_2)=(0,0)}$, $y^{\tau_i} := \left.\frac{\partial y}{\partial \tau_i}\right|_{(\tau_1,\tau_2)=(0,0)}$ and $z^{\tau_i} := \left.\frac{\partial z}{\partial \tau_i}\right|_{(\tau_1,\tau_2)=(0,0)}$. Notice that (y^{τ_i}, z^{τ_i}) is the solution of

$$\begin{cases} y_t^{\tau_i} - d_1 \Delta y^{\tau_i} = -\ell z_t^{\tau_i} & \text{in } Q, \\ z_t^{\tau_i} - d_2 \Delta z^{\tau_i} - (a + 2bu - 3u^2) z^{\tau_i} = y^{\tau_i} & \text{in } Q, \\ \dfrac{\partial y^{\tau_i}}{\partial n} = 0, \qquad \dfrac{\partial z^{\tau_i}}{\partial n} = 0 & \text{on } \Sigma, \\ y^{\tau_i}(0) = (2-i)\widehat{y}_0, \qquad z^{\tau_i}(0) = (i-1)\widehat{z}_0 & \text{in } \Omega. \end{cases}$$

Using (1.4)–(1.5), we find that

$$\left.\frac{\partial J_{\tau_1,\tau_2}(y,z)}{\partial \tau_i}\right|_{(\tau_1,\tau_2)=(0,0)} = (2-i)\int_\Omega \theta(0)\widehat{y}_0 \,\mathrm{d}x + (i-1)\int_\Omega q(0)\widehat{z}_0 \,\mathrm{d}x$$

for all $(\widehat{y}_0, \widehat{z}_0) \in L^2(\Omega)^2$ such that $\|\widehat{y}_0\|_{L^2(\Omega)} = \|\widehat{z}_0\|_{L^2(\Omega)} = 1$, from where we can conclude.

Let us remark that in system (1.4), we act in an indirect way on the state (θ, q) by a single control on the first equation. Indeed, the control h acts directly on the subsystem (w, u) and then, by means of the coupling, it acts on the subsystem (θ, q). The role of the observation region $\mathcal{O}$ is coupling both subsystems through the term $u\mathbb{1}_{\mathcal{O}}$ when they are linearized around the origin. Notice that no final state restriction on trajectories (w, u) has to be imposed.

As remarked in [14,15], for parabolic systems we cannot expect to find exact insensitizing controls for any initial data (y_0, z_0) if $\omega \neq \Omega$. For our system, we impose the condition $y_0 \equiv z_0 \equiv 0$, which is usual in the concerned literature. In addition to that, we also assume that $\omega \cap \mathcal{O} \neq \emptyset$. This condition is not known to be necessary for exact insensitizing controls of parabolic equations. In the case of approximate insensitizing controls, we find [30,26] where some unique continuation properties are proven for parabolic systems where $\omega \cap \mathcal{O} = \emptyset$.

Under these hypothesis we obtain our main result.

Theorem 1.1. *Assume $y_0 \equiv z_0 \equiv 0$ and $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exist $\delta > 0$ and $K > 0$ such that for any $f_1 \in L^2(Q)$, $f_2 \in L^p(Q)$, with $p > 5/2$, satisfying*

$$\|e^{K/t} f_1\|_{L^2(Q)} + \|f_2\|_{L^p(Q)} + \|e^{K/t} f_2\|_{L^2(Q)} \le \delta,$$

there is a control $h \in L^2(\omega \times (0,T))$ such that the associated solution (w, θ, u, q) of system (1.4) verifies $\theta(0) = 0$ and $q(0) = 0$ in Ω.

As already mentioned, the following result is a direct consequence of Theorem 1.1.

Corollary 1.2. *Assume $y_0 \equiv z_0 \equiv 0$ and $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exist $\delta > 0$ and $K > 0$ such that for any $f_1 \in L^2(Q)$, $f_2 \in L^p(Q)$, with $p > 5/2$, satisfying*

$$\|e^{K/t} f_1\|_{L^2(Q)} + \|f_2\|_{L^p(Q)} + \|e^{K/t} f_2\|_{L^2(Q)} \le \delta,$$

there are controls insensitizing the functional (1.2) for system (1.1).

Let us explain how we prove Theorem 1.1. By applying a classical strategy, we consider the following linearized system

$$\begin{cases} w_t - d_1\Delta w = -\ell u_t + f_1 + h\mathbb{1}_\omega & \text{in } Q, \\ -\theta_t - d_1\Delta\theta = -\ell\theta + q & \text{in } Q, \\ u_t - d_2\Delta u = g_1 u + w + f_2 & \text{in } Q, \\ -q_t - d_2\Delta q = -\ell d_2\Delta\theta + g_2\theta + g_3 q + u\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial w}{\partial n} = 0, \quad \dfrac{\partial \theta}{\partial n} = 0, \quad \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial q}{\partial n} = 0 & \text{on } \Sigma, \\ w(0) = 0, \quad \theta(T) = 0, \quad u(0) = 0, \quad q(T) = 0 & \text{in } \Omega, \end{cases} \tag{1.6}$$

with $g_1, g_2, g_3 \in L^\infty(Q)$. Later, they will be defined by $g_1 = a + bu - u^2$, $g_2 = -\ell(a + 2bu - 3u^2)$, and $g_3 = a + 2bu - 3u^2$, when fixed point arguments are applied to come back to the original nonlinear system. Notice that here, the only coupling between subsystem (w, u), where control h acts, and subsystem (θ, q), is through the subdomain $\mathcal{O}$.

In order to study the null controllability of the linearized system (1.6), we introduce its adjoint system, given by

$$\begin{cases} -\varphi_t - d_1\Delta\varphi = \sigma & \text{in } Q, \\ \psi_t - d_1\Delta\psi = -\ell\psi - \ell d_2\Delta\kappa + g_2\kappa & \text{in } Q, \\ -\sigma_t - d_2\Delta\sigma = \ell\varphi_t + g_1\sigma + \kappa\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ \kappa_t - d_2\Delta\kappa = \psi + g_3\kappa & \text{in } Q, \\ \dfrac{\partial \varphi}{\partial n} = 0, \quad \dfrac{\partial \psi}{\partial n} = 0, \quad \dfrac{\partial \sigma}{\partial n} = 0, \quad \dfrac{\partial \kappa}{\partial n} = 0 & \text{on } \Sigma, \\ \varphi(T) = 0, \quad \psi(0) = \psi_0, \quad \sigma(T) = 0, \quad \kappa(0) = \kappa_0 & \text{in } \Omega. \end{cases} \tag{1.7}$$

We obtain an appropriate Carleman estimate for system (1.7), and use it to prove the existence of constants $C, K > 0$ such that the solutions of system (1.7) satisfy the following observability inequality

$$\iint_Q e^{-K/t}\left(|\varphi|^2 + |\sigma|^2\right)\,\mathrm{d}x\,\mathrm{d}t \le C \iint_{\omega\times(0,T)} |\varphi|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{1.8}$$

By duality arguments, this observability inequality gives the null controllability result for system (1.6) provided that

$$\iint_Q e^{K/t}\left(|f_1|^2 + |f_2|^2\right)\,\mathrm{d}x\,\mathrm{d}t < \infty. \tag{1.9}$$

Finally, in order to prove the control result for the nonlinear equation (1.4), we apply a Kakutani fixed point theorem.

The rest of this paper is organized as follows. Section 2 is devoted to state a well-posedness framework for the system (1.6). In Section 3 we prove the desired control property for the linear system (1.6). As mentioned before, this is done by getting the observability inequality (1.8) thanks to some Carleman inequalities that we obtain. The nonlinear result stated in Theorem 1.1 is proven in Section 4 by applying a Kakutani fixed point theorem.

2. Well-posedness results

To obtain a well-posedness result for problem (1.6) we will use the existence, uniqueness and regularity result below, which is a particular case of a more general result for linear parabolic PDEs given in [27]. From now on, we use the notation

$$W^p_q(\Omega) := W^{p,q}(\Omega), \quad \text{and} \quad W^{2,1}_q(Q) = \{f \in L^q(0,T;W^{2,q}(\Omega)), f_t \in L^q(Q)\}.$$

Proposition 2.1. *Let $d > 0$ a constant. Then, for any $f \in L^q(Q)$ and any $v_0 \in W_q^{2-2/q}(\Omega)$ with $q > 1$ satisfying the compatibility condition $\frac{\partial v_0}{\partial n} = 0$ in $\partial\Omega$, the equation*

$$\begin{cases} v_t - d\Delta v = f & \text{in } Q \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Sigma \\ v(0) = v_0 & \text{in } \Omega \end{cases}$$

has a unique solution $v \in W_q^{2,1}(Q)$. This solution satisfies the estimate

$$\|v\|_{W_q^{2,1}(Q)} \leq C\left[\|f\|_{L^q(Q)} + \|v_0\|_{W_q^{2-2/q}(\Omega)}\right],$$

where C only depends on Ω, T and d.

The proof of Proposition 2.1 is almost completely identical to the case of Dirichlet boundary condition (see [27, Theorem 9.1, p. 341]) and therefore omitted here for the sake of simplicity.

Remark 2.2. In our case ($N \leq 3$), we have $W_p^2(\Omega) \subset W_q^{2-2/q}(\Omega)$ with a continuous embedding for all $p \in [3q/5, q]$, by the embedding result given in [1, Theorem 5.4, p. 97]. In particular, $H^2(\Omega) \subset W_q^{2-2/q}(\Omega)$ with a continuous embedding for all $2 \leq q \leq 10/3$.

Theorem 2.3. *Let us assume that d_1 and d_2 are positive constants, g_1, g_2, $g_3 \in L^\infty(Q)$, f_1, $f_2 \in L^2(Q)$ and $h \in L^2(\omega \times (0,T))$ are given functions. The following holds.*

(a) *Problem (1.6) possesses a unique solution $(w, \theta, u, q) \in [W_2^{2,1}(Q)]^4$ satisfying*

$$\|w\|_{W_2^{2,1}(Q)} + \|\theta\|_{W_2^{2,1}(Q)} + \|u\|_{W_2^{2,1}(Q)} + \|q\|_{W_2^{2,1}(Q)} \leq C(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} + \|h\|_{L^2(Q)}),$$

where $C = C(\Omega, \omega, T, d_1, d_2, \|g_1\|_{L^\infty(Q)}, \|g_2\|_{L^\infty(Q)}, \|g_3\|_{L^\infty(Q)})$.

(b) *The solution (w, θ, u, q) of (1.6) belongs to $W_2^{2,1}(Q) \times W_r^{2,1}(Q) \times W_2^{2,1}(Q) \times W_{10}^{2,1}(Q)$, for all $2 \leq r < \infty$, and satisfies*

$$\|w\|_{W_2^{2,1}(Q)} + \|\theta\|_{W_r^{2,1}(Q)} + \|u\|_{W_2^{2,1}(Q)} + \|q\|_{W_{10}^{2,1}(Q)} \leq C(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} + \|h\|_{L^2(Q)}).$$

(c) *If $f_2 \in L^p(Q)$, with $p > 2$, the solution (w, θ, u, q) of (1.6) belongs to $W_2^{2,1}(Q) \times W_r^{2,1}(Q) \times W_{\widetilde{p}}^{2,1}(Q) \times W_{10}^{2,1}(Q)$, with $\widetilde{p} = \min\{p, 10\}$ and $2 \leq r < +\infty$, and satisfies*

$$\|w\|_{W_2^{2,1}(Q)} + \|\theta\|_{W_r^{2,1}(Q)} + \|u\|_{W_{\widetilde{p}}^{2,1}(Q)} + \|q\|_{W_{10}^{2,1}(Q)} \leq C(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^p(Q)} + \|h\|_{L^2(Q)}).$$

Proof. Let us consider the following change of variables in problem (1.6): $\widehat{\theta}(x,t) = \theta(x, T-t)$ and $\widehat{q}(x,t) = q(x, T-t)$. Then, we obtain the linear parabolic system

$$\begin{cases} w_t - d_1\Delta w = -\ell u_t + f_1 + h\mathbb{1}_\omega & \text{in } Q, \\ \widehat{\theta}_t - d_1\Delta\widehat{\theta} = -\ell\widehat{\theta} + \widehat{q} & \text{in } Q, \\ u_t - d_2\Delta u = g_1 u + w + f_2 & \text{in } Q, \\ \widehat{q}_t - d_2\Delta\widehat{q} = -\ell d_2\Delta\widehat{\theta} + g_2\widehat{\theta} + g_3\widehat{q} + u(x, T-t)\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial w}{\partial n} = 0, \qquad \dfrac{\partial \widehat{\theta}}{\partial n} = 0, \qquad \dfrac{\partial u}{\partial n} = 0, \qquad \dfrac{\partial \widehat{q}}{\partial n} = 0 & \text{on } \Sigma, \\ w(0) = 0, \qquad \widehat{\theta}(0) = 0, \qquad u(0) = 0, \qquad \widehat{q}(0) = 0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

We work with system (2.1) instead of (1.6). To prove the first part of this theorem we will apply the Leray–Schauder fixed point theorem (see [19]) to the family of operators $\mathcal{T}_\lambda : \mathcal{B} \to \mathcal{B}$, where $\mathcal{B}$ is the Banach space $[L^2(Q)]^4$ and, for each $0 \le \lambda \le 1$, T_λ is given by:

$$\mathcal{T}_\lambda(\widetilde{w}, \widetilde{\theta}, \widetilde{u}, \widetilde{q}) = (w, \widehat{\theta}, u, \widehat{q}),$$

where $(w, \widehat{\theta}, u, \widehat{q})$ satisfies

$$\begin{cases} w_t - d_1\Delta w = -\ell u_t + f_1 + h\mathbb{1}_\omega & \text{in } Q,\\ \widehat{\theta}_t - d_1\Delta\widehat{\theta} + \ell\widehat{\theta} = \lambda\widetilde{q} & \text{in } Q,\\ u_t - d_2\Delta u - g_1 u = \lambda\widetilde{w} + f_2 & \text{in } Q,\\ \widehat{q}_t - d_2\Delta\widehat{q} - g_3\widehat{q} = -\ell d_2\Delta\widehat{\theta} + g_2\widehat{\theta} + u(x, T-t)\mathbb{1}_{\mathcal{O}} & \text{in } Q,\\ \dfrac{\partial w}{\partial n} = 0, \quad \dfrac{\partial \widehat{\theta}}{\partial n} = 0, \quad \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial \widehat{q}}{\partial n} = 0 & \text{on } \Sigma,\\ w(0) = 0, \quad \widehat{\theta}(0) = 0, \quad u(0) = 0, \quad \widehat{q}(0) = 0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

In order to apply Leray–Schauder fixed point theorem, we follow 5 steps.

Step 1: $\mathcal{T}_\lambda$ is well defined for each $0 \le \lambda \le 1$.

Indeed, if $(\widetilde{w}, \widetilde{\theta}, \widetilde{u}, \widetilde{q}) \in \mathcal{B}$ and $0 \le \lambda \le 1$, we have that $\tilde{q}$ and $\tilde{w} + f_2$ belong to $L^2(Q)$. Then, by applying Proposition 2.1 in second and third equations of (2.2) we obtain that they have unique solution θ and $u \in W_2^{2,1}(Q) \hookrightarrow L^2(Q)$, respectively. Now, we have that $-\ell u_t + f_1 + h\mathbb{1}_\omega$ and $-\ell d_2\Delta\widehat{\theta} + g_2\widehat{\theta} + u(x, T-t)\mathbb{1}_{\mathcal{O}}$ belong to $L^2(Q)$. Then, by applying again Proposition 2.1 in first and fourth equations of (2.2) we obtain that they have unique solution w and $q \in W_2^{2,1}(Q) \hookrightarrow L^2(Q)$, respectively.

Hence, for each $\lambda \in [0, 1]$, $\mathcal{T}_\lambda : \mathcal{B} \mapsto \mathcal{B}$ is well defined and $\mathcal{T}_\lambda(\mathcal{B}) \subset [W_2^{2,1}(Q)]^4$.

Step 2: For each fixed $0 \le \lambda \le 1$ the mapping $\mathcal{T}_\lambda : \mathcal{B} \to \mathcal{B}$ is continuous and compact.

We fix $\lambda \in [0, 1]$. Let $(\widetilde{w}_i, \widetilde{\theta}_i, \widetilde{u}_i, \widetilde{q}_i) \in \mathcal{B}$, for $i = 1, 2$, and $(\widetilde{w}, \widetilde{\theta}, \widetilde{u}, \widetilde{q}) = (\widetilde{w}_1, \widetilde{\theta}_1, \widetilde{u}_1, \widetilde{q}_1) - (\widetilde{w}_2, \widetilde{\theta}_2, \widetilde{u}_2, \widetilde{q}_2)$. Let us consider also $(w_i, \widehat{\theta}_i, u_i, \widehat{q}_i) = \mathcal{T}_\lambda(\widetilde{w}_i, \widetilde{\theta}_i, \widetilde{u}_i, \widetilde{q}_i)$, for $i = 1, 2$, and $(w, \widehat{\theta}, u, \widehat{q}) = (w_1, \widehat{\theta}_1, u_1, \widehat{q}_1) - (w_2, \widehat{\theta}_2, u_2, \widehat{q}_2)$.

From Proposition 2.1 applied to second, third, fourth and first equations of (2.2) and since g_1, g_2, $g_3 \in L^\infty(Q)$, we get

$$\begin{aligned} &\|\widehat{\theta}\|_{W_2^{2,1}(Q)} \le C\|\lambda\widetilde{q}\|_{L^2(Q)} \le C\|\widetilde{q}\|_{L^2(Q)},\\ &\|u\|_{W_2^{2,1}(Q)} \le C\|\lambda\widetilde{w}\|_{L^2(Q)} \le C\|\widetilde{w}\|_{L^2(Q)},\\ &\|w\|_{W_2^{2,1}(Q)} \le C\|-\ell u_t\|_{L^2(Q)} \le C\|u\|_{W_2^{2,1}(Q)} \le C\|\widetilde{w}\|_{L^2(Q)}, \end{aligned}$$

and

$$\begin{aligned} \|\widehat{q}\|_{W_2^{2,1}(Q)} &\le C\|-\ell d_2\Delta\widehat{\theta} + g_2\widehat{\theta} + u(x, T-t)\mathbb{1}_{\mathcal{O}}\|_{L^2(Q)}\\ &\le C\|\widehat{\theta}\|_{W_2^{2,1}(Q)} + C\|u\|_{L^2(Q)} \le C\|\widetilde{q}\|_{L^2(Q)} + C\|\widetilde{w}\|_{L^2(Q)}. \end{aligned} \tag{2.3}$$

Therefore,

$$\|(u, \widehat{\theta}, w, \widehat{q})\|_{[W_2^{2,1}(Q)]^4} \le C\|(\widetilde{u}, \widetilde{\theta}, \widetilde{w}, \widetilde{q})\|_{\mathcal{B}}.$$

This proves that $\mathcal{T}_\lambda$ is continuous as a mapping from the space $\mathcal{B}$ into the space $[W_2^{2,1}(Q)]^4$. From the compactness of the embedding $W_2^{2,1}(Q) \hookrightarrow L^2(Q)$, we have $[W_2^{2,1}(Q)]^4 \hookrightarrow \mathcal{B}$ compactly. Thus, $T_\lambda : \mathcal{B} \mapsto \mathcal{B}$ is continuous and compact for each $\lambda \in [0, 1]$.

Step 3: For any bounded set $\mathcal{A} \subset \mathcal{B}$, the mapping $\lambda \mapsto \mathcal{T}_\lambda(\widetilde{u}, \widetilde{\theta}, \widetilde{w}, \widetilde{q})$ is continuous, uniformly with respect to $(\widetilde{u}, \widetilde{\theta}, \widetilde{w}, \widetilde{q}) \in \mathcal{A}$.

Indeed, let $\mathcal{A} \subset \mathcal{B}$ be a bounded set, assume that $(\widetilde{u}, \widetilde{\theta}, \widetilde{w}, \widetilde{q}) \in \mathcal{A}$ and $\lambda_1,\ \lambda_2 \in [0,1]$ and set $(w_i, \widehat{\theta}_i, u_i, \widehat{q}_i) = \mathcal{T}_{\lambda_i}(\widetilde{u}, \widetilde{\theta}, \widetilde{w}, \widetilde{q})$, for $i = 1,\ 2$, and $(w, \widehat{\theta}, u, \widehat{q}) = (w_1, \widehat{\theta}_1, u_1, \widehat{q}_1) - (w_2, \widehat{\theta}_2, u_2, \widehat{q}_2)$. Proceeding as before for the system satisfied by $(w, \widehat{\theta}, u, \widehat{q})$, we find

$$\|(u, \widehat{\theta}, w, \widehat{q})\|_{[W_2^{2,1}(Q)]^4} \leq C|\lambda_1 - \lambda_2| \left(\|\widetilde{w}\|_{L^2(Q)} + \|\widetilde{q}\|_{L^2(Q)}\right).$$

Since $\mathcal{A} \subset \mathcal{B}$ is a bounded set and $(\widetilde{u}, \widetilde{\theta}, \widetilde{w}, \widetilde{q}) \in \mathcal{A}$, we have that

$$\|(u, \widehat{\theta}, w, \widehat{q})\|_{[W_2^{2,1}(Q)]^4} \leq C\|(u, \widehat{\theta}, w, \widehat{q})\|_{[W_2^{2,1}(Q)]^4} \leq C|\lambda_1 - \lambda_2|$$

where C only depends on $\sup_{\mathcal{A}} \|(\widetilde{u}, \widetilde{\theta}, \widetilde{w}, \widetilde{q})\|_{\mathcal{B}}$.

Step 4: $\mathcal{T}_0$ has a unique fixed point in $\mathcal{B}$.

In fact, $(w, \widehat{\theta}, u, \widehat{q}) \in \mathcal{B}$ is a fixed point of $\mathcal{T}_0$ if, and only if, it satisfies

$$\begin{cases} w_t - d_1\Delta w = -\ell u_t + f_1 + h\mathbb{1}_\omega & \text{in } Q,\\ \widehat{\theta}_t - d_1\Delta\widehat{\theta} + \ell\widehat{\theta} = 0 & \text{in } Q,\\ u_t - d_2\Delta u - g_1 u = f_2 & \text{in } Q,\\ \widehat{q}_t - d_2\Delta\widehat{q} - g_3\widehat{q} = -\ell d_2\Delta\widehat{\theta} + g_2\widehat{\theta} + u(x, T-t)\mathbb{1}_{\mathcal{O}} & \text{in } Q,\\ \dfrac{\partial w}{\partial n} = 0, \quad \dfrac{\partial\widehat{\theta}}{\partial n} = 0, \quad \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial\widehat{q}}{\partial n} = 0 & \text{on } \Sigma,\\ w(0) = 0, \quad \widehat{\theta}(0) = 0, \quad u(0) = 0, \quad \widehat{q}(0) = 0 & \text{in } \Omega. \end{cases}$$

Notice that the equations of $\widehat{\theta}$ and u can be solved independently (in fact, $\widehat{\theta} \equiv 0$). Then, we can solve the other two equations. Hence, this system has a unique solution and therefore the operator $\mathcal{T}_0$ has a unique fixed point.

Step 5: There exists a constant $K > 0$ (independent of λ) such that any fixed point $(w, \widehat{\theta}, u, \widehat{q})$ of $\mathcal{T}_\lambda$ satisfies $\|(w, \widehat{\theta}, u, \widehat{q})\|_{\mathcal{B}} \leq K$.

To prove this assertion, let us consider $(w, \widehat{\theta}, u, \widehat{q}) \in \mathcal{B}$ a fixed point of $\mathcal{T}_\lambda$ for some $\lambda \in [0,1]$. Then $(w, \widehat{\theta}, u, \widehat{q})$ satisfies

$$\begin{cases} w_t - d_1\Delta w = -\ell u_t + f_1 + h\mathbb{1}_\omega & \text{in } Q,\\ \widehat{\theta}_t - d_1\Delta\widehat{\theta} + \ell\widehat{\theta} = \lambda\widehat{q} & \text{in } Q,\\ u_t - d_2\Delta u - g_1 u = \lambda w + f_2 & \text{in } Q,\\ \widehat{q}_t - d_2\Delta\widehat{q} - g_3\widehat{q} = -\ell d_2\Delta\widehat{\theta} + g_2\widehat{\theta} + u(x, T-t)\mathbb{1}_{\mathcal{O}} & \text{in } Q,\\ \dfrac{\partial w}{\partial n} = 0, \quad \dfrac{\partial\widehat{\theta}}{\partial n} = 0, \quad \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial\widehat{q}}{\partial n} = 0 & \text{on } \Sigma,\\ w(0) = 0, \quad \widehat{\theta}(0) = 0, \quad u(0) = 0, \quad \widehat{q}(0) = 0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

We start by replacing u_t by $d_2\Delta u + g_1 u + \lambda w + f_2$ in the first equation of (2.4). Then, multiplying by w and integrating in Ω, we arrive at

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |w|^2\,\mathrm{d}x + d_1\int_\Omega |\nabla w|^2\,\mathrm{d}x &= \ell d_2\int_\Omega \nabla u\cdot\nabla w\,\mathrm{d}x - \ell\int_\Omega g_1 uw\,\mathrm{d}x \\ &\quad - \ell\lambda\int_\Omega |w|^2\,\mathrm{d}x - \ell\int_\Omega f_2 w\,\mathrm{d}x + \int_\Omega f_1 w\,\mathrm{d}x + \int_\Omega hw\mathbb{1}_\omega\,\mathrm{d}x. \end{aligned}$$

By using Young's inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |w|^2\,\mathrm{d}x + d_1\int_\Omega |\nabla w|^2\,\mathrm{d}x \le C\int_\Omega |\nabla u|^2\,\mathrm{d}x + C\int_\Omega \left(|w|^2+|u|^2\right)\mathrm{d}x + C\int_\Omega \left(|f_1|^2+|f_2|^2+|h|^2\mathbb{1}_\omega\right)\mathrm{d}x.$$

In a similar way, it is easy to see that from the equation satisfied by u in (2.4) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2\,\mathrm{d}x + d_2\int_\Omega |\nabla u|^2\,\mathrm{d}x \le C\int_\Omega \left(|w|^2+|u|^2\right)\mathrm{d}x + C\int_\Omega |f_2|^2\,\mathrm{d}x.$$

Adding these last two inequalities and integrating in $(0,t)$, we get

$$\int_\Omega \left(|w|^2+|u|^2\right)\mathrm{d}x + \int_0^t\int_\Omega \left(|\nabla w|^2+|\nabla u|^2\right)\mathrm{d}x\,\mathrm{d}s \le C\int_0^t\int_\Omega \left(|f_1|^2+|f_2|^2+|h|^2\mathbb{1}_\omega\right)\mathrm{d}x\,\mathrm{d}s \tag{2.5}$$

for every $t\in(0,T)$. Notice that the terms corresponding to ∇u in the right-hand side can be handled by multiplying the second estimate by a sufficiently large constant.

Analogously, we can obtain for the equations satisfied by $\widehat{\theta}$ and $\widehat{q}$ the estimates

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\widehat{\theta}|^2\,\mathrm{d}x + d_1\int_\Omega |\nabla\widehat{\theta}|^2\,\mathrm{d}x \le C\int_\Omega \left(|\widehat{\theta}|^2+|\widehat{q}|^2\right)\mathrm{d}x$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\widehat{q}|^2\,\mathrm{d}x + d_2\int_\Omega |\nabla\widehat{q}|^2\,\mathrm{d}x \le C\int_\Omega |\nabla\widehat{\theta}|^2\,\mathrm{d}x + C\int_\Omega \left(|\widehat{\theta}|^2+|\widehat{q}|^2\right)\mathrm{d}x + C\int_\Omega |u(x,T-t)|^2\,\mathrm{d}x,$$

respectively. Adding these two inequalities, we can integrate in time to obtain, together with (2.5),

$$\int_\Omega \left(|\widehat{\theta}|^2+|\widehat{q}|^2\right)\mathrm{d}x + \int_0^t\int_\Omega \left(|\nabla\widehat{\theta}|^2+|\nabla\widehat{q}|^2\right)\mathrm{d}x\,\mathrm{d}s \le C\iint_Q \left(|f_1|^2+|f_2|^2+|h|^2\mathbb{1}_\omega\right)\mathrm{d}x\,\mathrm{d}t. \tag{2.6}$$

Now, we multiply the first equation in (2.4) by w_t and perform similar computations. In this way, we have

$$\int_\Omega |w_t|^2\,\mathrm{d}x + d_1\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^2\,\mathrm{d}x \le C\int_\Omega |u_t|^2\,\mathrm{d}x + C\int_\Omega \left(|f_1|^2+|h|^2\mathbb{1}_\omega\right)\mathrm{d}x.$$

On the other hand, we have that

$$\int_\Omega |u_t|^2\,\mathrm{d}x + d_2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2\,\mathrm{d}x \le C\int_\Omega \left(|w|^2+|u|^2\right)\mathrm{d}x + C\int_\Omega |f_2|^2\,\mathrm{d}x.$$

Adding the last two inequalities and integrating in time, we obtain from (2.5)

$$\iint_Q \left(|w_t|^2+|u_t|^2\right)\mathrm{d}x\,\mathrm{d}t \le C\iint_Q \left(|f_1|^2+|f_2|^2+|h|^2\mathbb{1}_\omega\right)\mathrm{d}x\,\mathrm{d}t.$$

Furthermore, from this estimate, (2.5) and the equations satisfied by w and u, we readily have

$$\iint_Q \left(|w_t|^2+|\Delta w|^2+|u_t|^2+|\Delta u|^2\right)\mathrm{d}x\,\mathrm{d}t \le C\iint_Q \left(|f_1|^2+|f_2|^2+|h|^2\mathbb{1}_\omega\right)\mathrm{d}x\,\mathrm{d}t. \tag{2.7}$$

Now, from the equation satisfied by $\widehat{\theta}$ (multiplying by $\widehat{\theta}_t$ and integrating) and using (2.6), it is easy to verify that

$$\iint_Q \left(|\widehat{\theta}_t|^2+|\Delta\widehat{\theta}|^2\right)\mathrm{d}x\,\mathrm{d}t \le C\iint_Q \left(|f_1|^2+|f_2|^2+|h|^2\mathbb{1}_\omega\right)\mathrm{d}x\,\mathrm{d}t. \tag{2.8}$$

Finally, we multiply the fourth equation in (2.4) by $\widehat{q}_t$ and integrate in space. The same process as before leads to

$$\int_\Omega |\widehat{q}_t|^2 \,\mathrm{d}x + d_2 \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \widehat{q}|^2 \,\mathrm{d}x \le C \int_\Omega \left(|\widehat{\theta}|^2 + |\Delta\widehat{\theta}|^2 + |\widehat{q}|^2\right) \mathrm{d}x + C \int_\Omega |u(x, T-t)|^2 \,\mathrm{d}x.$$

Integrating in time this inequality, from (2.5), (2.6) and (2.8) we get

$$\iint_Q \left(|\widehat{q}_t|^2 + |\Delta\widehat{q}|^2\right) \mathrm{d}x\,\mathrm{d}t \le C \iint_Q \left(|f_1|^2 + |f_2|^2 + |h|^2 \mathbb{1}_\omega\right) \mathrm{d}x\,\mathrm{d}t. \tag{2.9}$$

Gathering inequalities (2.5)–(2.9), we conclude that

$$\begin{aligned} &\|w\|_{W_2^{2,1}(Q)} + \|\widehat{\theta}\|_{W_2^{2,1}(Q)} + \|u\|_{W_2^{2,1}(Q)} + \|\widehat{q}\|_{W_2^{2,1}(Q)} \\ &\quad \le C(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} + \|h\|_{L^2(\omega\times(0,T))}). \end{aligned} \tag{2.10}$$

Since $W_2^{2,1}(Q) \subset L^2(Q)$ with compact embedding, we have $[W_2^{2,1}(Q)]^4 \subset \mathcal{B}$ and

$$\|(w, \widehat{\theta}, u, \widehat{q})\|_{\mathcal{B}} \le C\|(w, \widehat{\theta}, u, \widehat{q})\|_{[W_2^{2,1}(Q)]^4} \le C.$$

By Steps 1–5, we can apply Leray–Schauder fixed point theorem. Then, there exists a unique $(w, \widehat{\theta}, u, \widehat{q}) \in \mathcal{B}$ fixed point of $\mathcal{T}_1$. Thus, by taking $\theta(x,t) = \widehat{\theta}(x, T-t)$ and $q(x,t) = \widehat{q}(x, T-t)$, we have that (w, θ, u, q) is the unique solution of (1.6) in $\mathcal{B}$. Moreover, by (2.10) the solution belongs to $[W_2^{2,1}(Q)]^4$ and satisfies

$$\|w\|_{W_2^{2,1}(Q)} + \|\theta\|_{W_2^{2,1}(Q)} + \|u\|_{W_2^{2,1}(Q)} + \|q\|_{W_2^{2,1}(Q)} \le C(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} + \|h\|_{L^2(\omega\times(0,T))}).$$

This completes the proof of part (a).

Since $W_2^{2,1}(Q) \hookrightarrow L^{10}(Q)$, the right-hand side of the second equation in (2.4) belongs to $L^{10}(Q)$. Hence, by Proposition 2.1, it follows that $\widehat{\theta} \in W_{10}^{2,1}(Q)$ and

$$\|\widehat{\theta}\|_{W_{10}^{2,1}(Q)} \le C\|\widehat{q}\|_{L^{10}(Q)} \le C\|\widehat{q}\|_{W_2^{2,1}(Q)} \le C(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} + \|h\|_{L^2(\omega\times(0,T))}).$$

Due to the facts that $\widehat{\theta} \in W_{10}^{2,1}(Q)$ and $u \in W_2^{2,1}(Q) \hookrightarrow L^{10}(Q)$, the right-hand side of the fourth equation in (2.4) belongs to $L^{10}(Q)$. Therefore, by applying again Proposition 2.1, we obtain $\widehat{q} \in W_{10}^{2,1}(Q)$ and

$$\begin{aligned} \|\widehat{q}\|_{W_{10}^{2,1}(Q)} &\le C\| -\ell d_2 \Delta\widehat{\theta} + g_2\widehat{\theta} + u\mathbb{1}_{\mathcal{O}}\|_{L^{10}(Q)} \le C\left[\|\widehat{\theta}\|_{W_{10}^{2,1}(Q)} + \|u\|_{W_2^{2,1}(Q)}\right] \\ &\le C(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} + \|h\|_{L^2(\omega\times(0,T))}). \end{aligned}$$

As $W_{10}^{2,1}(Q) \hookrightarrow L^\infty(Q)$, the right-hand side of the second equation in (2.4) belongs to $L^\infty(Q)$. Thus, by Proposition 2.1, we get that $\widehat{\theta} \in W_r^{2,1}(Q)$ and

$$\|\widehat{\theta}\|_{W_r^{2,1}(Q)} \le C\|\widehat{q}\|_{L^r(Q)} \le C\|\widehat{q}\|_{W_{10}^{2,1}(Q)} \le C(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} + \|h\|_{L^2(\omega\times(0,T))}),$$

for any $2 \le r < \infty$.

By taking $\theta(x,t) = \widehat{\theta}(x, T-t)$ and $q(x,t) = \widehat{q}(x, T-t)$ we obtain part (b).

Finally, we assume that $f_2 \in L^p(Q)$, with $p > 2$. As $w \in W_2^{2,1}(Q) \subset L^{10}(Q)$, the right-hand side of the third equation in (2.4) belongs to $L^{\widetilde{p}}(Q)$, where $\widetilde{p} = \min\{p, 10\}$. From Proposition 2.1, it follows that $\widehat{u} \in W_{\widetilde{p}}^{2,1}(Q)$ and

$$\begin{aligned} \|u\|_{W_{\widetilde{p}}^{2,1}(Q)} &\le C\|\lambda w + f_2\|_{L^{\widetilde{p}}(Q)} \le C(\|w\|_{W_2^{2,1}(Q)} + \|f_2\|_{L^p(Q)}) \\ &\le C(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^p(Q)} + \|h\|_{L^2(\omega\times(0,T))}). \end{aligned}$$

This completes the proof of part (c). $\square$

3. Controllability of the linearized system

The goal of this section is to prove the following result.

Proposition 3.1. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, for any $f_1, f_2 \in L^2(Q)$ such that (1.9) holds and any $g_1, g_2, g_3 \in L^\infty(Q)$, there exists a control $h \in L^2(\omega \times (0,T))$ such that $\theta(0) = 0$ and $q(0) = 0$ in Ω, where (w, θ, u, q) is the solution of system (1.6). Furthermore, there exist constants $K, C > 0$ such that*

$$\|h\|_{L^2(\omega\times(0,T))} \leq C\big(\|e^{K/t} f_1\|_{L^2(Q)} + \|e^{K/t} f_2\|_{L^2(Q)}\big).$$

As it is stated in the introduction, by following the Hilbert Uniqueness Method (HUM) [28], to prove this null controllability result, it suffices to prove the existence of constants $C, K > 0$ such that every solution of (1.7) satisfies the observability inequality

$$\iint_Q e^{-K/t}\big(|\varphi|^2 + |\sigma|^2\big)\,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\omega\times(0,T)} |\varphi|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{3.1}$$

Thus, once (3.1) is established, the proof of Proposition 3.1 will be complete.

We first define some weight functions. Let us fix an open set $\omega_0 \subset \omega \cap \mathcal{O}$ and let $\eta(x) \in C^2(\overline{\Omega})$ be a function satisfying

$$\eta(x) > 0 \quad \forall x \in \Omega, \qquad \eta(x) = 0 \quad \forall x \in \partial\Omega, \qquad \text{and} \quad |\nabla\eta(x)| > 0 \quad \forall x \in \overline{\Omega \setminus \omega_0}. \tag{3.2}$$

The existence of such a function η is given in [20]. Now, let us define for $\lambda > 1$,

$$\begin{aligned}
&\alpha(x,t) := \frac{e^{4\lambda\|\eta\|_\infty} - e^{\lambda(2\|\eta\|_\infty + \eta(x))}}{t(T-t)}, \qquad \xi(x,t) := \frac{e^{\lambda(2\|\eta\|_\infty + \eta(x))}}{t(T-t)},\\
&\alpha^*(t) := \max_{x\in\overline{\Omega}} \alpha(x,t), \qquad \xi^*(t) := \min_{x\in\overline{\Omega}} \xi(x,t),\\
&\widehat{\alpha}(t) := \min_{x\in\overline{\Omega}} \alpha(x,t), \qquad \widehat{\xi}(t) := \max_{x\in\overline{\Omega}} \xi(x,t).
\end{aligned} \tag{3.3}$$

Given a function v and $r \in \mathbb{R}$, we call

$$I^r_{s,\lambda}(v) := \iint_Q e^{-2s\alpha}\Big(s^{-1+r}\lambda^r\xi^{-1+r}(|v_t|^2 + |\Delta v|^2) + s^{1+r}\lambda^{2+r}\xi^{1+r}|\nabla v|^2 + s^{3+r}\lambda^{4+r}\xi^{3+r}|v|^2\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

With these elements, we can state the following Carleman estimate valid for the solution of the heat equation with homogeneous Neumann boundary conditions. This result is essentially proved in [20] (see also [17]).

Lemma 3.2 (*[20]*)**.** *Let $r \geq 0$, $d > 0$, $v_0 \in L^2(\Omega)$ and $f \in L^2(Q)$ be given. Consider also any open set $\widetilde{\omega}$ such that $\omega_0 \subset \widetilde{\omega} \subset \Omega$. There exist positive constants λ_0, s_0 and C, depending only on $\widetilde{\omega}$ and Ω, such that the solution v of*

$$\begin{cases} v_t - d\Delta v = f & \text{in } Q,\\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Sigma,\\ v(0) = v_0 & \text{in } \Omega, \end{cases}$$

satisfies

$$I^r_{s,\lambda}(v) \leq C s^r\lambda^r \iint_Q e^{-2s\alpha}\xi^r|f|^2\,dx\,dt + C s^{3+r}\lambda^{4+r}\iint_{\widetilde{\omega}\times(0,T)} e^{-2s\alpha}\xi^{3+r}|v|^2\,dx\,dt \tag{3.4}$$

for every $\lambda \geq \lambda_0$ and every $s \geq s_0(d^{-1}T + T^2)$.

Now, we prove the following Carleman inequality.

Proposition 3.3. *Let $g_1, g_2, g_3 \in L^\infty(Q)$, $\psi_0, \kappa_0 \in L^2(\Omega)$ and assume $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exist constants $\lambda_0, s_0, C_0 > 0$, independent of T, such that every solution $(\varphi, \psi, \sigma, \kappa)$ of system (1.7) satisfies*

$$I^1_{s,\lambda}(\varphi) + I^0_{s,\lambda}(\psi) + I^0_{s,\lambda}(\sigma) + I^1_{s,\lambda}(\kappa) \leq C_0 s^{29}\lambda^{30} \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^{29}|\varphi|^2 \, dx\, dt \tag{3.5}$$

for every $\lambda \geq \lambda_0$ and every $s \geq s_0(T + T^2 + \|g_1\|_\infty^{2/3}T^2 + \|g_2\|_\infty^{1/2}T^2 + \|g_3\|_\infty^{2/3}T^2)$.

Proof. The general idea to prove (3.5) is to combine the Carleman inequality (3.4) for each equation in (1.7) choosing carefully the parameter r. Then, we use the couplings to eliminate the local terms not corresponding to the right-hand side of (3.5). This procedure being technical, we divide it in several steps.

We start by setting some open sets $\omega_1, \omega_2, \omega_3, \omega_4$ such that $\omega_0 \Subset \omega_1 \Subset \omega_2 \Subset \omega_3 \Subset \omega_4 \Subset \omega \cap \mathcal{O}$, where $A \Subset B$ means that $\overline{A} \subset B$.

Step 1: Estimate for ψ and κ.

In this step we prove

$$I^0_{s,\lambda}(\psi) + I^1_{s,\lambda}(\kappa) \leq C s^7\lambda^8 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^7|\kappa|^2 \, \mathrm{d}x\, \mathrm{d}t \tag{3.6}$$

for every $\lambda \geq C$ and $s \geq C(T + T^2 + \|g_2\|_\infty^{1/2}T^2 + \|g_3\|_\infty^{2/3}T^2)$.

We apply Lemma 3.2 to the equations satisfied by ψ and κ with $(\widetilde{\omega} = \omega_1, r = 0)$ and $(\widetilde{\omega} = \omega_2, r = 1)$, respectively. This gives

$$I^0_{s,\lambda}(\psi) \leq C \iint_Q e^{-2s\alpha}\Big(|\psi|^2 + |\Delta\kappa|^2 + \|g_2\|^2_\infty|\kappa|^2\Big) \, \mathrm{d}x\, \mathrm{d}t + C s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2 \, \mathrm{d}x\, \mathrm{d}t$$

and

$$I^1_{s,\lambda}(\kappa) \leq C s\lambda \iint_Q e^{-2s\alpha}\xi\Big(|\psi|^2 + \|g_3\|^2_\infty|\kappa|^2\Big) \, \mathrm{d}x\, \mathrm{d}t + C s^4\lambda^5 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^4|\kappa|^2 \, \mathrm{d}x\, \mathrm{d}t$$

for every $\lambda \geq C$ and $s \geq C(T + T^2)$. Adding these inequalities, it is easy to see that

$$I^0_{s,\lambda}(\psi) + I^1_{s,\lambda}(\kappa) \leq C s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2 \, \mathrm{d}x\, \mathrm{d}t + C s^4\lambda^5 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^4|\kappa|^2 \, \mathrm{d}x\, \mathrm{d}t \tag{3.7}$$

for every $\lambda \geq C$ and $s \geq C(T + T^2 + \|g_2\|_\infty^{1/2}T^2 + \|g_3\|_\infty^{2/3}T^2)$.

Now, we need to estimate the first local term in the right-hand side of (3.7). To do so, let $\rho_1(x) \in C_0^\infty(\omega_2)$ be a non-negative function such that $\rho_1 = 1$ in ω_1. Using the last equation in (1.7), we have

$$\begin{aligned} C s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2 \, \mathrm{d}x\, \mathrm{d}t &\leq C s^3\lambda^4 \iint_{\omega_2\times(0,T)} \rho_1(x) e^{-2s\alpha}\xi^3|\psi|^2 \, \mathrm{d}x\, \mathrm{d}t \\ &= C s^3\lambda^4 \iint_{\omega_2\times(0,T)} \rho_1(x) e^{-2s\alpha}\xi^3\psi\Big(\kappa_t - d_2\Delta\kappa - g_3\kappa\Big) \, \mathrm{d}x\, \mathrm{d}t. \end{aligned}$$

It is easy to verify, by means of integration by parts and Young's inequality, that

$$C s^3\lambda^4 \iint_{\omega_2\times(0,T)} \rho_1(x) e^{-2s\alpha}\xi^3\psi\Big(\kappa_t - d_2\Delta\kappa\Big) \, \mathrm{d}x\, \mathrm{d}t \leq \frac{1}{4} I^0_{s,\lambda}(\psi) + C s^7\lambda^8 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^7|\kappa|^2 \, \mathrm{d}x\, \mathrm{d}t$$

for every $\lambda \geq C$ and $s \geq C(T + T^2)$. On the other hand,

$$-C s^3\lambda^4 \iint_{\omega_2\times(0,T)} \rho_1(x) e^{-2s\alpha}\xi^3\psi g_3\kappa \, \mathrm{d}x\, \mathrm{d}t \leq \frac{1}{4} I^0_{s,\lambda}(\psi) + s^3\lambda^4\|g_3\|^2_\infty \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^3|\kappa|^2 \, \mathrm{d}x\, \mathrm{d}t.$$

Going back to (3.7), we obtain (3.6).

Step 2: Estimate of the local term of κ.

Let us now estimate the local term in (3.6). More precisely, let us prove

$$Cs^7\lambda^8 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^7|\kappa|^2 \,\mathrm{d}x\,\mathrm{d}t \leq Cs^{14}\lambda^{15}\iint_{\omega_3\times(0,T)} e^{-2s\alpha}\xi^{14}|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + Cs^{14}\lambda^{15}\iint_{\omega_3\times(0,T)} e^{-2s\alpha}\xi^{14}|\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{3.8}$$

for every $\lambda \geq C$ and $s \geq C(T+T^2+\|g_1\|_\infty^{1/2}T^2)$.

Just as we did in the last part of the previous step, we consider $\rho_2(x)\in C_0^\infty(\omega_3)$ a non-negative function such that $\rho_2=1$ in ω_2. Then, since $\omega_3\subset\omega\cap\mathcal{O}$,

$$\begin{aligned}Cs^7\lambda^8 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^7|\kappa|^2 \,\mathrm{d}x\,\mathrm{d}t &\leq Cs^7\lambda^8 \iint_{\omega_3\times(0,T)} \rho_2(x)e^{-2s\alpha}\xi^7|\kappa|^2 \,\mathrm{d}x\,\mathrm{d}t\\ &= Cs^7\lambda^8 \iint_{\omega_3\times(0,T)} \rho_2(x)e^{-2s\alpha}\xi^7\kappa\Big(-\sigma_t - d_2\Delta\sigma - \ell\varphi_t - g_1\sigma\Big) \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Here, we have used the third equation in (1.7). Integrating by parts and using Young's inequality, we get

$$\begin{aligned}&Cs^7\lambda^8 \iint_{\omega_3\times(0,T)} \rho_2(x)e^{-2s\alpha}\xi^7\kappa\Big(-\sigma_t - d_2\Delta\sigma - \ell\varphi_t\Big) \,\mathrm{d}x\,\mathrm{d}t\\ &\leq \frac{1}{4}I^1_{s,\lambda}(\kappa) + Cs^{14}\lambda^{15}\iint_{\omega_3\times(0,T)} e^{-2s\alpha}\xi^{14}\Big(|\varphi|^2+|\sigma|^2\Big) \,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for every $\lambda \geq C$ and $s \geq C(T+T^2)$, and

$$-Cs^7\lambda^8 \iint_{\omega_3\times(0,T)} \rho_2(x)e^{-2s\alpha}\xi^7\kappa g_1\sigma \,\mathrm{d}x\,\mathrm{d}t \leq \frac{1}{4}I^1_{s,\lambda}(\kappa) + s^{10}\lambda^{11}\|g_1\|^2_\infty \iint_{\omega_3\times(0,T)} e^{-2s\alpha}\xi^{10}|\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Hence, inequality (3.8) is readily deduced.

Step 3: Estimate for φ and σ, and conclusion.

First, let us gather the estimates from the previous steps. From (3.6) and (3.8) we have

$$I^0_{s,\lambda}(\psi) + I^1_{s,\lambda}(\kappa) \leq Cs^{14}\lambda^{15}\iint_{\omega_3\times(0,T)} e^{-2s\alpha}\xi^{14}|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + Cs^{14}\lambda^{15}\iint_{\omega_3\times(0,T)} e^{-2s\alpha}\xi^{14}|\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{3.9}$$

for every $\lambda \geq C$ and $s \geq C(T+T^2+\|g_1\|_\infty^{1/2}T^2+\|g_2\|_\infty^{1/2}T^2+\|g_3\|_\infty^{2/3}T^2)$.

Once again, we apply Lemma 3.2, this time for the equations satisfied by φ and σ with $(\widetilde{\omega}=\omega_4, r=1)$ and $(\widetilde{\omega}=\omega_3, r=0)$, respectively. This gives

$$I^1_{s,\lambda}(\varphi) \leq Cs\lambda \iint_Q e^{-2s\alpha}\xi|\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t + Cs^4\lambda^5\iint_{\omega_4\times(0,T)} e^{-2s\alpha}\xi^4|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

and

$$I^0_{s,\lambda}(\sigma) \leq C\iint_Q e^{-2s\alpha}\Big(|\varphi_t|^2 + \|g_1\|^2_\infty|\sigma|^2 + |\kappa|^2\Big) \,\mathrm{d}x\,\mathrm{d}t + Cs^3\lambda^4\iint_{\omega_3\times(0,T)} e^{-2s\alpha}\xi^3|\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t$$

for every $\lambda \geq C$ and $s \geq C(T+T^2)$. Adding up these two estimates yields

$$\begin{aligned}I^1_{s,\lambda}(\varphi) + I^0_{s,\lambda}(\sigma) &\leq Cs^4\lambda^5\iint_{\omega_4\times(0,T)} e^{-2s\alpha}\xi^4|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + C\iint_Q e^{-2s\alpha}|\kappa|^2 \,\mathrm{d}x\,\mathrm{d}t\\ &\quad + Cs^3\lambda^4\iint_{\omega_3\times(0,T)} e^{-2s\alpha}\xi^3|\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.10}$$

for every $\lambda \geq C$ and $s \geq C(T + T^2 + \|g_1\|_\infty^{2/3} T^2)$. Now, we combine (3.10) with (3.9) to obtain

$$I^1_{s,\lambda}(\varphi) + I^0_{s,\lambda}(\psi) + I^0_{s,\lambda}(\sigma) + I^1_{s,\lambda}(\kappa) \leq Cs^{14}\lambda^{15} \iint_{\omega_4\times(0,T)} e^{-2s\alpha}\xi^{14}|\varphi|^2 \, \mathrm{d}x\,\mathrm{d}t$$
$$+ Cs^{14}\lambda^{15} \iint_{\omega_3\times(0,T)} e^{-2s\alpha}\xi^{14}|\sigma|^2 \, \mathrm{d}x\,\mathrm{d}t \tag{3.11}$$

for every $\lambda \geq C$ and $s \geq C(T+T^2+\|g_1\|_\infty^{2/3}T^2+\|g_2\|_\infty^{1/2}T^2+\|g_3\|_\infty^{2/3}T^2)$. To finish the proof of (3.5), we only need to estimate the last term in the right-hand side of (3.11). Notice that this is completely analogous to the last part of Step 1 (with no g_3). Thus, the proof of (3.5) is done by taking $\lambda \geq C$ and $s \geq C(T+T^2)$. □

In order to be able to prove (3.1), we need to deduce a Carleman estimate similar to (3.5) with weights that do not vanish at $t = T$. They are given by

$$\begin{aligned}
&\beta(x,t) := \frac{e^{4\lambda\|\eta\|_\infty} - e^{\lambda(2\|\eta\|_\infty+\eta(x))}}{\tau(t)}, \qquad \gamma(x,t) := \frac{e^{\lambda(2\|\eta\|_\infty+\eta(x))}}{\tau(t)},\\
&\beta^*(t) := \max_{x\in\overline{\Omega}} \beta(x,t), \qquad \gamma^*(t) := \min_{x\in\overline{\Omega}} \gamma(x,t),\\
&\widehat{\beta}(t) := \min_{x\in\overline{\Omega}} \beta(x,t), \qquad \widehat{\gamma}(t) := \max_{x\in\overline{\Omega}} \gamma(x,t)
\end{aligned} \tag{3.12}$$

where $\tau \in C^1([0,T])$ is defined by $\tau(t) := t(T-t)\mathbb{1}_{[0,T/2)} + (T^2/4)\mathbb{1}_{[T/2,T]}$. We have the following result.

Proposition 3.4. *Assume hypothesis of Proposition 3.3, and suppose that λ and s are fixed so that (3.5) holds. Then, there exists $C_0 > 0$ such that every solution $(\varphi,\psi,\sigma,\kappa)$ of system (1.7) satisfies*

$$\iint_Q e^{-2s\beta^*}\Big((\gamma^*)^4|\varphi|^2 + (\gamma^*)^3|\psi|^2 + (\gamma^*)^3|\sigma|^2 + (\gamma^*)^4|\kappa|^2\Big)\, dx\, dt$$
$$\leq C_0 \iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}}\widehat{\gamma}^{29}|\varphi|^2 \, dx\, dt. \tag{3.13}$$

Proof. The key to prove (3.13) is to find $C > 0$ such that

$$\|\varphi\|^2_{L^2(Q)} + \|\psi\|^2_{L^2(Q)} + \|\sigma\|^2_{L^2(Q)} + \|\kappa\|^2_{L^2(Q)} \leq C\Big(\|f_1\|^2_{L^2(Q)} + \|f_2\|^2_{L^2(Q)} + \|f_3\|^2_{L^2(Q)} + \|f_4\|^2_{L^2(Q)}$$
$$+ \|\varphi_0\|^2_{L^2(\Omega)} + \|\psi_0\|^2_{L^2(\Omega)} + \|\sigma_0\|^2_{L^2(\Omega)} + \|\kappa_0\|^2_{L^2(\Omega)}\Big) \tag{3.14}$$

for any solution of

$$\begin{cases}
-\varphi_t - d_1\Delta\varphi = \sigma + f_1 & \text{in } Q,\\
\psi_t - d_1\Delta\psi = -\ell\psi - \ell d_2\Delta\kappa + g_2\kappa + f_2 & \text{in } Q,\\
-\sigma_t - d_2\Delta\sigma = \ell\varphi_t + g_1\sigma + \kappa\mathbb{1}_{\mathcal{O}} + f_3 & \text{in } Q,\\
\kappa_t - d_2\Delta\kappa = \psi + g_3\kappa + f_4 & \text{in } Q,\\
\dfrac{\partial\varphi}{\partial n} = 0, \quad \dfrac{\partial\psi}{\partial n} = 0, \quad \dfrac{\partial\sigma}{\partial n} = 0, \quad \dfrac{\partial\kappa}{\partial n} = 0 & \text{on } \Sigma,\\
\varphi(T) = \varphi_0, \quad \psi(0) = \psi_0, \quad \sigma(T) = \sigma_0, \quad \kappa(0) = \kappa_0 & \text{in } \Omega.
\end{cases} \tag{3.15}$$

Actually, by similar computations to those in Step 5 (Proof of Theorem 2.3), we obtain

$$\|\psi\|^2_{L^2(0,T;H^1(\Omega))} + \|\kappa\|^2_{L^2(0,T;H^1(\Omega))} \leq C\Big(\|f_2\|^2_{L^2(Q)} + \|f_4\|^2_{L^2(Q)} + \|\psi_0\|^2_{L^2(\Omega)} + \|\kappa_0\|^2_{L^2(\Omega)}\Big).$$

By writing $-\sigma_t - d_2\Delta\sigma = -\ell\sigma + g_1\sigma - \ell d_1\Delta\varphi + \kappa\mathbb{1}_{\mathcal{O}} - \ell f_1 + f_3$, we get

$$\|\varphi\|^2_{L^2(0,T;H^1(\Omega))} + \|\sigma\|^2_{L^2(0,T;H^1(\Omega))} \leq C\Big(\|f_1\|^2_{L^2(Q)} + \|f_3\|^2_{L^2(Q)} + \|\kappa\|^2_{L^2(Q)} + \|\varphi_0\|^2_{L^2(\Omega)} + \|\sigma_0\|^2_{L^2(\Omega)}\Big).$$

The combination of these inequalities gives (3.14).

Now, by a very well known strategy (see for instance, [10, Lemma 3.1]), we obtain (3.13). Notice that conditions $\varphi(T) = \sigma(T) = 0$ in (1.7) are essential to get (3.13). □

From (3.13) it is easy to deduce (3.1). Indeed, we have that

$$\exp\big(-2s\widehat{\beta}\big)\widehat{\gamma}^{29} \le \frac{C}{T^{58}} \exp\big(-C(1/T + \|g_1\|_\infty^{2/3} + \|g_2\|_\infty^{1/2} + \|g_3\|_\infty^{2/3})\big) \le CT^{-58} \quad \text{in } Q.$$

On the other hand,

$$\exp\big(-2s\beta^*\big)(\gamma^*)^4 \ge \frac{C}{T^8} \exp\Big(\frac{-CT(1 + \|g_1\|_\infty^{2/3} + \|g_2\|_\infty^{1/2} + \|g_3\|_\infty^{2/3})}{t}\Big) \quad \text{in } Q$$

and

$$\exp\big(-2s\beta^*\big)(\gamma^*)^3 \ge \frac{C}{T^6} \exp\Big(\frac{-CT(1 + \|g_1\|_\infty^{2/3} + \|g_2\|_\infty^{1/2} + \|g_3\|_\infty^{2/3})}{t}\Big) \quad \text{in } Q.$$

This completes the proof of Proposition 3.1.

4. Controllability of the non-linear system

In this section we prove Theorem 1.1. Before starting the proof, let us observe that system (1.4) with $y_0 \equiv z_0 \equiv 0$ can be rewritten as

$$\begin{cases} w_t - d_1\Delta w = -\ell u_t + f_1 + h\mathbb{1}_\omega & \text{in } Q,\\ -\theta_t - d_1\Delta\theta = -\ell\theta + q & \text{in } Q,\\ u_t - d_2\Delta u = g_1(u)u + w + f_2 & \text{in } Q,\\ -q_t - d_2\Delta q = -\ell d_2\Delta\theta + g_2(u)\theta + g_3(u)q + u\mathbb{1}_{\mathcal{O}} & \text{in } Q,\\ \dfrac{\partial w}{\partial n} = 0, \quad \dfrac{\partial \theta}{\partial n} = 0, \quad \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial q}{\partial n} = 0 & \text{on } \Sigma,\\ w(0) = 0, \quad \theta(T) = 0, \quad u(0) = 0, \quad q(T) = 0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

with $g_1(v) = \big(a + bv - v^2\big)$, $g_2(v) = -\ell\left(a + 2bv - 3v^2\right)$, and $g_3(v) = \big(a + 2bv - 3v^2\big)$.

We introduce the spaces

$$X := L^2(0,T;H^1(\Omega)) \times [C([0,T];H^1(\Omega)) \cap L^4(Q)] \times L^2(0,T;H^1(\Omega)) \times C([0,T];H^1(\Omega))$$
$$Y := W_2^{2,1}(Q) \times W_{10}^{2,1}(Q) \times W_p^{2,1}(Q) \times W_{10}^{2,1}(Q)$$

and the function

$$\Pi_R(v) = \begin{cases} v & \text{if } |v| \le R,\\ R\ sgn(v) & \text{if } |v| > R, \end{cases} \tag{4.2}$$

where $R > 0$ is an arbitrary constant.

For each $\xi = (\widetilde{w}, \widetilde{\theta}, \widetilde{u}, \widetilde{q}) \in X$, let us consider the linear system

$$\begin{cases} w_t - d_1\Delta w = -\ell u_t + f_1 + h\mathbb{1}_\omega & \text{in } Q,\\ -\theta_t - d_1\Delta\theta = -\ell\theta + q & \text{in } Q,\\ u_t - d_2\Delta u = \alpha_\xi u + w + f_2 & \text{in } Q,\\ -q_t - d_2\Delta q = -\ell d_2\Delta\theta - \ell\beta_\xi\theta + \beta_\xi q + u\mathbb{1}_{\mathcal{O}} & \text{in } Q,\\ \dfrac{\partial w}{\partial n} = 0, \quad \dfrac{\partial \theta}{\partial n} = 0, \quad \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial q}{\partial n} = 0 & \text{on } \Sigma,\\ w(0) = 0, \quad \theta(T) = 0, \quad u(0) = 0, \quad q(T) = 0 & \text{in } \Omega, \end{cases} \tag{4.3}$$

where $\alpha_\xi = g_1(\Pi_R(\widetilde{u}))$ and $\beta_\xi = g_3(\Pi_R(\widetilde{u}))$. Notice that $-\ell\beta_\xi = -\ell g_3(\Pi_R(\widetilde{u})) = g_2(\Pi_R(\widetilde{u}))$, and α_ξ, $\beta_\xi \in L^\infty(Q)$.

Proof of Theorem 1.1. First of all, for each $\xi = (\widetilde{w}, \widetilde{\theta}, \widetilde{u}, \widetilde{q}) \in X$ let us apply Proposition 3.1 to find a control h_ξ such that the solution $(w_h, \theta_h, u_h, q_h)$ of system (4.3) with control h_ξ and potentials α_ξ and β_ξ is such that $\theta_\xi(0) = q_\xi(0) = 0$ in Ω. Moreover, there exists $C_\xi(\|\alpha_\xi\|_{L^\infty(Q)}, \|\beta_\xi\|_{L^\infty(Q)}) > 0$ such that

$$\|h_\xi\|_{L^2(0,T;L^2(\omega))} \leq C_\xi\big(\|e^{K/t} f_1\|_{L^2(Q)} + \|e^{K/t} f_2\|_{L^2(Q)}\big). \tag{4.4}$$

Besides, by Theorem 2.3, part (c), we obtain that $(w_h, \theta_h, u_h, q_h) \in Y$ and the estimate

$$\|w_h\|_{W_2^{2,1}(Q)} + \|\theta_h\|_{W_{10}^{2,1}(Q)} + \|u_h\|_{W_p^{2,1}(Q)} + \|q_h\|_{W_{10}^{2,1}(Q)} \leq C\big(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^p(Q)} + \|h_\xi\|_{L^2(\omega\times(0,T))}\big).$$

Then, by this inequality and (4.4), it follows

$$\begin{aligned} &\|w_h\|_{W_2^{2,1}(Q)} + \|\theta_h\|_{W_{10}^{2,1}(Q)} + \|u_h\|_{W_p^{2,1}(Q)} + \|q_h\|_{W_{10}^{2,1}(Q)} \\ &\quad \leq C_\xi\big(\|e^{K/t} f_1\|_{L^2(Q)} + \|f_2\|_{L^p(Q)} + \|e^{K/t} f_2\|_{L^2(Q)}\big). \end{aligned} \tag{4.5}$$

Now, for each $\xi \in X$, we define the family of control functions

$$\mathcal{A}_R(\xi) = \{h_\xi \in L^2(Q) : (w_h, \theta_h, u_h, q_h) \in Y,\ \theta_h(0) = q_h(0) = 0 \text{ in } \Omega \text{ and } h_\xi \text{ satisfies (4.4)}\}. \tag{4.6}$$

Then, we can consider the multi-valued mapping $\Lambda_R : X \to 2^X$ defined by

$$\Lambda_R(\xi) = \{(w_h, \theta_h, u_h, q_h) \in Y :\ h_\xi \in \mathcal{A}_R(\xi) \text{ and } (w_h, \theta_h, u_h, q_h) \text{ satisfies (4.5)}\}. \tag{4.7}$$

This map is well defined since $Y \subset X$. Let us apply Kakutani's fixed point theorem to Λ_R (see [5, Chapter 9] for the statement and [16,18] for applications in control theory). For this, we observe that:

(i) $\Lambda_R(\xi) \neq \emptyset$ for all $\xi \in X$.
See the first paragraph in this proof.
(ii) $\Lambda_R(\xi)$ is a convex set for all $\xi \in X$.
This follows directly from the linearity of system (4.3).
(iii) $\Lambda_R(\xi)$ is closed for all $\xi \in X$.

For this, let us fix an arbitrary $\xi \in X$. Let $\{(w^n, \theta^n, u^n, q^n)\}_{n\in\mathbb{N}}$ be a sequence in $\Lambda_R(\xi)$ such that $(w^n, \theta^n, u^n, q^n) \to (w, \theta, u, q)$ in X. Let us denote by h^n the control function associated to $(w^n, \theta^n, u^n, q^n)$ for each $n \in \mathbb{N}$. We have that $(w^n, \theta^n, u^n, q^n)$ satisfies (4.5) and consequently there exists a subsequence (still denoted by $\{(w^n, \theta^n, u^n, q^n)\}_{n\in\mathbb{N}}$) such that

$$(w^n, \theta^n, u^n, q^n) \rightharpoonup (w, \theta, u, q) \quad \text{weakly in } Y.$$

Since h^n satisfies (4.4) $\forall n \in \mathbb{N}$, we have the existence of a subsequence (still denoted by $\{h^n\}_{n\in\mathbb{N}}$) such that

$$h^n \rightharpoonup h \quad \text{weakly in } L^2(\omega \times (0,T)).$$

By using the previous convergences to pass to the limit $n \to \infty$ in (4.3) (in a subsequence if necessary) we obtain that (w, θ, u, q) is the solution of (4.3) with control h and potentials α_ξ and β_ξ. Furthermore, we get $\theta(0) = q(0) = 0$ in Ω, and consequently, $(w, \theta, u, q) \in \Lambda_R(\xi)$. So we conclude that $\Lambda_R(\xi)$ is closed.

(iv) $\Lambda_R(\xi)$ is uniformly bounded for all $\xi \in X$.
This follows from estimate (4.5).
(v) $\Lambda_R : X \to 2^X$ is a compact mapping.

Let $\mathcal{B} \subset X$ be a bounded set and $\Lambda_R(\mathcal{B}) = \cup\{\Lambda_R(\xi) : \xi \in \mathcal{B}\}$. Since $\mathcal{B}$ is bounded, it follows, by estimate (4.5), that

$$\|w_h\|_{W_2^{2,1}(Q)} + \|\theta_h\|_{W_{10}^{2,1}(Q)} + \|u_\xi\|_{W_p^{2,1}(Q)} + \|q_\xi\|_{W_{10}^{2,1}(Q)} \leq C\big(\|e^{K/t} f_1\|_{L^2(Q)} + \|f_p\|_{L^2(Q)} + \|e^{K/t} f_2\|_{L^2(Q)}\big),$$

with $C = C(\mathcal{B})$. This implies that $\Lambda_R(\mathcal{B})$ is bounded in Y. Moreover, $W_2^{2,1}(Q)$, $W_p^{2,1}(Q) \subset L^2(0,T;H^1(\Omega))$ and $W_{10}^{2,1}(Q) \subset C([0,T];H^1(\Omega))$ with compact embeddings. Consequently, $\Lambda_R(\mathcal{B})$ is relatively compact in X. Thus, $\Lambda_R : X \to 2^X$ is compact.

(vi) $\Lambda_R : X \to 2^X$ is upper hemicontinuous.

We have to prove that for each bounded linear functional μ on X ($\mu \in X'$), the function

$$\begin{aligned} X &\to \mathbb{R} \\ \xi &\mapsto \sup_{(w,\theta,u,q)\in\Lambda_R(\xi)} \langle \mu, (w,\theta,u,q)\rangle \end{aligned}$$

is upper semicontinuous. In other words, we need to show that the set

$$\mathcal{B}_{k,\mu} = \left\{ \xi \in X : \sup_{(w,\theta,u,q)\in\Lambda_R(\xi)} \langle \mu, (w,\theta,u,q)\rangle \geq k \right\}$$

is closed in X, for all $\mu \in X'$ and all $k \in \mathbb{R}$.

For this, let us fix $\mu \in X'$ and $k \in \mathbb{R}$. Let $\{\xi^n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}_{k,\mu}$ such that $\xi^n = (\widetilde{w}^n, \widetilde{\theta}^n, \widetilde{u}^n, \widetilde{q}^n) \to \xi = (\widetilde{w}, \widetilde{\theta}, \widetilde{u}, \widetilde{q})$ in X. Consequently, we have

$$\begin{aligned} &\widetilde{u}^n \to \widetilde{u} \quad \text{strongly in } L^4(Q), \\ &\alpha_{\xi^n} = \left(a + b\Pi_R(\widetilde{u}^n) - \Pi_R(\widetilde{u}^n)^2\right) \to \alpha_\xi \quad \text{strongly in } L^2(Q), \\ &\beta_{\xi^n} = \left(a + 2b\Pi_R(\widetilde{u}^n) - 3\Pi_R(\widetilde{u}^n)^2\right) \to \beta_\xi \quad \text{strongly in } L^2(Q). \end{aligned}$$

Observe that α_ξ, $\beta_\xi \in L^\infty(Q)$.

Now, for each $n \in \mathbb{N}$, $\Lambda_R(\xi^n)$ is closed and relatively compact in X, then it is a compact set. Thus, there exists $(w^n, \theta^n, u^n, q^n) \in \Lambda_R(\xi^n)$ such that

$$k \leq \sup_{(w,\theta,u,q)\in\Lambda_R(\xi^n)} \langle \mu, (w,\theta,u,q)\rangle = \langle \mu, (w^n,\theta^n,u^n,q^n)\rangle. \tag{4.8}$$

By definitions of $\Lambda_R(\xi^n)$ and $\mathcal{A}_R(\xi^n)$, there exists a control $h^n \in L^2(Q)$ such that $(w^n,\theta^n,u^n,q^n) \in Y$ is solution of (4.3) with control h^n and potentials $\alpha_{\xi^n} = g_1(\Pi_R(\widetilde{u}^n))$ and $\beta_{\xi^n} = g_3(\Pi_R(\widetilde{u}^n))$. Furthermore $\theta^n(0) = q^n(0) = 0$ in Ω and h^n, (w^n,θ^n,u^n,q^n) satisfy (4.4) and (4.5), respectively.

Since $\{h^n\}_{n\in\mathbb{N}}$ and $\{(w^n,\theta^n,u^n,q^n)\}_{n\in\mathbb{N}}$ are bounded sequences in $L^2(\omega\times(0,T))$ and Y, respectively, we get the following convergences (up to a subsequence),

$$\begin{aligned} &(w^n,\theta^n,u^n,q^n) \rightharpoonup (\widehat{w},\widehat{\theta},\widehat{u},\widehat{q}) \quad \text{weakly in } Y, \\ &h^n \rightharpoonup \widehat{h} \quad \text{weakly in } L^2(\omega\times(0,T)). \end{aligned}$$

Using the previous convergences to pass to the limit, as $n \to \infty$, in (4.3), we obtain that $(\widehat{w},\widehat{\theta},\widehat{u},\widehat{q})$ satisfies (4.3) with control $\widehat{h}$ and potentials $\alpha_\xi = g_1(\Pi_R(\widetilde{u}))$ and $\beta_\xi = g_3(\Pi_R(\widetilde{u}))$. Moreover, $\widehat{\theta}(0) = \widehat{q}(0) = 0$ in Ω and the estimates (4.4) and (4.5) hold for $\widehat{h}$ and $(\widehat{w},\widehat{\theta},\widehat{u},\widehat{q})$, respectively. Thus,

$$\widehat{h} \in \mathcal{A}_R(\xi) \quad \text{and} \quad (\widehat{w},\widehat{\theta},\widehat{u},\widehat{q}) \in \Lambda_R(\xi). \tag{4.9}$$

From (4.9) and passing to the limit in (4.8), we obtain

$$k \leq \langle \mu, (\widehat{w},\widehat{\theta},\widehat{u},\widehat{q})\rangle \leq \sup_{(w,\theta,u,q)\in\Lambda_R(\xi)} \langle \mu, (w,\theta,u,q)\rangle.$$

Thus $\xi \in \mathcal{B}_{k,\mu}$, which implies that $\mathcal{B}_{k,\mu}$ is a closed set and then Λ_R is upper hemicontinuous.

We apply now Kakutani's fixed point theorem and conclude that there exists a fixed point $(w,\theta,u,q) \in X$ of Λ_R. Therefore, there exists a control $h \in L^2(\omega\times(0,T))$ such that $(w,\theta,u,q) \in X$ is the corresponding solution of (4.3) with potentials $\alpha_\xi = g_1(\Pi_R(u))$ and $\beta_\xi = g_3(\Pi_R(u))$. Moreover, h and (w, θ, u, q) satisfy (4.4), (4.5) and $\theta(0) = q(0) = 0$ in Ω.

The last point in the proof is to show that $\Pi_R(u) = u$. Using (4.5) and the compact embedding $W_p^{2,1}(Q) \subset L^\infty(Q)$ (recall that $p > 5/2$), we have

$$\|u\|_{L^\infty(Q)} \le \widehat{C}\|u\|_{W_p^{2,1}(Q)} \le C\widehat{C}\big(\|e^{K/t}f_1\|_{L^2(Q)} + \|f_2\|_{L^p(Q)} + \|e^{K/t}f_2\|_{L^2(Q)}\big).$$

We take $\delta = R/\widehat{C}C > 0$ and obtain that $\Pi_R(u) = u$ provided that

$$\|e^{K/t}f_1\|_{L^2(Q)} + \|f_2\|_{L^p(Q)} + \|e^{K/t}f_2\|_{L^2(Q)} \le \delta.$$

This ends the proof of Theorem 1.1. $\square$

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