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On the determination of the principal coefficient from boundary measurements in a KdV equation

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Abstract. This paper concerns the inverse problem of retrieving the principal coefficient in a Korteweg–de Vries (KdV) equation from boundary measurements of a single solution. The Lipschitz stability of this inverse problem is obtained using a new global Carleman estimate for the linearized KdV equation. The proof is based on the Bukhgeĭm–Klibanov method.

Keywords. Inverse problem, Korteweg–de Vries equation, Carleman estimate.

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1 Introduction

The Korteweg–de Vries (KdV) equation

$$y_t(t,x) + y_{xxx}(t,x) + y_x(t,x) + y(t,x)y_x(t,x) = 0$$

is a nonlinear dispersive equation that serves as a mathematical model to study the propagation of long water waves in channels of relatively shallow depth and flat bottom [36]. In this model, the function $y = y(t,x)$ represents the surface elevation of the water wave at time t and at position x. Since the works by Johnson [32] and Grimshaw [24] (see also the recent survey [25]), the study of water waves moving over variable topography has been considered. If we denote $h = h(x)$ the function describing the variations in depth of the channel, then the proposed model becomes (after scaling)

$$\begin{aligned} y_t(t,x) + h^2(x)y_{xxx}(t,x) + (\sqrt{h(x)}\,y(t,x))_x \\ + \frac{1}{\sqrt{h(x)}}y(t,x)y_x(t,x) = 0. \end{aligned} \tag{1.1}$$

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In a recent paper, Israwi [31] proposes a correction to this model by saying that the main coefficient is not $h^2(x)y_{xxx}(t,x)$ but $h^{\frac{5}{2}}(x)y_{xxx}(t,x)$. Thus, we are led to consider variable coefficients KdV equations to model the water wave propagation in non-flat channels.

The original inverse problem which motivated this paper can be stated in a informal way as

> "to recover information on the topography of a channel (denoted by h) by means of boundary measurements of a water wave (given by y)".

This kind of coefficient inverse problem is by now classical and referred to as the determination of coefficients in evolution partial differential equations (PDEs) by means of the partial knowledge of a single solution.

Many other inverse problems on KdV equations could be studied with possible applications. For example the pulsatile blood flow in an arterial compartment can be modelized by KdV equation, see [20] for details. It leads to several open inverse problems, as the determination of the flow at the input of the arteria with only one measure at the output, or the determination of physical coefficients like stiffness of arterias. Those open inverse problems are very relevant for cardiovascular medicine.

Since the pioneering works by Bukhgeĭm, Klibanov and Malinsky [13,34], there has been a number of papers devoted to the proof of uniqueness and stability for inverse problems concerning the determination of source or coefficients in PDEs by means of the localized knowledge of a single solution.

The problem of recovering sources has been first addressed for hyperbolic equations in [39, 40] and for parabolic equations in [27]. The determination of coefficients has also drawn the attention of many authors. Among the most classical results for the wave equation, we can mention the determination of a potential from boundary [45] or internal [28] measurements and the determination of the main coefficient from boundary [7, 35] or internal [29] observation. Regarding the problem of recovering coefficients in parabolic equations, we send the interested reader to the survey [44] by Yamamoto and the references therein.

Concerning dispersive equations, a stability result was obtained in [5] for the problem of recovering the potential in a Schrödinger equation. That paper motivated several further works as [14,37], where appropriate Carleman estimates were proven in different contexts. To the best of our knowledge, there are no previous works concerning coefficient inverse problems for KdV equations.

Most of the references given above are inspired by what is often called the Bukhgeĭm–Klibanov method (see the recent book [6] or the survey [33]), which was meant to prove global uniqueness results for coefficient inverse problems with data resulting from a single measurement event. Roughly speaking, it consists in two steps. The first step comprises the time differentiation of the governing PDE, getting an auxiliary system, in which the unknown coefficient appears now in the initial condition. The second step relies on the application of a Carleman estimate whose parameters and weight are crucial to drive to the conclusion. An idea of Imanuvilov and Yamamoto, presented for the first time in [27] (for parabolic equation) and then often used (for other PDEs), allows to use the appropriate global Carleman inequality to estimate the unknown coefficient in terms of internal or boundary measurements of the solution. One should point out that in this method, there is a constraint on the initial condition (or some of its derivatives) which should never vanish in the domain where the PDE is posed. The interested reader can find in [33, Section 1] some details on the reason why such an assumption is not a real problem anymore.

This article is a first step in order to address the problem of recovering the topography of the channel by means of boundary measurements of the water wave. Indeed, we will deal with a KdV equation posed on a bounded interval $[0, L]$ with an unknown third-order coefficient:

$$\begin{cases} y_t + a(x)y_{xxx} + y_x + yy_x = 0, & \forall (x,t) \in (0,L) \times (0,T), \\ y(0,t) = y(L,t) = y_x(L,t) = 0, & \forall t \in (0,T), \\ y(x,0) = y_0(x), & \forall x \in (0,L), \end{cases} \tag{1.2}$$

where the initial data y_0 is known. The unknown coefficient $a = a(x)$ is assumed to be time independent. The case where we have several unknown coefficients (as in the equation (1.1)) is very hard to deal with. There are few results in this direction and they impose many conditions on the coefficients. See for instance [15] where a Schrödinger equation is considered on an unbounded strip.

Up to our knowledge, this paper presents the first stability and uniqueness theorem for a coefficient inverse problem for the KdV equation. As mentioned before, the Bukhgeĭm–Klibanov method requires a Carleman estimate for the equation. This type of inequalities has already been proven for the KdV equation on a bounded interval in the framework of control problems. The control of this equation from the left Dirichlet boundary condition has been studied in [43] and [23], where the authors get the null-controllability of this equation by proving one-parameter Carleman estimates. Unlike those papers, here we will prove a two-parameter estimate in order to be able to deal with a space dependent main coefficient.

The inverse problem we are interested in can be stated as follows.

Inverse problem. Retrieve the principal coefficient $a = a(x)$ of equation (1.2) from the measurement of $y_x(0,t)$, $y_{xx}(0,t)$ and $y_{xx}(L,t)$ on $(0,T)$, where y is the solution of equation (1.2).

A partial and local answer for this nonlinear inverse problem will be proved in this paper. To be more specific, let us denote by y and $\tilde{y}$ the solutions of equation (1.2) corresponding to coefficients a and $\tilde{a}$ respectively (we shall also use the notation $y[a]$, $y[\tilde{a}]$).

Stability. Estimate $\|\tilde{a}-a\|_{L^2(0,L)}$ by suitable norms of the three measurements $\|\tilde{y}_x(0,t)-y_x(0,t)\|$, $\|\tilde{y}_{xx}(0,t)-y_{xx}(0,t)\|$ and $\|\tilde{y}_{xx}(L,t)-y_{xx}(L,t)\|$.

The next theorem is the main result of this article and gives the stability result for the inverse problem under consideration.

Theorem 1.1. *Let r_0, a_0, α and K be given positive constants. Assume that the initial data $y_0 \in \{w \in H^7(0,L) : w(0)=w(L)=w'(L)=0\}$ satisfies for every $x \in [0,L]$*

$$y_0'''(x) = y_0'''(L-x) \tag{1.3}$$

and

$$|y_0'''(x)| \geq r_0 > 0. \tag{1.4}$$

Define the set

$$\Sigma(a_0,\alpha) = \{a \in W^{6,\infty}(0,L) : \forall x \in [0,L],\ a(x) \geq a_0 > 0, a(x) = a(L-x), \\ \text{and } \|a\|_{W^{6,\infty}(0,L)} \leq \alpha\}. \tag{1.5}$$

Then, there exists a constant $C = C(L,T,r_0,a_0,\alpha,K) > 0$ such that

$$\begin{aligned} C\|a-\tilde{a}\|_{L^2(0,L)} \leq \|y_x(0,t)-\tilde{y}_x(0,t)\|_{H^1(0,T)} \\ + \|y_{xx}(0,t)-\tilde{y}_{xx}(0,t)\|_{H^1(0,T)} \\ + \|y_{xx}(L,t)-\tilde{y}_{xx}(L,t)\|_{H^1(0,T)} \end{aligned} \tag{1.6}$$

for every $a,\tilde{a}$ in $\Sigma(a_0,\alpha)$ such that the corresponding solutions of equation (1.2) *satisfy*

$$\max\{\|y\|_{W^{1,\infty}(0,T;W^{3,\infty}(0,L))}, \|\tilde{y}\|_{W^{1,\infty}(0,T;W^{3,\infty}(0,L))}\} \leq K.$$

Remark 1.2. In the Appendix, we will sketch the proof of the fact that the required hypotheses on $a \in W^{6,\infty}(0,L)$ and

$$y_0 \in \{w \in H^7(0,L) : w(0)=w(L)=w'(L)=0\}$$

are enough to guarantee that the solution of (1.2) lies in $W^{1,\infty}(0,T;W^{3,\infty}(0,L))$ (see Proposition A.6). Thus, the set of trajectories where Theorem 1.1 is valid is not empty. We mention that hypothesis $y_0 \in H^7(0,L)$ is not sharp, see Remark A.7 in the Appendix.

Remark 1.3. The symmetry hypothesis on the initial data and the main coefficient are technical points coming from our proof. More precisely, they allow us to extend the solution of KdV equation to negative times, in order to apply the Bukhgeĭm–Klibanov method. We can emphasize that this extension will be crucial to avoid an observation of the solution in a given time $T_0 > 0$, as in the case of parabolic equations, see e.g. [1] or [44]. Therefore, it is an important new point to be able to extend the solution of KdV equation to negative times.

Remark 1.4. Concerning the question of the minimal number of boundary measurement required in Theorem 1.1, one could expect to get estimate (1.6) with either one measurement at $x = L$ or two measurements at $x = 0$, but not with just one measurement at $x = 0$. This is based on the fact that Carleman estimates imply some observability inequalities, which do not hold for the linear KdV equation with only one observation at $x = 0$. This lack of observability holds for some critical lengths L, as one can read in [16, 17, 19, 42]. On the other hand, we need in this paper three measurements because of the symmetry assumptions and the extension to negative times used.

Other related papers on this type of inverse problem for PDEs can be listed: [1, 26] (one-dimensional fourth order parabolic equation), [3, 4, 10] (discontinuous coefficients), [2] (network of one-dimensional waves), [8, 9] (logarithmic stability estimates), [21] (parabolic system), [41] (unknown coefficient in the nonlinearity), [15, 21, 46] (two unknown coefficients). We also mention some important books which can be the starting point to study inverse problems [30], Carleman estimates [22], and control theory for partial differential equations [18].

Outline. In Section 2, a two-parameters global Carleman estimate for the linearized KdV equation with non-constant main coefficient is obtained. It is then used in Section 3, following the Bukhgeĭm–Klibanov method, to prove the desired stability result. As mentioned before, the Appendix A is concerned with the Cauchy problem.

2 Carleman estimate

In this section, a global Carleman inequality will be proven for the linearized KdV equation on the domain denoted here by $Q := (0, L) \times (-T, T)$. As mentioned in the introduction, there are in the literature some Carleman estimates for the KdV equation with constant main coefficient posed on a bounded interval [23, 43]. In this work we deal with non-constant coefficients $a = a(x)$. We prove a two-parameter (s, λ) Carleman estimate in order to avoid extra hypothesis on the coefficients when solving the inverse problem.

Let us consider $d \in L^\infty((-T,T)\times(0,L))$, $b \in L^\infty(-T,T;W^{1,\infty}(0,L))$ and $a \in W^{3,\infty}(0,L)$, with

$$a(x) \geq a_0, \quad \forall x \in [0,L], \quad \text{and} \quad \|a\|_{W^{3,\infty}(0,L)} \leq \alpha,$$

where $a_0, \alpha > 0$. We define the operator

$$P = \partial_t + a(x)\partial_{xxx} + b(x,t)\partial_x + d(x,t) \tag{2.1}$$

and the space

$$\mathcal{V} = \{v \in L^2(-T,T;H^3 \cap H_0^1(0,L)) : Pv \in L^2((0,L)\times(-T,T))\}. \tag{2.2}$$

Consider $\beta \in C^3([0,L])$ such that there exist $r > 0$ and $\kappa \in (1,2)$ satisfying

$$0 < r \leq \beta(x), \quad 0 < r \leq \beta'(x), \quad \forall x \in (0,L), \tag{2.3}$$

and

$$\kappa \max_{[0,L]} \beta < 2 \min_{[0,L]} \beta. \tag{2.4}$$

Remark 2.1. We can consider $\beta(x) = x + 2L$. However, we make all the computations with a generic function β, in order to reveal its potential role in the application of the Carleman estimate in another context.

Given $\lambda > 0$, we define the functions ϕ, θ in $[0,L]\times(-T,T)$ by

$$\begin{aligned} \phi(x,t) &= \frac{e^{\kappa\lambda\|\beta\|_\infty} - e^{\lambda\beta(x)}}{(T+t)(T-t)}, \\ \theta(x,t) &= \frac{e^{\lambda\beta(x)}}{(T+t)(T-t)}. \end{aligned} \tag{2.5}$$

Let us notice that there exists a constant $C = C(T)$, independent of λ, such that

$$\begin{aligned} 0 < \frac{1}{C} &\leq \theta(x,t), \\ |\theta_t(x,t)| &\leq C\theta^2(x,t), \\ |\phi_t(x,t)| &\leq C\theta^2(x,t) \end{aligned} \tag{2.6}$$

for any $(x,t) \in [0,L]\times(-T,T)$. Indeed, the first two inequalities in (2.6) follow directly from the definitions of ϕ and θ. The third one comes from condition (2.4).

We will prove the following Carleman estimate.

Theorem 2.2. *Let ϕ and θ be defined by* (2.5)*, and let P be the linearized KdV operator defined in* (2.1)*. There exist $s_0 > 0$, $\lambda_0 > 0$ and a constant $C > 0$ depending on $L, T, s_0, \lambda_0, \alpha, a_0, r$ and $\|\beta\|_{C^3([0,L])}$ such that for every $s \geq s_0$ and $\lambda \geq \lambda_0$, one has*

$$\begin{aligned}
&\int_{-T}^{T}\int_0^L e^{-2s\phi}(s^5\lambda^6\theta^5|u|^2 + s^3\lambda^4\theta^3|u_x|^2 + s\lambda^2\theta|u_{xx}|^2)dxdt \\
&\quad \leq C\int_{-T}^{T}\int_0^L e^{-2s\phi}|Pu|^2dxdt \\
&\qquad + Cs^3\lambda^3\int_{-T}^{T} e^{-2s\phi(L,t)}\theta^3(L,t)|u_x(L,t)|^2dt \\
&\qquad + Cs\lambda\int_{-T}^{T} e^{-2s\phi(L,t)}\theta(L,t)|u_{xx}(L,t)|^2dt
\end{aligned} \tag{2.7}$$

for all $u \in \mathcal{V}$ defined by (2.2)*.*

Remark 2.3. When looking for a one-parameter Carleman estimate (as in [23, 43] where $\phi = \frac{\beta(x)}{(t+T)(T-t)}$), some additional hypotheses on the second derivative of the function β have to be imposed. The use of the second parameter λ allows to omit them.

Proof. Let $s > 0$ and define $\mathcal{W}_s = \{e^{-s\phi}u : u \in \mathcal{V}\}$. For $u \in \mathcal{V}$, we set $w = e^{-s\phi}u$ and we define the operator P_ϕ from $\mathcal{W}_s$ to $L^2((0,L)\times(-T,T))$ by

$$P_\phi w = e^{-s\phi}P(e^{s\phi}w).$$

After straightforward computations, we write

$$P_\phi w = P_1 w + P_2 w + Rw,$$

where, for a constant $m > 0$ to be chosen later,

$$P_1 w = w_t + 3as^2\phi_x^2 w_x + aw_{xxx} + 3ams^2\phi_x\phi_{xx}w, \tag{2.8}$$

$$P_2 w = as^3\phi_x^3 w + 3as\phi_x w_{xx} + 3sw_x(a\phi_x)_x, \tag{2.9}$$

and

$$\begin{aligned}
Rw = {}& bs\phi_x w + bw_x + as\phi_{xxx}w + 3as^2\phi_x\phi_{xx}w + dw \\
&+ s\phi_t w - 3sa_x\phi_x w_x - 3ams^2\phi_x\phi_{xx}w.
\end{aligned} \tag{2.10}$$

Therefore,

$$\|P_\phi w - Rw\|^2_{L^2(Q)} = \|P_1 w\|^2_{L^2(Q)} + 2\langle P_1 w, P_2 w\rangle + \|P_2 w\|^2_{L^2(Q)}, \tag{2.11}$$

where $\langle\cdot,\cdot\rangle$ is the $L^2(Q)$ scalar product.

Step 1: Explicit calculations. Notice that we have $w(0,t) = w(L,t) = 0$ for all $t \in (-T,T)$ and $w(x,\pm T) = 0$ for all $x \in (0,L)$. This is due to the fact that $w = e^{-s\phi}u$, with $u \in L^2(-T,T;H_0^1(0,L))$ and $\lim_{t\to\pm T}\phi(x,t) = +\infty$.

By developing $\langle P_1 w, P_2 w\rangle$, we get

$$\langle P_1 w, P_2 w\rangle = \sum_{\substack{i=1,\dots,4\\ j=1,\dots,3}} I_{i,j},$$

where $I_{i,j}$ is the $L^2(Q)$ scalar product of the i-th term in (2.8) with the j-th term in (2.9). By integrations by parts, we obtain

$$I_{1,1} = -\frac{3s^3}{2}\iint_Q a\phi_x^2\phi_{xt}|w|^2dxdt = \frac{3}{2}s^3\lambda^3\iint_Q a\beta_x^3\theta^2\theta_t|w|^2dxdt,$$

$$\begin{aligned} I_{1,2} &= -3s\iint_Q (a\phi_x)_x w_x w_t dxdt + \frac{3s}{2}\iint_Q a\phi_{xt}|w_x|^2dxdt \\ &= -3s\iint_Q (a\phi_x)_x w_x w_t dxdt - \frac{3}{2}s\lambda\iint_Q a\beta_x\theta_t|w_x|^2dxdt, \end{aligned}$$

$$I_{1,3} = 3s\iint_Q (a\phi_x)_x w_x w_t dxdt,$$

$$\begin{aligned} I_{2,1} &= -\frac{3s^5}{2}\iint_Q (a^2\phi_x^5)_x|w|^2dxdt \\ &= \frac{3}{2}s^5\lambda^5\iint_Q (a^2\beta_x^5)_x\theta^5|w|^2dxdt + \frac{15}{2}s^5\lambda^6\iint_Q a^2\beta_x^6\theta^5|w|^2dxdt, \end{aligned}$$

$$\begin{aligned} I_{2,2} &= -\frac{9s^3}{2}\iint_Q (a^2\phi_x^3)_x|w_x|^2dxdt + \frac{9s^3}{2}\int_{-T}^{T} a^2\phi_x^3|w_x|^2\Big|_{x=0}^{x=L}dt \\ &= \frac{9}{2}s^3\lambda^3\iint_Q (a^2\beta_x^3)_x\theta^3|w_x|^2dxdt + \frac{27}{2}s^3\lambda^4\iint_Q a^2\beta_x^4\theta^3|w_x|^2dxdt \\ &\qquad - \frac{9}{2}s^3\lambda^3\int_{-T}^{T} a^2\beta_x^3\theta^3|w_x|^2\Big|_{x=0}^{x=L}dt, \end{aligned}$$

$$\begin{aligned} I_{2,3} &= 9s^3\iint_Q a\phi_x^2(a\phi_x)_x|w_x|^2dxdt \\ &= -9s^3\lambda^3\iint_Q a\beta_x^2(a\beta_x)_x\theta^3|w_x|^2dxdt - 9s^3\lambda^4\iint_Q a^2\beta_x^4\theta^3|w_x|^2dxdt, \end{aligned}$$

$$
\begin{aligned}
I_{3,1} &= \frac{3s^3}{2}\iint_Q (a^2\phi_x^3)_x|w_x|^2dxdt - \frac{s^3}{2}\iint_Q (a^2\phi_x^3)_{xxx}|w|^2dxdt \\
&\quad - \frac{s^3}{2}\int_{-T}^{T} a^2\phi_x^3|w_x|^2\Big|_{x=0}^{x=L}dt \\
&= -\frac{3}{2}s^3\lambda^3\iint_Q (a^2\beta_x^3)_x\theta^3|w_x|^2dxdt \\
&\quad - \frac{9}{2}s^3\lambda^4\iint_Q a^2\beta_x^4\theta^3|w_x|^2dxdt \\
&\quad + \frac{s^3\lambda^3}{2}\iint_Q (a^2\beta_x^3\theta^3)_{xxx}|w|^2dxdt \\
&\quad + \frac{s^3\lambda^3}{2}\int_{-T}^{T} a^2\beta_x^3\theta^3|w_x|^2\Big|_{x=0}^{x=L}dt, \\
I_{3,2} &= -\frac{3s}{2}\iint_Q (a^2\phi_x)_x|w_{xx}|^2dxdt + \frac{3s}{2}\int_{-T}^{T} a^2\phi_x|w_{xx}|^2\Big|_{x=0}^{x=L}dt \\
&= \frac{3}{2}s\lambda\iint_Q (a^2\beta_x)_x\theta|w_{xx}|^2dxdt + \frac{3}{2}s\lambda^2\iint_Q a^2\beta_x^2\theta|w_{xx}|^2dxdt \\
&\quad - \frac{3}{2}s\lambda\int_{-T}^{T} a^2\beta_x\theta|w_{xx}|^2\Big|_{x=0}^{x=L}dt, \\
I_{3,3} &= -3s\iint_Q a(a\phi_x)_x|w_{xx}|^2dxdt + \frac{3s}{2}\iint_Q [a(a\phi_x)_x]_{xx}|w_x|^2dxdt \\
&\quad - \frac{3s}{2}\int_{-T}^{T} (a(a\phi_x)_x)_x|w_x|^2\Big|_{x=0}^{x=L}dt \\
&\quad + 3s\int_{-T}^{T} a(a\phi_x)_x w_x w_{xx}\Big|_{x=0}^{x=L}dt \\
&= 3s\lambda\iint_Q a(a\beta_x)_x\theta|w_{xx}|^2dxdt + 3s\lambda^2\iint_Q a^2\beta_x^2\theta|w_{xx}|^2dxdt \\
&\quad + \frac{3}{2}s\lambda\int_{-T}^{T} (a(a\beta_x\theta)_x)_x|w_x|^2\Big|_{x=0}^{x=L}dt \\
&\quad - \frac{3}{2}s\lambda\iint_Q (a(a\beta_x\theta)_x)_{xx}|w_x|^2dxdt \\
&\quad - 3s\lambda\int_{-T}^{T} a(a\beta_x\theta)_x w_x w_{xx}\Big|_{x=0}^{x=L}dt,
\end{aligned}
$$

$$\begin{aligned}
I_{4,1} &= 3ms^5 \iint_Q a^2\phi_x^4\phi_{xx}|w|^2 dxdt \\
&= -3ms^5\lambda^5 \iint_Q a^2\beta_x^4\beta_{xx}\theta^5|w|^2 dxdt \\
&\qquad - 3ms^5\lambda^6 \iint_Q a^2\beta_x^6\theta^5|w|^2 dxdt, \\
I_{4,2} &= -9ms^3 \iint_Q a^2\phi_x^2\phi_{xx}|w_x|^2 dxdt + \frac{9ms^3}{2}\iint_Q (a^2\phi_x^2\phi_{xx})_{xx}|w|^2 dxdt \\
&= 9ms^3\lambda^3 \iint_Q a^2\beta_x^2\beta_{xx}\theta^3|w_x|^2 dxdt \\
&\qquad + 9ms^3\lambda^4 \iint_Q a^2\beta_x^4\theta^3|w_x|^2 dxdt \\
&\qquad - \frac{9m}{2}s^3\lambda^3 \iint_Q (a^2\beta_x^2\beta_{xx}\theta^3)_{xx}|w|^2 dxdt \\
&\qquad - \frac{9m}{2}s^3\lambda^4 \iint_Q (a^2\beta_x^4\theta^3)_{xx}|w|^2 dxdt, \\
I_{4,3} &= -\frac{9ms^3}{2}\iint_Q (a\phi_x\phi_{xx}(a\phi_x)_x)_x|w|^2 dxdt \\
&= \frac{9m}{2}s^3\lambda^3 \iint_Q (a\beta_x\beta_{xx}\theta^2(a\beta_x\theta)_x)_x|w|^2 dxdt \\
&\qquad + \frac{9m}{2}s^3\lambda^4 \iint_Q (a\beta_x^3\theta^2(a\beta_x\theta)_x)_x|w|^2 dxdt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle P_1 w, P_2 w\rangle = \left(\frac{15}{2} - 3m\right) s^5\lambda^6 &\iint_Q a^2\beta_x^6\theta^5|w|^2 dxdt \\
&+ 9ms^3\lambda^4 \iint_Q a^2\beta_x^4\theta^3|w_x|^2 dxdt \\
&+ \frac{9}{2}s\lambda^2 \iint_Q a^2\beta_x^2\theta|w_{xx}|^2 dxdt + B + X_1,
\end{aligned} \tag{2.12}$$

where X_1 and B satisfy the following. The term X_1 gathers the non-dominating terms and satisfies the following: there exists a positive constant $C = C(L, T, \alpha, r, \|\beta\|_{C^3([0,L])})$ independent of s and λ, (which may change from line to line) such that, using estimates (2.6), we get

$$
\begin{aligned}
|X_1| \le C(s^3\lambda^6 + s^5\lambda^5) & \iint_Q \theta^5 |w|^2 dx dt \\
& + C(s\lambda^4 + s^3\lambda^3) \iint_Q \theta^3 |w_x|^2 dx dt \\
& + C s\lambda \iint_Q \theta |w_{xx}|^2 dx dt.
\end{aligned}
\tag{2.13}
$$

The term B gathers the boundary terms. We split them as follows:

$$
B := B^+ + B^- + B^* \tag{2.14}
$$

with the notations

$$
\begin{aligned}
B^+ &= 4s^3\lambda^3 \int_{-T}^{T} \left(a^2\beta_x^3\theta^3\right)\Big|_{x=0} |w_x(0,t)|^2 dt \\
&\quad + \frac{3}{2}s\lambda \int_{-T}^{T} \left(a^2\beta_x\theta\right)\Big|_{x=0} |w_{xx}(0,t)|^2 dt, \\
B^- &= -4s^3\lambda^3 \int_{-T}^{T} \left(a^2\beta_x^3\theta^3\right)\Big|_{x=L} |w_x(L,t)|^2 dt \\
&\quad - \frac{3}{2}s\lambda \int_{-T}^{T} \left(a^2\beta_x\theta\right)\Big|_{x=L} |w_{xx}(L,t)|^2 dt, \\
B^* &= \frac{3}{2}s\lambda \int_{-T}^{T} (a(a\beta_x\theta)_x)_x\Big|_{x=L} |w_x(L,t)|^2 dt \\
&\quad - \frac{3}{2}s\lambda \int_{-T}^{T} (a(a\beta_x\theta)_x)_x\Big|_{x=0} |w_x(0,t)|^2 dt \\
&\quad - 3s\lambda \int_{-T}^{T} a(a\beta_x\theta)_x\Big|_{x=L} w_x(L,t) w_{xx}(L,t) dt \\
&\quad + 3s\lambda \int_{-T}^{T} a(a\beta_x\theta)_x\Big|_{x=0} w_x(0,t) w_{xx}(0,t) dt.
\end{aligned}
$$

Step 2: Dominating terms. By choosing, and fixing, m such that

$$
0 < m < \frac{5}{2}, \tag{2.15}
$$

we obtain that there exists a constant $C = C(L, T, a_0, r) > 0$ such that

$$
\left(\frac{15}{2} - 3m\right) s^5\lambda^6 \iint_Q a^2\beta_x^6\theta^5 |w|^2 dx dt \ge C s^5\lambda^6 \iint_Q \theta^5 |w|^2 dx dt,
$$

$$9ms^3\lambda^4 \iint_Q a^2\beta_x^4\theta^3|w_x|^2dxdt \geq Cs^3\lambda^4 \iint_Q \theta^3|w_x|^2dxdt,$$
$$\frac{9}{2}s\lambda^2 \iint_Q a^2\beta_x^2\theta|w_{xx}|^2dxdt \geq Cs\lambda^2 \iint_Q \theta|w_{xx}|^2dxdt.$$

We introduce the notation given by

$$\|w\|_{s,\lambda,\theta}^2 = s^5\lambda^6 \iint_Q \theta^5|w|^2dxdt + s^3\lambda^4 \iint_Q \theta^3|w_x|^2dxdt + s\lambda^2 \iint_Q \theta|w_{xx}|^2dxdt.$$

From (2.13) we obtain that

$$|X_1| \leq C\left(\frac{1}{s^2}+\frac{1}{\lambda}\right)\|w\|_{s,\lambda,\theta}^2.$$

Therefore, choosing s and λ large enough, from (2.12) we get some $C>0$ depending only on $L, T, \alpha, a_0, r, \|\beta\|_{C^3([0,L])}, s_0$ and λ_0 and such that, for all $s>s_0$ and $\lambda>\lambda_0$,

$$C\|w\|_{s,\lambda,\theta}^2 \leq \langle P_1w, P_2w\rangle - B. \tag{2.16}$$

Step 3: Boundary terms. From (2.14) and (2.16), taking into account that B^+ and $-B^-$ are non-negative, we write

$$C\|w\|_{s,\lambda,\theta}^2 + B^+ \leq \langle P_1w, P_2w\rangle - B^- - B^*. \tag{2.17}$$

Now, we shall deal with the boundary terms in B^*, that can alternate in sign. We will bound $|B^*|$ from above. For the four integrals one by one, using (2.6) and Cauchy–Schwarz inequality, this gives

$$\left|\frac{3}{2}s\lambda \int_{-T}^T (a(a\beta_x\theta)_x)_x\Big|_{x=L}|w_x(L)|^2dt\right| \leq Cs\lambda^3 \int_{-T}^T a^2(L)\beta_x^3(L)\theta(L)|w_x(L)|^2dt,$$
$$\left|\frac{3}{2}s\lambda \int_{-T}^T (a(a\beta_x\theta)_x)_x\Big|_{x=0}|w_x(0)|^2dt\right| \leq Cs\lambda^3 \int_{-T}^T a^2(0)\beta_x^3(0)\theta(0)|w_x(0)|^2dt,$$

$$\begin{aligned}
&\left|3s\lambda \int_{-T}^{T} a(a\beta_x\theta)_x\Big|_{x=L} w_x(L)w_{xx}(L)dt\right| \\
&\qquad \le Cs^2\lambda^3 \int_{-T}^{T} a^2(L)\beta_x^3(L)\theta(L)|w_x(L)|^2 dt \\
&\qquad\qquad + C\lambda \int_{-T}^{T} a^2(L)\beta_x(L)\theta(L)|w_{xx}(L)|^2 dt, \\
&\left|3s\lambda \int_{-T}^{T} a(a\beta_x\theta)_x\Big|_{x=0} w_x(0)w_{xx}(0)dt\right| \\
&\qquad \le Cs^2\lambda^3 \int_{-T}^{T} a^2(0)\beta_x^3(0)\theta(0)|w_x(0)|^2 dt \\
&\qquad\qquad + C\lambda \int_{-T}^{T} a^2(0)\beta_x(0)\theta(0)|w_{xx}(0)|^2 dt.
\end{aligned}$$

From these last four inequalities we get that

$$|B^*| \le \frac{C}{s}B^+ + \frac{C}{s}|B^-|.$$

Therefore, taking s_0 large enough, inequality (2.17) becomes, for all $s > s_0$,

$$\|w\|_{s,\lambda,\theta}^2 + B^+ \le C\langle P_1 w, P_2 w\rangle + C|B^-|. \tag{2.18}$$

Step 4: Carleman estimate for w. Since $B^+ \ge 0$, inequality (2.18) yields

$$\begin{aligned}
\|w\|_{s,\lambda,\theta}^2 \le C\langle P_1 w, P_2 w\rangle &+ Cs^3\lambda^3 \int_{-T}^{T} a^2(L)\beta_x^3(L)\theta^3(L,t)|w_x(L,t)|^2 dt \\
&+ Cs\lambda \int_{-T}^{T} a^2(L)\beta_x(L)\theta(L,t)|w_{xx}(L,t)|^2 dt.
\end{aligned}\tag{2.19}$$

Moreover, from (2.6) and (2.10) we have

$$\begin{aligned}
\|P_\phi w - Rw\|_{L^2(Q)}^2 &\le 2\|P_\phi w\|_{L^2(Q)}^2 + 2\|Rw\|_{L^2(Q)}^2 \\
&\le 2\|P_\phi w\|_{L^2(Q)}^2 + Cs^4\lambda^6 \iint_Q \theta^4|w|^2 dxdt \\
&\qquad + Cs^2\lambda^2 \iint_Q \theta^2|w_x|^2 dxdt.
\end{aligned}$$

Therefore, choosing again s_0 and λ_0 large enough, we have proved the following.

Proposition 2.4. *There exist $s_0, \lambda_0 > 0$ and a constant $C > 0$ depending on L, T, s_0, λ_0, α, a_0, r and $\|\beta\|_{C^3([0,L])}$ such that for all $s \geq s_0$, for all $\lambda \geq \lambda_0$,*

$$
\begin{aligned}
s^5\lambda^6 \int_{-T}^{T}\int_0^L \theta^5|w|^2dxdt + s^3\lambda^4 \int_{-T}^{T}\int_0^L \theta^3|w_x|^2dxdt \\
+ s\lambda^2 \int_{-T}^{T}\int_0^L \theta|w_{xx}|^2dxdt + \int_{-T}^{T}\int_0^L |P_1w|^2dxdt \\
+ \int_{-T}^{T}\int_0^L |P_2w|^2dxdt \qquad (2.20)\\
\leq C \int_{-T}^{T}\int_0^L |P_\phi w|^2dxdt + Cs^3\lambda^3 \int_{-T}^{T} \theta^3(L,t)|w_x(L,t)|^2dt \\
+ Cs\lambda \int_{-T}^{T} \theta(L,t)|w_{xx}(L,t)|^2dt
\end{aligned}
$$

for all $w \in \mathcal{W}_s := \{e^{-s\phi}v \,:\, v \in \mathcal{V}\}$.

Step 5: Back to the variable u. The Carleman estimate stated in Theorem 2.2 will now be deduced from Proposition 2.4. It is only a matter of going back to the variable $u \in \mathcal{V}$. Recall that

$$u = e^{s\phi}w \quad \text{and} \quad u(0,t) = u(L,t) = 0, \quad \forall t \in (-T,T).$$

One easily checks that there exists a constant $C = C(L,T,s_0,\lambda_0,\alpha,a_0,r) > 0$ such that for all $x \in (0,L)$ and $t \in (-T,T)$,

$$
\begin{aligned}
e^{-2s\phi}|u_x|^2 &\leq 2|w_x|^2 + Cs^2\lambda^2\theta^2|w|^2,\\
e^{-2s\phi}|u_{xx}|^2 &= 2|w_{xx}|^2 + Cs^2\lambda^2\theta^2|w_x|^2 + Cs^4\lambda^4\theta^2|w|^2.
\end{aligned}
$$

Hence the left hand side of (2.7) is estimated by the left hand side of (2.20).

Moreover, we have

$$P_\phi w = e^{-s\phi}Pu,$$

and concerning the boundary terms, using $u(0,t) = u(L,t) = 0$ for all $t \in (-T,T)$ we obtain that in $\{0,L\} \times (-T,T)$,

$$
\begin{aligned}
|w_x|^2 &\leq Ce^{-2s\phi}|u_x|^2,\\
|w_{xx}|^2 &\leq Ce^{-2s\phi}(|u_{xx}|^2 + s^2\lambda^2\theta^2|u_x|^2).
\end{aligned}
$$

Therefore, for s_0 and λ_0 large enough, Theorem 2.2 directly follows from Proposition 2.4. □

3 Inverse problem

This section is devoted to the proof of the Lipschitz stability result stated in Theorem 1.1 about the inverse problem of retrieving the main coefficient a in equation (1.2) from boundary measurements of the solution. We recall here that we will use the Bukhgeĭm–Klibanov method. For the sake of clarity, we divide the proof in several steps.

Step 1: Local solution of the inverse problem. We consider two coefficients $a, \tilde{a}$ and the corresponding solutions of (1.2), y and $\tilde{y}$, and we define

$$u(x,t) := y(x,t) - \tilde{y}(x,t),$$
$$\sigma(x) := \tilde{a}(x) - a(x).$$

The function u satisfies

$$\begin{cases} u_t + a(x)u_{xxx} + (1+\tilde{y})u_x + y_x u = \sigma \tilde{y}_{xxx}, & \forall (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0, & \forall t \in (0,T), \\ u(x,0) = 0, & \forall x \in (0,L). \end{cases} \tag{3.1}$$

We consider now $z(x,t) := u_t(x,t)$, which satisfies the system

$$\begin{cases} z_t + a(x)z_{xxx} + (1+\tilde{y})z_x + y_x z = f, & \forall (x,t) \in (0,L) \times (0,T), \\ z(0,t) = 0, \quad z(L,t) = 0, \quad z_x(L,t) = 0, & \forall t \in (0,T), \\ z(x,0) = \sigma(x) y_0'''(x), & \forall x \in (0,L), \end{cases} \tag{3.2}$$

where

$$f = \sigma(x)\tilde{y}_{xxxt} - y_{xt}u - \tilde{y}_t u_x.$$

One should notice that we have now σ, that we seek to estimate, appearing not only in the source term of the equation, but also in the initial condition.

Step 2: Extension to negative time. In order to estimate σ through $z(x,0)$ using the Carleman estimate of the previous section, we shall extend the partial differential equations (3.1) in u and (3.2) in z to negative times. In order to extend these linearized KdV equations to the interval $(-T,T)$, we define the *symmetric extension* of any function g defined on $[0,L] \times [0,T]$ by

$$\hat{g}(x,t) = \begin{cases} g(x,t) & \text{if } x \in [0,L],\ t \in [0,T], \\ g(L-x,-t) & \text{if } x \in [0,L],\ t \in [-T,0). \end{cases} \tag{3.3}$$

One should notice that this extension satisfies $\hat{f}\hat{g} = \widehat{fg}$. We also define the following *anti-symmetric extension* to $[0, L] \times [-T, T]$ of any function g defined on $[0, L] \times [0, T]$ by

$$\check{g}(x,t) = \begin{cases} g(x,t) & \text{if } x \in [0,L],\ t \in [0,T], \\ -g(L-x,-t) & \text{if } x \in [0,L],\ t \in [-T,0). \end{cases} \tag{3.4}$$

Therefore, we first assume that the initial data $z_0(x) = z(x,0)$ of equation (3.2) satisfies

$$z_0(x) = z_0(L-x), \quad \forall x \in [0,L], \tag{3.5}$$

meaning that we have to assume this for a, $\tilde{a}$ and y_0''', as we did in assumptions (1.3) and (1.5). We define the extension of y (resp. $\tilde{y}$) solution of equation (1.2) by $\hat{y}$ (resp. $\hat{\tilde{y}}$). Now we can define $v := \hat{z}$ on $[0, L] \times [-T, T]$, which satisfies the equation given by

$$\begin{cases} v_t + a(x)v_{xxx} + (1+\hat{\tilde{y}})v_x + \hat{y}_x v = \check{f}, & \forall x \in (0,L),\ t \in (-T,T), \\ v(0,t) = 0, \quad v(L,t) = 0, & \forall t \in (-T,T), \\ v_x(L,t) = 0, & \forall t \in (0,T), \\ v_x(L,t) = -z_x(0,-t), & \forall t \in (-T,0), \\ v(x,0) = \sigma(x)y_0'''(x), & \forall x \in (0,L). \end{cases} \tag{3.6}$$

The partial differential operator of this equation is eligible for the Carleman estimate of Theorem 2.2, according to (2.1) with $b = 1 + \hat{\tilde{y}}$ and $d = \hat{y}_x$, provided that $b \in L^\infty(-T,T;W^{1,\infty}(0,L))$ and $d \in L^\infty((-T,T)\times(0,L))$. This regularity is fulfilled thanks to the hypothesis on the regularity of y_0, a and $\tilde{a}$, which ensures the corresponding solutions y and $\tilde{y}$ to be regular enough. See Proposition A.6 in the Appendix.

Step 3: First use of the Carleman estimate. Following the definitions stated in the proof of Theorem 2.4, we set $w = e^{-s\phi}v$ and since $w(0,t) = w(L,t) = 0$ and $w(x,\pm T) = 0$ (because of the weight function ϕ), we can either compute or estimate the following integral:

$$\begin{aligned} \mathcal{I} &:= \int_{-T}^{0}\int_0^L w\, P_1 w\, dxdt \\ &= \int_{-T}^{0}\int_0^L w(w_t + 3as^2\phi_x^2 w_x + aw_{xxx} + 3as^2\phi_x\phi_{xx}w)dxdt, \end{aligned} \tag{3.7}$$

where P_1 has been defined in (2.8) and $m = 1$ chosen according to (2.15).

First of all, making integrations by parts, it is not difficult to see that

$$\mathcal{I} = \frac{1}{2}\int_0^L |w(x,0)|^2 dx + \mathcal{R},$$

where

$$\begin{aligned}
\mathcal{R} &= \int_{-T}^0 \int_0^L (3as^2\phi_x^2 w_x w + a w_{xxx} w + 3as^2\phi_x\phi_{xx}|w|^2)dxdt \\
&= -\frac{3}{2}s^2\lambda^2 \int_{-T}^0 \int_0^L (a\beta_x^2\theta^2)_x |w|^2 dxdt - \int_{-T}^0 \int_0^L a w_{xx} w_x dxdt \\
&\quad + \int_{-T}^0 \int_0^L a_x |w_x|^2 dxdt - \frac{1}{2}\int_{-T}^0 \int_0^L a_{xxx}|w|^2 dxdt \\
&\quad + 3s^2\lambda^2 \int_{-T}^0 \int_0^L a\beta_x(\beta_{xx} + \lambda\beta_x^2)\theta^2 |w|^2 dxdt.
\end{aligned}$$

Using $a \in \Sigma(a_0, \alpha)$, $\beta \in C^3([0,L])$, the property (2.6) of θ and the Cauchy–Schwartz inequality, one obtains

$$|\mathcal{R}| \le C \int_{-T}^T \int_0^L (s^2\lambda^3\theta^3|w|^2 + |w_x|^2 + |w_{xx}|^2)dxdt$$

for the same generic constant $C > 0$ as in the previous section.

Hence, estimating the quantity $s\mathcal{I}$ by Carleman inequality (2.20), and choosing s_0 large enough to absorb the terms of $\mathcal{R}$ by the dominant ones of the left hand side of (2.20), we prove

$$\begin{aligned}
s\int_0^L |w(x,0)|^2 dx &\le 2s^2 \int_{-T}^T \int_0^L |w|^2 dxdt + 2\int_{-T}^T \int_0^L |P_1 w|^2 dxdt + 2s|\mathcal{R}| \\
&\le C \int_{-T}^T \int_0^L e^{-2s\phi} |\check{f}(x,t)|^2 dxdt \\
&\quad + Cs^3\lambda^3 \int_{-T}^T \theta^3(L,t)|w_x(L,t)|^2 dt \\
&\quad + Cs\lambda \int_{-T}^T \theta(L,t)|w_{xx}(L,t)|^2 dt.
\end{aligned}$$

On the one hand, since we assume that $|y_0'''(x)| \ge r_0 > 0$, we have

$$\begin{aligned}
s\int_0^L |w(x,0)|^2 dx &= s\int_0^L e^{-2s\phi(0)}|\sigma(x) y_0'''(x)|^2 dx \\
&\ge s r_0^2 \int_0^L e^{-2s\phi(0)} |\sigma(x)|^2 dx.
\end{aligned}$$

Therefore, by these two last estimates, fixing $\lambda > \lambda_0$, we get

$$s\int_0^L e^{-2s\phi(x,0)}|\sigma(x)|^2dx \le C\int_{-T}^{T}\int_0^L e^{-2s\phi}|\check{f}(x,t)|^2dxdt + Cs^3\int_{-T}^{T}\theta^3(L,t)|w_x(L,t)|^2dt + Cs\int_{-T}^{T}\theta(L,t)|w_{xx}(L,t)|^2dt$$

and using the definition of $\check{f}$ from (3.4), and the fact that θ and ϕ are even in time, $w = e^{-s\phi}v = e^{-s\phi}\hat{z}$ with definition (3.3), and the boundary properties of z in (3.2), we obtain

$$\begin{aligned} s\int_0^L & e^{-2s\phi(x,0)}|\sigma(x)|^2dx \\ &\le C\int_0^T\int_0^L \big(e^{-2s\phi(x,t)} + e^{-2s\phi(L-x,t)}\big)|f(x,t)|^2dxdt \\ &\quad + Cs^3\int_0^T e^{-2s\phi(L,t)}\theta^3(L,t)|z_x(0,t)|^2dt \\ &\quad + Cs\int_0^T e^{-2s\phi(L,t)}\theta(L,t)|z_{xx}(0,t)|^2dt \\ &\quad + Cs\int_0^T e^{-2s\phi(L,t)}\theta(L,t)|z_{xx}(L,t)|^2dt. \end{aligned} \tag{3.8}$$

On the other hand, since $\phi(x,t) \ge \phi(x,0)$ for all $(x,t) \in (0,L)\times(-T,T)$, and since the maps $\tilde{y}_t$, y_{xt} and $\tilde{y}_{xxxt}$ belong to $L^\infty((0,L)\times(0,T))$ (recall that $y_0 \in \{w \in H^7(0,L) : w(0) = w(L) = w'(L) = 0\}$ and $a, \tilde{a} \in \Sigma(a_0,\alpha)$, which implies that $y, \tilde{y} \in W^{1,\infty}(0,T;W^{3,\infty}(0,L))$), we can write

$$\begin{aligned} \int_0^T\int_0^L & e^{-2s\phi(x,t)}|f(x,t)|^2dxdt \\ &= \int_0^T\int_0^L e^{-2s\phi(x,t)}|\sigma(x)\tilde{y}_{xxxt} - y_{xt}u - \tilde{y}_t u_x|^2dxdt \\ &\le C\int_0^L e^{-2s\phi(x,0)}|\sigma(x)|^2dx \\ &\qquad + C\int_0^T\int_0^L e^{-2s\phi(x,t)}(|u|^2 + |u_x|^2)dxdt \end{aligned}$$

and also

$$
\begin{aligned}
&\int_0^T \int_0^L e^{-2s\phi(L-x,t)} |f(x,t)|^2 dx dt \\
&\quad = \int_0^T \int_0^L e^{-2s\phi(L-x,t)} |\sigma(x)\tilde{y}_{xxxt} - y_{xt}u - \tilde{y}_t u_x|^2 dx dt \\
&\quad \le C \int_0^L e^{-2s\phi(L-x,0)} |\sigma(x)|^2 dx \\
&\qquad\qquad + C \int_0^T \int_0^L e^{-2s\phi(L-x,t)} (|u|^2 + |u_x|^2) dx dt \\
&\quad \le C \int_0^L e^{-2s\phi(x,0)} |\sigma(x)|^2 dx + C \int_{-T}^0 \int_0^L e^{-2s\phi(x,t)} (|\hat{u}|^2 + |\hat{u}_x|^2) dx dt.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&\int_{-T}^T \int_0^L e^{-2s\phi} |\check{f}(x,t)|^2 dx dt \\
&\qquad \le C \int_0^L e^{-2s\phi(x,0)} |\sigma(x)|^2 dx \\
&\qquad\qquad + C \int_{-T}^T \int_0^L e^{-2s\phi(x,t)} (|\hat{u}|^2 + |\hat{u}_x|^2) dx dt.
\end{aligned} \tag{3.9}
$$

Here, the second term on the right hand side has to be estimated, and this is done using again a Carleman inequality.

Step 4: Second use of the Carleman estimate. Applying now Carleman estimate (2.7) to the equation satisfied by $\hat{u}$ that is the extension of (3.1) to negative times, and where the first and zero-th order potentials are $b(x,t) = 1 + \hat{\tilde{y}}(x,t)$, and $d(x,t) = \hat{y}_x(x,t)$, we obtain

$$
\begin{aligned}
&\int_{-T}^T \int_0^L e^{-2s\phi} (|\hat{u}|^2 + |\hat{u}_x|^2) dx dt \\
&\qquad \le C \int_{-T}^T \int_0^L e^{-2s\phi} |\sigma \tilde{y}_{xxx}|^2 dx dt \\
&\qquad\qquad + C s^3 \int_{-T}^T e^{-2s\phi(L,t)} \theta^3(L,t) |\hat{u}_x(L,t)|^2 dt \\
&\qquad\qquad + C s \int_{-T}^T e^{-2s\phi(L,t)} \theta(L,t) |\hat{u}_{xx}(L,t)|^2 dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \int_0^L e^{-2s\phi(x,0)} |\sigma(x)|^2 dx dt \\
&\qquad + C s^3 \int_0^T e^{-2s\phi(L,t)} \theta(L,t) |u_x(0,t)|^2 dt \\
&\qquad + C s \int_0^T e^{-2s\phi(L,t)} \theta(L,t) |u_{xx}(0,t)|^2 dt \\
&\qquad + C s \int_0^T e^{-2s\phi(L,t)} \theta(L,t) |u_{xx}(L,t)|^2 dt.
\end{aligned}
$$

Thus, from (3.9), we can write

$$
\begin{aligned}
&\int_0^T \int_0^L e^{-2s\phi} |\check{f}(x,t)|^2 dx dt \\
&\quad \leq C \int_0^L e^{-2s\phi(x,0)} |\sigma(x)|^2 dx dt \\
&\qquad + C s^3 \int_0^T e^{-2s\phi(L,t)} \theta(L,t) |u_x(0,t)|^2 dt \\
&\qquad + C s \int_0^T e^{-2s\phi(L,t)} \theta(L,t) |u_{xx}(0,t)|^2 dt \\
&\qquad + C s \int_0^T e^{-2s\phi(L,t)} \theta(L,t) |u_{xx}(L,t)|^2 dt
\end{aligned}
\tag{3.10}
$$

and from (3.8) and (3.10), choosing s_0 large enough, we deduce that

$$
\begin{aligned}
&s \int_0^L e^{-2s\phi(x,0)} |\sigma(x)|^2 dx \\
&\quad \leq C s^3 \int_0^T e^{-2s\phi(L,t)} \theta(L,t) (|u_x(0,t)|^2 + |z_x(0,t)|^2) dt \\
&\qquad + C s \int_0^T e^{-2s\phi(L,t)} \theta(L,t) (|u_{xx}(0,t)|^2 + |z_{xx}(0,t)|^2) dt \\
&\qquad + C s \int_0^T e^{-2s\phi(L,t)} (|u_{xx}(L,t)|^2 + |z_{xx}(L,t)|^2) dt.
\end{aligned}
\tag{3.11}
$$

Taking into account that

$$
z = u_t = \partial_t (y - \tilde{y})
$$

the result of Theorem 1.1 directly follows from (3.11).

A Cauchy problem

In this appendix, we state the well-posedness results for the KdV equation considered in the paper. We only give the main ideas in the proofs because the tools are pretty similar to those applied for the constant coefficient case, which is already standard in the literature (see [11,12,18,19,42]). It is worthwhile to emphasize that these results allow to get, under smoothness hypothesis on the data, the existence of solutions for the KdV equation with the regularity required for our approach in order to prove the stability of the inverse problem.

We consider a main coefficient satisfying

$$\exists a_0 \in \mathbb{R},\ \forall x \in [0, L]: \quad a(x) \geq a_0 > 0. \tag{A.1}$$

Let us introduce, for any $s \geq 0$, the space

$$\mathcal{B}^s = C([0,T], H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L))$$

and consider only the principal part of the equation, i.e., study the problem

$$\begin{cases} y_t + a(x) y_{xxx} = f, & \forall (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0, \quad y(L,t) = 0, & \forall t \in (0,T), \\ y_x(L,t) = 0, & \forall t \in (0,T), \\ y(0,x) = y_0(x), & \forall x \in (0,L). \end{cases} \tag{A.2}$$

In order to take into account the coefficient $a = a(x)$ we work in the space $X = L^2(0,L)$ endowed with the inner product

$$\langle w_1, w_2 \rangle_X := \int_0^L \frac{1}{a(x)} w_1(x) w_2(x) dx.$$

Notice that since $a \in L^\infty(0,L)$ satisfies (A.1), the norm defined by the inner product $\langle \cdot, \cdot \rangle_X$ is equivalent to the L^2-norm.

In the domain $D(A) = \{w \in H^3(0,L) : w(0) = w(L) = w'(L) = 0\}$, we define the operator $A : D(A) \subset X \to X$ as

$$A(w) = -a w'''.$$

It is easy to see that both A and its adjoint operator A^* are dissipative, and therefore, by standard semigroup theory (for instance a corollary of the Lumer–Phillips Theorem, see [38, Chapter 1]), we get that A generates a strongly continuous semigroup in $L^2(0,L)$.

Thus, if we have $y_0 \in L^2(0, L)$, $a \in L^\infty(0, L)$ satisfies condition (A.1) and $f \in L^1(0, T, L^2(0, L))$, then the linear KdV equation (A.2) has a unique solution (called mild solution) in the space $C([0, T], L^2(0, L))$. Applying the multipliers technique, we get a Kato smoothing effect, which implies $y \in L^2(0, T; H^1(0, L))$ and thus $y \in \mathcal{B}^0$. This regularity is indeed obtained by multiplying formally equation (A.2) by $\frac{x}{a(x)} y(t, x)$ and integrating in $(0, L) \times (0, T)$. This proves the following result.

Proposition A.1. *Let $y_0 \in L^2(0, L)$, $a \in L^\infty(0, L)$ verifying condition* (A.1) *and $f \in L^1(0, T, L^2(0, L))$. Then the linear KdV equation* (A.2) *has a unique solution in the space $\mathcal{B}^0$. Moreover, there exists a constant $C > 0$ such that*

$$\|y\|_{\mathcal{B}^0} \le C(\|y_0\|_{L^2(0,T)} + \|f\|_{L^1(0,T,L^2(0,L))}).$$

Depending on the regularity of the data, we can prove the existence of more regular solutions. Let us consider y, a solution of (A.2). The function $u := y_t$ satisfies

$$\begin{cases} u_t + a(x) u_{xxx} = f_t, & \forall (x, t) \in (0, L) \times (0, T), \\ u(0, t) = 0, \quad u(L, t) = 0, & \forall t \in (0, T), \\ u_x(L, t) = 0, & \forall t \in (0, T), \\ u(0, x) = f(0, x) - a(x) y_{0xxx}, & \forall x \in (0, L). \end{cases} \tag{A.3}$$

Assuming that $f_t \in L^1(0, T; L^2(0, L))$ and $y_0 \in D(A)$, we have that

$$(f(0, x) - a(x) y_{0xxx}) \in L^2(0, L),$$

and then Proposition A.1 implies that $u = y_t \in \mathcal{B}^0$.

If additionally we ask $f \in \mathcal{B}^0$, then by equation (A.2) we have

$$a(x) y_{xxx} = f - y_t \in \mathcal{B}^0,$$

and if $a \in W^{1,\infty}(0, L)$ satisfies condition (A.1), we get that the original solution satisfies $y \in \mathcal{B}^3$.

Analyzing in the same way each one of the equations fulfilled by y_{tt} and y_{ttt}, we get the following result.

Proposition A.2. *Let $y_0 \in H^9(0, L) \cap D(A)$, $a \in W^{6,\infty}(0, L)$ verifying* (A.1), *and $f \in \mathcal{B}^6$ such that $f_t \in \mathcal{B}^3$, $f_{tt} \in \mathcal{B}^0$, and $f_{ttt} \in L^1(0, T, L^2(0, L))$. Then the linear KdV equation* (A.2) *has a unique solution y in the space $\mathcal{B}^9$. This solution also satisfies $y_t \in \mathcal{B}^6$, $y_{tt} \in \mathcal{B}^3$ and $y_{ttt} \in \mathcal{B}^0$.*

Remark A.3. The hypothesis $a \in W^{6,\infty}(0, L)$ is used to ensure that the initial condition of the problem solved by y_{ttt} belongs to the space $L^2(0, L)$.

In order to prove a similar result for the nonlinear system

$$\begin{cases} y_t + a(x)y_{xxx} + y_x + yy_x = 0, & \forall (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0, \quad y(L,t) = 0, & \forall t \in (0,T), \\ y_x(L,t) = 0, & \forall t \in (0,T), \\ y(0,x) = y_0(x), & \forall x \in (0,L), \end{cases} \tag{A.4}$$

we define the map

$$\Pi : \tilde{y} \in \mathcal{B}^0 \to \Pi(\tilde{y}) = y \in \mathcal{B}^0,$$

where $\Pi(\tilde{y}) = y$ is the solution of (A.2) with f replaced by $(-\tilde{y}_x - \tilde{y}\tilde{y}_x)$. For $\tilde{y} \in \mathcal{B}^0$, we have $\tilde{y} \in L^2(0,T;H^1(0,L))$, and therefore

$$\tilde{y}\tilde{y}_x \in L^1(0,T;L^2(0,L)).$$

Thus we are able to use Proposition A.1 with the right hand side $f = (-\tilde{y}_x - \tilde{y}\tilde{y}_x)$ to define $\Pi(\tilde{y}) = y \in \mathcal{B}^0$. Applying a fixed point argument in small time and global a priori estimates to deal with the long-time case, we obtain the following.

Proposition A.4. *Let $y_0 \in L^2(0,L)$ and $a \in L^\infty(0,L)$ verifying condition* (A.1). *Then, the nonlinear KdV equation* (A.4) *has a unique solution in the space $\mathcal{B}^0$.*

We can also prove the nonlinear result in the more regular framework.

Proposition A.5. *Let $y_0 \in H^9(0,L) \cap D(A)$ and $a \in W^{6,\infty}(0,L)$ verifying condition* (A.1). *Then, the nonlinear KdV equation* (A.4) *has a unique solution in the space $\mathcal{B}^9$.*

The proof of Proposition A.5 can be done as before (with a fixed point argument and global a priori estimates) with the same map Π but now defined in the space

$$\mathcal{Y} := \{y \in \mathcal{B}^9 : y_t \in \mathcal{B}^6,\ y_{tt} \in \mathcal{B}^3 \text{ and } y_{ttt} \in \mathcal{B}^0\}$$

instead of $\mathcal{B}^0$. Let us notice that the nonlinearity $\tilde{y}\tilde{y}_x$ satisfies all the hypothesis required in Proposition A.2 for the right hand side f provided that $\tilde{y} \in \mathcal{Y}$.

Using some interpolation arguments, we can prove the following result, which gives us the existence of solutions as we require in order to solve our Inverse Problem in Theorem 1.1.

Theorem A.6. *Let $y_0 \in H^7(0,L) \cap D(A)$ and $a \in W^{6,\infty}(0,L)$ satisfying* (A.1). *Then the nonlinear KdV equation* (A.4) *has a unique solution in the space $\mathcal{B}^7$. Furthermore $y_t \in \mathcal{B}^4$ and therefore $y \in W^{1,\infty}(0,T;W^{3,\infty}(0,L))$.*

Indeed, the proof of this result can be deduced from the Interpolation Theorem by Bona and Scott ([11], see [12, Theorem 4.3]). This theorem is applied in the following way. Denoting by S the mapping which sends the initial condition y_0 to the solution y of the equation (A.4), by Propositions A.4 and A.5 we have that

$$S : L^2(0,L) \to \mathcal{B}^0 \quad \text{and} \quad S : H^9(0,L) \cap D(A) \to \mathcal{B}^9$$

are well defined. Additionally, two inequalities are needed (see [12, statements (i) and (ii) of Theorem 4.3]):

- The first one is the estimation of $\|S(y_{01}) - S(y_{02})\|_{\mathcal{B}^0}$ in terms of the norm $\|y_{01} - y_{02}\|_{L^2(0,L)}$, which can be proven exactly in the same way as in the proof of [12, Proposition 4.2], but now using our Proposition A.1.
- The second one consists in estimating $\|S(y_0)\|_{\mathcal{B}^9}$ by $\|y_0\|_{H^9(0,L)}$, which can be proven from Proposition A.5 and following [19, Proposition 15].

From the Interpolation Theorem we get that $S : H^7(0,L) \cap D(A) \to \mathcal{B}^7$ is well defined, which gives us the proof of Theorem A.6.

Remark A.7. A sharper result can be obtained with $y_0 \in H^s(0,L) \cap D(A)$ where $s > 6 + \frac{1}{2}$. In that case, previous arguments give the existence of a solution $y \in \mathcal{B}^s$ with $y_t \in \mathcal{B}^{s-3}$, and then $y_{txxx} \in C([0,T], H^q(0,L))$ with $q > \frac{1}{2}$. Thus, we obtain for system (A.4) a solution $y \in W^{1,\infty}(0,T; W^{3,\infty}(0,L))$.

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