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Lipschitz stability in an inverse problem for the Kuramoto–Sivashinsky equation

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In this article, we present an inverse problem for the nonlinear 1D Kuramoto–Sivashinsky (KS) equation. More precisely, we study the nonlinear inverse problem of retrieving the anti-diffusion coefficient from the measurements of the solution on a part of the boundary and also at some positive time in the whole space domain. The Lipschitz stability for this inverse problem is our main result and it relies on the Bukhgeĭm–Klibanov method. The proof is indeed based on a global Carleman estimate for the linearized KS equation.

Keywords: inverse problem; Kuramoto–Sivashinsky equation; Carleman estimate

AMS Subject Classifications: 35R30; 35K55

1. Introduction

We focus in this article on an inverse problem that consists the determination of a coefficient in a partial differential equation (pde) from the partial knowledge of a given single solution of the equation. This class of problems (single-measurement coefficient inverse problems) was investigated using Carleman estimates for the first time by Bukhgeĭm and Klibanov [1]. See [2–4] for details about the so-called Bukhgeĭm–Klibanov method. This method was initially used to prove the uniqueness for inverse problems (i.e. each measurement corresponds to only one coefficient) from local Carleman estimates (valid for solutions with compact support in the interior of the domain), as in [1]. Regarding the continuity of the inverse problem of recovering the source term, the first Lipschitz stability result for a multidimensional wave equation was obtained by Puel and Yamamoto [5] using the uniqueness result and a compactness–uniqueness argument.

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Global Carleman estimates (valid for solutions considered in the whole domain and satisfying boundary conditions) were applied to parabolic equations for the first time in [6], where the Lipschitz stability of an inverse problem was established. Since then, this type of inverse problems for parabolic equations has received a large amount of attention. The primary difference with respect to hyperbolic inverse problems is that parabolic problems are not time-reversible: therefore, an additional measurement must be added if that method is applied. As one can read in the discussion of the introduction of [6], the knowledge of the full-state of the solution for some positive time is required. Proving the Lipschitz stability without this assumption, which is usually needed when global Carleman inequalities are used, is still an open problem. Nevertheless, there are some uniqueness results with less assumptions on the measurements, that can be found in the literature, such as [7] or some other inversion method in [3,8].

Recent results regarding linear parabolic problems, can be found in [9] (discontinuous coefficient), [10] (systems), [11] (network) and the references therein. In [7,12,13], even nonlinear parabolic equations were considered.

Among others pde's coefficient/source inverse problems, where Carleman estimates have been used, we can mention, without being exhaustive, logarithmic stability [14], Calderón problem [15] or the Schrödinger equation [16].

In this article, we consider a 1D nonlinear fourth-order parabolic equation called the Kuramoto–Sivashinsky (KS) equation. This equation was proposed independently by Kuramoto and Tsuzuki [17] as a model for the phase turbulence in reaction diffusion systems, and by Sivashinsky [18], as a model of the physical phenomena of plane flame propagation, were the combined influence of diffusion and thermal conduction of a gas is described.

The KS equation with non-constant coefficients describing the diffusion $\sigma = \sigma(x)$ and the anti-diffusion $\gamma = \gamma(x)$ is given as

$$\begin{cases} y_t + (\sigma(x)\, y_{xx})_{xx} + \gamma(x)\, y_{xx} + y y_x = g, & \forall\, (t, x) \in Q, \\ y(t, 0) = h_1(t), \quad y(t, 1) = h_2(t), & \forall t \in (0, T), \\ y_x(t, 0) = h_3(t), \quad y_x(t, 1) = h_4(t), & \forall t \in (0, T), \\ y(0, x) = y_0(x), & \forall x \in (0, 1), \end{cases} \tag{1}$$

where $Q := (0, T) \times (0, 1)$, $\sigma\colon [0, 1] \to \mathbb{R}_+^*$ and the functions y_0, g, h_j are the initial condition, the source term and the boundary data, respectively. All these terms are assumed to be known and compatible.

In this nonlinear pde, the fourth-order term models the diffusion and the second-order term models the incipient instabilities. We consider the inverse problem of retrieving the anti-diffusion coefficient γ from boundary measurements of the solution. This corresponds, for instance, to getting information on the instability of a reaction–diffusion media by measuring a single solution, which could represent a flame propagating on the domain. Concerning the boundary measurements we will make, it is worth to mention that in a fourth-order parabolic problem like KS, boundary data u_{xx} and u_{xxx} are referred to as Neumann data, which in fact represent the heat flux [19] in this kind of models.

To the knowledge of the authors, there are no results in the literature concerning the determination of coefficients for this nonlinear equation. However, a Carleman estimate has been used to obtain the null-controllability of the KS equation in [20] for the constant coefficient case. Other results on the control of the KS equation can be found in [19,21–24].

Since the linearized equation is parabolic, we know that boundary measurements will not be sufficient to prove stability and we must consider an additional measurement of the full solution for a given time T_0 (as in [6,9] among others).

Our first result involves the local well-posedness of the nonlinear Equation (1). A less regular framework can be used for this equation but the method applied in this article requires the solution and its time-derivative to be at least in $L^2(0, T; H^4(0, 1))$. Therefore, let us introduce the following notations for the functional spaces appearing in this article:

$$\begin{aligned} &\mathcal{Y}_k := C([0, T]; H^k(0, 1)) \cap L^2(0, T; H^{k+2}(0, 1)), \quad \text{for } k \in \mathbb{N};\\ &\mathcal{F} := \{f \in L^2(0, T; H^4(0, 1)) / f_t \in L^2(0, T; L^2(0, 1))\};\\ &\mathcal{Z} := \{z \in \mathcal{Y}_6 / z_t \in \mathcal{Y}_2\}. \end{aligned} \tag{2}$$

THEOREM 1.1 *Let* $\gamma \in H^4(0, 1)$ *and* $\sigma \in H^4(0, 1)$ *be such that*

$$\sigma(x) \geq \sigma_0 > 0, \quad \forall x \in (0, 1). \tag{3}$$

There exists an $\varepsilon > 0$ *such that if* $y_0 \in H^6(0, 1)$, $g \in \mathcal{F}$ *and* $h_j \in H^2(0, T)$ *for* $j = 1, \ldots, 4$ *satisfy the compatibility conditions*

$$y_0(0) = h_1(0), \quad y_{0,x}(0) = h_3(0), \quad y_0(1) = h_2(0), \quad y_{0,x}(1) = h_4(0), \tag{4}$$

and

$$\| y_0 \|_{H^6(0,1)} \leq \varepsilon, \quad \| g \|_{\mathcal{F}} \leq \varepsilon, \quad \| h_j \|_{H^2(0,T)} \leq \varepsilon \text{ for } j = 1, \ldots, 4, \tag{5}$$

then the KS equation (1) *has a unique solution* $y \in \mathcal{Z}$.

Once the existence of solutions to the KS equation has been established (Section 2), the following inverse problem is addressed:

> Is it possible to retrieve the anti-diffusion coefficient $\gamma = \gamma(x)$ from the measurement of $y_{xx}(t, 0)$ and $y_{xxx}(t, 0)$ on $(0, T)$ and from the measurement of $y(T_0, x)$ on $(0, 1)$, where y is the solution to Equation (1) and $T_0 \in (0, T)$?

A local answer for this nonlinear inverse problem is given (Section 4). To be more specific, let $\tilde{\gamma}$ be given and fixed. We denote by $\tilde{y}$ the solution to Equation (1) with γ replaced by $\tilde{\gamma}$. This article focuses on the following question concerning the unknown γ and y.

Stability: Is it possible to estimate $\|\tilde{\gamma} - \gamma\|_{L^2(0,1)}$ by suitable norms $\|\tilde{y}(T_0, x) - y(T_0, x)\|$ in space and $\|\tilde{y}_{xx}(t, 0) - y_{xx}(t, 0)\|$, $\|\tilde{y}_{xxx}(t, 0) - y_{xxx}(t, 0)\|$ in time?

Of course, a positive answer implies the usual uniqueness result.

Uniqueness: Do the equalities of the measurements $\tilde{y}_{xx}(t, 0) = y_{xx}(t, 0)$ and $\tilde{y}_{xxx}(t, 0) = y_{xxx}(t, 0)$ for $t \in (0, T)$ and $\tilde{y}(T_0, x) = y(T_0, x)$ for $x \in (0, 1)$ imply $\tilde{\gamma} = \gamma$ on $(0, 1)$?

In order to answer these questions, we use the Bukhgeĭm–Klibanov method. First, a global Carleman estimate for the linearized KS equation with nonconstant coefficients is obtained. It is then used to prove the main result, which can be stated as follows.

To precisely state the results we prove in this article, we introduce, for $m > 0$, the set

$$L^\infty_{\leq m}(0,1) = \left\{\gamma \in L^\infty(0,1) s.t. \ \|\gamma\|_{L^\infty(0,1)} \leq m\right\}.$$

THEOREM 1.2 *Let us consider* $\sigma \in H^4(0,1)$ *satisfying* (3), $\gamma \in H^4(0,1)$, $g \in \mathcal{F}$ *and the data* $y_0 \in H^6(0,1)$ *and* $h_j \in H^2(0,T)$ *for* $j = 1, \dots, 4$ *under the compatibility conditions* (4). *Let* $y \in \mathcal{Z}$ *be the solution of* (1), *and* $\tilde{y} \in \mathcal{Z}$ *the solution corresponding to a given* $\tilde{\gamma} \in H^4(0,1)$ *instead of* γ. *We assume that there exists* $\eta > 0$ *and* $T_0 \in (0,T)$ *such that*

$$\inf\left\{\left|\tilde{y}_{xx}(T_0, x)\right|, x \in (0,1)\right\} \geq \eta. \tag{6}$$

Then, given $M > 0$, *there exists a positive constant* C *depending on the parameters* (T, m, M, η) *such that for every* $\gamma \in L^\infty_{\leq m}(0,1)$,

$$\begin{aligned}\|\gamma - \tilde{\gamma}\|^2_{L^2(0,1)} &\leq C\| y_{xx}(\cdot,0) - \tilde{y}_{xx}(\cdot,0)\|^2_{H^1(0,T)} + C\| y_{xxx}(\cdot,0) - \tilde{y}_{xxx}(\cdot,0)\|^2_{H^1(0,T)} \\ &+ C\| y(T_0,\cdot) - \tilde{y}(T_0,\cdot)\|^2_{H^4(0,1)} + C\| y(T_0,\cdot) - \tilde{y}(T_0,\cdot)\|^4_{H^1(0,1)} \end{aligned} \tag{7}$$

for all y *satisfying*

$$\| y\|_{\mathcal{Z}} \leq M.$$

This inequality states the stability of the inverse problem. Before giving the outline of our article and the proofs of the different steps, we want to give several comments on this result.

Remark 1.3 For numerical purposes, it would be interesting to know explicitly how the constant C in (7) depends on the diffusion σ or on the time T. This kind of question has been addressed in [25,26] for the observability constant in the framework of second-order parabolic equations. In those papers the authors got an exponential dependence on both the constant diffusion and the time.

Remark 1.4 One can show that there exist solutions satisfying assumption (6). We present two different arguments:

(1) We take $\varepsilon > 0$ given by Theorem 1.1, and some $y^0 \in H^6(0,1)$ such that $\inf_{x\in(0,1)}\left|y^0_{xx}\right| \geq \varepsilon/2$. For arbitrary boundary data and a source term belonging to the corresponding spaces, by Theorem 1.1 there exists a solution $\tilde{y} \in C([0,1]; H^6(0,1))$ with $\tilde{y}(0,\cdot) = y^0$. Using the Sobolev injection and continuity, we obtain the existence of a time $T_0 > 0$ such that (6) is fulfilled with $\eta = \varepsilon/4$.

(2) We can also prove that there exist solutions satisfying (6) without asking T_0 to be small, but instead, constraining the source term and boundary data as follows: let y_0 be the initial data and T_0 belong to $(0, T)$. Let us pick up a state $y_1 = y_1(x)$ strictly convex. We consider the trajectory $\tilde{y}(t,x) = \frac{T_0 - t}{T_0} y_0(x) + \frac{t}{T_0} y_1(x)$, which is the solution of Equation (1) with a source term given by $g = \tilde{y}_t + (\sigma(x)\tilde{y}_{xx})_{xx} + \gamma(x)\tilde{y}_{xx} + \tilde{y}\tilde{y}_x$ and the boundary data given by the traces of $\tilde{y}$. Thus, $\tilde{y}(T_0, x) = y_1(x)$ and hence the trajectory $\tilde{y}$ satisfies (6).

Therefore, the set of data and solutions where our stability result is valid and is not empty.

Remark 1.5 We obtain the same result if $\tilde{y}$ has a different initial condition than y. See in Section 4, that the term $v(x, 0)$ of system (46) does no play any role in the result.

Remark 1.6 We can complete inequality (7) by the following:

$$\begin{aligned}
&\| y_{xx}(\cdot, 0) - \tilde{y}_{xx}(\cdot, 0)\|^2_{H^1(0,T)} + \| y_{xxx}(\cdot, 0) - \tilde{y}_{xxx}(\cdot, 0)\|^2_{H^1(0,T)} \\
&\quad + \| y(T_0, \cdot) - \tilde{y}(T_0, \cdot)\|^2_{H^4(0,1)} + \| y(T_0, \cdot) - \tilde{y}(T_0, \cdot)\|^4_{H^1(0,1)} \\
&\quad \le C\Big(\| y - \tilde{y}\|^2_{H^1(0,T;H^4(0,1))} + \| y - \tilde{y}\|^4_{C([0,T];H^1(0,1))}\Big).
\end{aligned}$$

This inequality follows directly from standard Sobolev injections. It indicates that the required measurements are finite if y and $\tilde{y}$ belong to the space $H^1(0, T; H^4(0, 1))$ and this is true if y and $\tilde{y}$ are solutions in $\mathcal{Z}$ provided by Theorem 1.1.

Remark 1.7 As stated in the introduction, an internal measurement at $t = T_0$ is required if this method, using Carleman estimates, is used to prove the stability for this type of inverse problem for parabolic equations. Nevertheless, this is probably a technical point since there is no counter-example that demonstrates whether this assumption is required for stability. In [7], uniqueness (but not stability) is proven using a very different technique in an inverse problem for a parabolic equation and without any internal measurements in the whole space domain. One can also mention a method in [8] that can deliver uniqueness from hyperbolic equations to parabolic ones.

Remark 1.8 In this article, the boundary measurements are located at $x = 0$, but the result would be the same if we measure at $x = 1$ instead. Indeed, the choice of a suitable weight function in the proof of the Carleman estimate in Section 3 is critical to impose the side of measurement.

This article is organized as follows. The well-posedness result stated in Theorem 1.1 is proved in Section 2. A global Carleman estimate for a general KS equation is given and proved in Section 3. Finally, Section 4 contains the use of the Bukhgeĭm–Klibanov method to prove the Lipschitz stability of the inverse problem stated in Theorem 1.2.

2. On the Cauchy problem for the KS equation

This section presents a proof of Theorem 1.1 in a more general case including time-dependent lower-order coefficients. We consider the following KS system:

$$\begin{cases}
y_t + (\sigma(x)\, y_{xx})_{xx} + \gamma(x)\, y_{xx} + G_1 y_x + G_2 y + y y_x = g, & \forall\, (t, x) \in Q, \\
y(t, 0) = h_1(t), \quad y(t, 1) = h_2(t), & \forall t \in (0, T), \\
y_x(t, 0) = h_3(t), \quad y_x(t, 1) = h_4(t), & \forall t \in (0, T), \\
y(0, x) = y_0(x), & \forall x \in (0, 1),
\end{cases} \tag{8}$$

where G_1, G_2 belong to $H^1(0, T; H^4(0, 1))$, $g \in \mathcal{F}$ and $y_0 \in H^6(0, 1)$ is compatible with $h_j \in H^2(0, T)$ for $j = 1, \ldots, 4$. Recall that the coefficients satisfy $\gamma \in H^4(0, 1)$, $\sigma \in H^4(0, 1)$ and hypothesis (3).

First, we only consider the main part of the linear differential operator in the next proposition.

PROPOSITION 2.1 *Let $z_0 \in H^6 \cap H_0^2(0, 1)$ and $f \in \mathcal{F}$. Then, the following equation*

$$\begin{cases} z_t + (\sigma(x) z_{xx})_{xx} = f, & \forall (t, x) \in Q, \\ z(t, 0) = 0, \quad z(t, 1) = 0, & \forall t \in (0, T), \\ z_x(t, 0) = 0, \quad z_x(t, 1) = 0, & \forall t \in (0, T), \\ z(0, x) = z_0(x), & \forall x \in (0, 1), \end{cases} \tag{9}$$

has a unique solution $z \in \mathcal{Z}$ and there exists a $C > 0$ such that

$$\|z\|_{\mathcal{Z}} \leq C\big(\|f\|_{\mathcal{F}} + \|z_0\|_{H^6}\big).$$

Proof The operator

$$\begin{aligned} H^4 \cap H_0^2(0, 1) \subset L^2(0, 1) &\longrightarrow L^2(0, 1) \\ z &\longmapsto (\sigma(x) z''(x))'', \end{aligned}$$

is simultaneously positive, coercive and self-adjoint. By the Hille–Yosida–Phillips theorem [27], it generates a strongly continuous semigroup in $L^2(0, 1)$. Therefore, for each $z_0 \in H^4 \cap H_0^2(0, 1)$ and $f \in C^1([0, T]; L^2(0, 1))$, Equation (9) has a unique solution $z \in C([0, T]; H^4 \cap H_0^2(0, 1)) \cap C^1([0, T]; L^2(0, 1))$.

We will demonstrate that the solutions $z \in \mathcal{Z}$ (refer to the notation introduced in (2)), can be obtained by taking z_0 and f sufficiently regular. We now search for some energy estimates that indicate the space where the solutions lie on depending on the regularity of the data. Suppose that there are solutions sufficiently regular to perform the following computations. Equation (9) is multiplied by z and integrated over (0, 1) in space. Some integrations by parts give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 |z(t, x)|^2 \,\mathrm{d}x\right) + \int_0^1 |z_{xx}(t, x)|^2 \,\mathrm{d}x \leq C\left(\int_0^1 |f(t, x)|^2 \,\mathrm{d}x + \int_0^1 |z(t, x)|^2 \,\mathrm{d}x\right). \tag{10}$$

Throughout this article, C denotes a positive constant that may vary from line to line. To make the reading easier, we denote for any function u of x and t,

$$\iint_Q u = \int_0^T \int_0^1 u(t, x) \,\mathrm{d}x \,\mathrm{d}t.$$

Using Gronwall's lemma, we first obtain that for all $t > 0$,

$$\int_0^1 |z(t, x)|^2 \,\mathrm{d}x \leq C\left(\iint_Q |f|^2 + \int_0^1 |z_0|^2 \,\mathrm{d}x\right). \tag{11}$$

Then, (10) is integrated over $[0, T]$ and (11) is used to get

$$\iint_Q |z_{xx}|^2 \leq C\left(\iint_Q |f|^2 + \int_0^1 |z_0|^2 \,\mathrm{d}x\right). \tag{12}$$

Inequalities (11) and (12) finally imply that

$$\|z\|^2_{\mathcal{Y}_0} \le C \iint_Q |f|^2 + C \int_0^1 |z_0|^2 \, \mathrm{d}x. \tag{13}$$

Now, Equation (9) is multiplied by $(\sigma z_{xx})_{xx}$ and integrated over (0, 1) in space. Some integrations by parts also give

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \sigma |z_{xx}(t,x)|^2 \, \mathrm{d}x\right) + \int_0^1 |(\sigma z_{xx}(t,x))_{xx}|^2 \, \mathrm{d}x = \int_0^1 f(t,x)(\sigma z_{xx}(t,x))_{xx} \, \mathrm{d}x.$$

Using the inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \sigma |z_{xx}(t,x)|^2 \, \mathrm{d}x\right) + \int_0^1 |(\sigma z_{xx}(t,x))_{xx}|^2 \, \mathrm{d}x \le \int_0^1 |f(t,x)|^2 \, \mathrm{d}x. \tag{14}$$

Using Gronwall's lemma, from(14) and (3) we obtain that for all $t > 0$,

$$\int_0^1 |z_{xx}(t,x)|^2 \, \mathrm{d}x \le C\left(\iint_Q |f|^2 + \int_0^1 |z_0''|^2 \, \mathrm{d}x\right). \tag{15}$$

Then, (14) is integrated over [0, T] and (15) is used to get

$$\iint_Q |(\sigma z_{xx})_{xx}|^2 \le C\left(\iint_Q |f|^2 + \int_0^1 |z_0''|^2 \, \mathrm{d}x\right), \tag{16}$$

and then taking into account that $\sigma \in H^4$, we get

$$\iint_Q |z_{xxxx}|^2 \le C\left(\iint_Q |f|^2 + \int_0^1 |z_0''|^2 \, \mathrm{d}x\right) + C\|z\|_{L^2(0,T;H^3(0,1))}. \tag{17}$$

For any $\varepsilon > 0$, from Ehrling's Lemma (see Theorem 7.30 in [28]) and (11), we have

$$\begin{aligned} \|z\|_{L^2(0,T;H^3(0,1))} &\le \varepsilon \|z\|_{L^2(0,T;H^4(0,1))} + C\|z\|_{L^2(0,T;L^2(0,1))} \\ &\le \varepsilon \|z\|_{L^2(0,T;H^4(0,1))} + C\left(\iint_Q |f|^2 + \int_0^1 |z_0|^2 \, \mathrm{d}x\right). \end{aligned} \tag{18}$$

Taking $\varepsilon > 0$ small enough, inequalities (15), (17) and (18) imply that

$$\|z\|^2_{\mathcal{Y}_2} \le C \iint_Q |f|^2 + C\|z_0\|^2_{L^2(0,T;H^2(0,1))}. \tag{19}$$

On the other hand, Equation (9) is derived with respect to time. Thus, $q := z_t$ satisfies

$$\begin{cases} q_t + (\sigma(x) q_{xx})_{xx} = f_t, & \forall (t,x) \in Q, \\ q(t,0) = 0, \quad q(t,1) = 0, & \forall t \in (0,T), \\ q_x(t,0) = 0, \quad q_x(t,1) = 0, & \forall t \in (0,T), \\ q(0,x) = f(0,x) - (\sigma z_0''(x))'', & \forall x \in (0,1). \end{cases} \tag{20}$$

Using estimate (19), we obtain $q \in \mathcal{Y}_2$ if $(f(0,x) - (\sigma z_0''(x))'') \in H^2(0,1)$ and $f_t \in L^2(0,\, T;\, L^2(0,\, 1))$. These hypotheses are fulfilled if $z_0 \in H^6 \cap H_0^2(0,1)$ and $f \in \mathcal{F}$.

Note that $\mathcal{F} \subset C([0,T];\ H^2(0,1))$. From the equation satisfied by z and the fact that $f \in \mathcal{F}$ and $z_t \in \mathcal{Y}_2$, we determine that $z \in \mathcal{Y}_6$, which concludes the proof of Proposition 2.1. ■

Then, we focus on the linear problem with non-homogenous boundary conditions and low-order coefficients that depend on time.

Proposition 2.2 *Let* $z_0 \in H^6(0,1)$, $\hat{f} \in \mathcal{F}$, G_1, $G_2 \in H^1(0,T;\ H^4(0,1))$ *and* $h_j \in H^2(0,T)$ *for* $j = 1, \ldots, 4$ *satisfy the compatibility conditions with* z_0. *Then, the equation*

$$\begin{cases} z_t + (\sigma(x) z_{xx})_{xx} + \gamma(x) z_{xx} + G_1 z_x + G_2 z = \hat{f}, & \forall\, (t,x) \in Q, \\ z(t,0) = h_1(t), \quad z(t,1) = h_2(t), & \forall t \in (0,T), \\ z_x(t,0) = h_3(t), \quad z_x(t,1) = h_4(t), & \forall t \in (0,T), \\ z(0,x) = z_0(x), & \forall x \in (0,1), \end{cases} \tag{21}$$

has a unique solution $z \in \mathcal{Z}$ *and there exists a* $C > 0$ *such that*

$$\|z\|_{\mathcal{Z}} \le C\left(\|\hat{f}\|_{\mathcal{F}} + \|z_0\|_{H^6} + \sum_{j=1}^{4} \|h_j\|_{H^2} \right).$$

Proof We first prove this result for null boundary data (i.e. for $h_j = 0$ for $j = 1, \ldots, 4$ and therefore $z_0 \in H^6 \cap H_0^2(0,1)$).

For any $\hat{w} \in \mathcal{Z}$, $\Pi(\hat{w})$ is defined as the solution of (9) with $f = (\hat{f} - \gamma(x)\hat{w}_{xx} - G_1 \hat{w}_x - G_2 \hat{w})$. Note that $f \in \mathcal{F}$ and therefore $\Pi(\hat{w}) \in \mathcal{Z}$ is well defined.

If T is small enough, then Π is a contraction. Indeed, for any w, $\hat{w} \in \mathcal{Z}$, we have

$$\begin{aligned} \|\Pi(\hat{w}) - \Pi(w)\|_{\mathcal{Z}} &\le C\|\gamma(x)(w_{xx} - \hat{w}_{xx}) + G_1(w_x - \hat{w}_x) + G_2(w - \hat{w})\|_{\mathcal{F}} \\ &\le C\|w - \hat{w}\|_{L^2(H^6)} + C\|w_t - \hat{w}_t\|_{L^2(H^2)} \end{aligned} \tag{22}$$

$$\begin{aligned} &\le CT^{\frac{1}{4}}\|w - \hat{w}\|_{L^4(H^6)} + CT^{\frac{1}{4}}\|w_t - \hat{w}_t\|_{L^4(H^2)} \\ &\le CT^{\frac{1}{4}}\|w - \hat{w}\|_{\mathcal{Y}_6} + CT^{\frac{1}{4}}\|w_t - \hat{w}_t\|_{\mathcal{Y}_2} \\ &\le CT^{\frac{1}{4}}\|w - \hat{w}\|_{\mathcal{Z}}, \end{aligned} \tag{23}$$

where the space $L^m(0,T;\ H^n(0,1))$ is denoted as $L^m(H^n)$.

Hence, the operator Π has a unique fixed point in $\mathcal{Z}$, which is the solution of (21) with $h_j = 0$ for $j = 1, \ldots, 4$. Using standard arguments and the linearity of this equation, the solution can be extended to a larger time interval.

In order to prove the general case, take $h_j \in H^2(0,T)$, $j = 1, \ldots, 4$ compatible with z_0. It is not difficult to find a function $\psi \in H^2(0,T;\ C^\infty([0,1]))$ satisfying the boundary conditions of (21). For instance, take $\psi(x,t) = \sum_{j=1}^{4} p_j(x) h_j(t)$ where $p_1(x) = 2x^3 - 3x^2 + 1$, $p_2(x) = -2x^3 + 3x^2$, $p_3(x) = x^3 - 2x^2 + x$ and $p_4(x) = x^3 - x^2$. In particular, we have $L\psi := \psi_t + (\sigma(x)\psi_{xx})_{xx} + \gamma(x)\psi_{xx} + G_1\psi_x + G_2\psi \in \mathcal{F}$. Then, if w is the solution of Equation (21) with null boundary data, initial condition $w_0 - \psi(\cdot, 0)$

and right-hand side equal to $\hat{f} - L\psi$, let us define $z = w + \psi$. It is not difficult to see that z is the required solution. ■

Remark 2.3 The third-order term z_{xxx} can be added to Equation (21). Indeed, in that case (22) becomes $C\|w - \hat{w}\|_{L^2(H^7)} + C\|w_t - \hat{w}_t\|_{L^2(H^3)}$, which is bounded by

$$CT^{\frac{1}{4}}\|w - \hat{w}\|^{1/2}_{L^\infty(H^6)}\|w - \hat{w}\|^{1/2}_{L^2(H^8)} + CT^{\frac{1}{4}}\|w_t - \hat{w}_t\|^{1/2}_{L^\infty(H^2)}\|w_t - \hat{w}_t\|^{1/2}_{L^2(H^4)}.$$

This last expression is bounded by (23). The remainder of the proof is the same.

Again, by using a fixed-point theorem, we can prove Theorem 1.1 for Equation (8).

Let $y_0 \in H^6(0, 1)$, $h_j \in H^2(0, 1)$ compatible with y_0 and $g \in \mathcal{F}$. For any $v \in \mathcal{Z}$, we define $\Lambda(v)$ as the solution of (21) with $\hat{f} = (g - vv_x)$ and $z_0 = y_0$. Note that $\hat{f} \in \mathcal{F}$ and therefore $\Lambda(v) \in \mathcal{Z}$ is well defined. Indeed, if $v \in \mathcal{Y}_3$ and $v_t \in \mathcal{Y}_0$, then we have

$$(vv_x)_{xxxx} = (10v_{xx}v_{xxx} + 5v_x v_{xxxx} + vv_{xxxxx}) \in L^2(0, T; L^2(0, 1))$$

and

$$(vv_x)_t = v_t v_x + vv_{xt} \in L^2(0, T; L^2(0, 1)).$$

Furthermore, we can prove

$$\begin{aligned}\|\Lambda(v)\|_{\mathcal{Z}} &\le C\left(\|g\|_{\mathcal{F}} + \|vv_x\|_{\mathcal{F}} + \|y_0\|_{H^6} + \sum_{j=1}^{4}\|h_j\|_{H^2}\right)\\ &\le C\left(\|g\|_{\mathcal{F}} + \|v\|^2_{\mathcal{Z}} + \|y_0\|_{H^6} + \sum_{j=1}^{4}\|h_j\|_{H^2}\right). \qquad (24)\end{aligned}$$

Let $\varepsilon > 0$ and suppose that y_0, h_j and g satisfy (5). Consider v such that $\|v\|_{\mathcal{Z}} \le R$ with $R > 0$ satisfying $C(6\varepsilon + R^2) < R$. From (24), we obtain $\|\Lambda(v)\|_{\mathcal{Z}} < R$. Thus, the application Λ maps the ball $B_R := \{v \in \mathcal{Z}/\|v\|_{\mathcal{Z}} \le R\}$ into itself.

We will now prove that Λ: $B_R \to B_R$ is a contraction. For any z, $v \in B_R$, $\Lambda(z) - \Lambda(v)$ is the solution of (21) with $z_0 = 0$, $h_j = 0$ for $j = 1, \dots, 4$ and $\hat{f} = vv_x - zz_x$. We obtain the estimate

$$\|\Lambda(z) - \Lambda(v)\|_{\mathcal{Z}} \le C\|vv_x - zz_x\|_{\mathcal{F}} \le C\big(\|(v - z)v_x\|_{\mathcal{F}} + \|z(v_x - z_x)\|_{\mathcal{F}}\big).$$

Using the definition (2) of the space $\mathcal{F}$, v, $z \in C([0, 1]; H^6(0, 1)) \hookrightarrow L^\infty(0, T; W^{5,\infty}(0, 1))$ and v_t, $z_t \in C([0, 1]; H^2(0, 1)) \hookrightarrow L^\infty(0, T; W^{1,\infty}(0, 1))$, we obtain

$$\|\Lambda(z) - \Lambda(v)\|_{\mathcal{Z}} \le C(\|v\|_{\mathcal{Z}} + \|z\|_{\mathcal{Z}})\|v - z\|_{\mathcal{Z}} \le 2CR\|v - z\|_{\mathcal{Z}},$$

which implies that Λ is a contraction if R is chosen small enough. More precisely, we can choose R, ε such that $2CR < 1$ and $C(6\varepsilon + R^2) < R$. Hence, the map Λ has a unique fixed point $y \in \mathcal{Z}$, which is the unique solution of (8). Thus, we have proven Theorem 1.1. ■

3. Global Carleman inequality

In this section, a global Carleman inequality will be proved for the linearized KS equation. We define the space

$$\mathcal{V} = \{v \in L^2(0, T; H^4 \cap H_0^2(0,1)) \big/ Lv \in L^2((0,T) \times (0,1))\}, \tag{25}$$

where

$$Lv = v_t + (\sigma v_{xx})_{xx} + q_2 v_{xx} + q_1 v_x + q_0 v$$

with $q_j \in L^\infty(\Omega)$ for $j = 0, 1, 2$.

Consider a $\beta \in C^4([0,1])$ such that for some $r > 0$ we have, for all $x \in (0, 1)$:

$$\begin{aligned} &0 < r \le \beta(x), \\ &0 < r \le \beta'(x), \\ &\beta''(x) \le -r < 0, \\ &|\sigma'(x)\beta'(x)| \le \frac{r}{4} \min_{z \in [0,1]} \{\sigma(z)\}. \end{aligned} \tag{26}$$

For instance, if σ is constant, we can consider $\beta(x) = \sqrt{1+x}$.

On the other hand, given $T_0 \in (0, T)$, we can choose $\phi_0 \in C^1([0, T])$ such that

$$\begin{aligned} &\phi_0(0) = \phi_0(T) = 0, \quad \text{and} \\ &0 < \phi_0(t) \le \phi_0(T_0) \quad \text{for each } t \in (0, T). \end{aligned} \tag{27}$$

For example, if $T_0 = T/2$, we can use $\phi_0(t) = t(T - t)$.

We finally define the function

$$\phi(t, x) = \frac{\beta(x)}{\phi_0(t)}, \tag{28}$$

for $(t, x) \in (0, T) \times [0, 1]$, which is the weight function of the Carleman estimate. From (26) and (27) it is not difficult to see that ϕ satisfies the following properties:

$$\begin{aligned} &\exists C > 0 \text{ such that } \phi \le C\phi_x \text{ and} \\ &\phi^n \le C\phi^m \quad \text{for each positive integers } n < m. \end{aligned} \tag{29}$$

THEOREM 3.1 *Let ϕ be a function defined by* (28) *and $m > 0$. Then there exist $\lambda_0 > 0$ and a constant $C = C(T, \lambda_0, r, m) > 0$ such that if $\|q_i\|_{L^\infty((0,T)\times(0,1))} \le m$ for $i = 0, 1, 2$ then we have*

$$\begin{aligned} &\int_0^T \int_0^1 e^{-2\lambda\phi} \left(\frac{|v_t|^2 + |(\sigma v_{xx})_{xx}|^2}{\lambda\phi} + \lambda^7\phi^7|v|^2 + \lambda^5\phi^5|v_x|^2 + \lambda^3\phi^3|v_{xx}|^2 + \lambda\phi|v_{xxx}|^2 \right) \mathrm{d}x\,\mathrm{d}t \\ &\le C \int_0^T \int_0^1 e^{-2\lambda\phi}|Lv|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + C \int_0^T e^{-2\lambda\phi(t,0)} \left(\lambda^3\phi_x^3(t,0)\sigma(0)^2|v_{xx}(t,0)|^2 + \lambda\phi_x(t,0)\sigma^2(0)|v_{xxx}(t,0)|^2 \right) \mathrm{d}t \end{aligned} \tag{30}$$

for all $v \in \mathcal{V}$, for all $\lambda \ge \lambda_0$.

As we pointed out in Section 1, a Carleman estimate for the KS equation with constant coefficients σ and γ was previously obtained in [20]. The final goal in that work was to prove null-controllability with boundary controls. Thus, (30) is a generalization to the case of non-constant coefficients.

Proof Consider the following operator P defined in $\mathcal{W}_\lambda := \{e^{-\lambda\phi}v\colon v \in \mathcal{V}\}$ by

$$Pw = e^{-\lambda\phi}L(e^{\lambda\phi}w).$$

We then obtain the decomposition $Pw = P_1w + P_2w + Rw$, where

$$P_1w = 6\lambda^2\phi_x^2\sigma w_{xx} + \lambda^4\phi_x^4\sigma w + (\sigma w_{xx})_{xx} + 6\lambda^2(\phi_x^2\sigma)_x w_x \tag{31}$$

$$P_2w = w_t + 4\lambda^3\phi_x^3\sigma w_x + 4\lambda\phi_x\sigma w_{xxx} + 4\lambda^3\phi_x(\phi_x^2\sigma)_x w \tag{32}$$

$$\begin{aligned} Rw = {} & \lambda\phi_t w + 2\lambda\phi_x\sigma_{xx}w_x + \lambda^2\phi_x^2\sigma_{xx}w + \lambda\phi_{xx}\sigma_{xx}w \\ & + 6\lambda\phi_x\sigma_x w_{xx} + 6\lambda^2\phi_x\phi_{xx}\sigma_x w + 6\lambda\phi_{xx}\sigma_x w + 2\lambda\phi_{xxx}\sigma_x w \\ & + 4\lambda^2\phi_x\phi_{xxx}\sigma w + 6\lambda\phi_{xx}\sigma w_{xx} + 3\lambda^2\phi_{xx}^2\sigma w + 4\lambda\phi_{xxx}\sigma w_x \\ & + \lambda\phi_{xxxx}\sigma w + q_0w + q_1w_x + q_1\lambda\phi_x w \\ & + q_2w_{xx} + 2\lambda q_2\phi_x w_x + \lambda^2 q_2\phi_x^2 w + \lambda\phi_{xx}q_2 w \\ & - 2\lambda^3\phi_x^2\phi_{xx}\sigma w - 2\lambda^3\phi_x^3\sigma_x w. \end{aligned} \tag{33}$$

Thus

$$\|Pw - Rw\|_{L^2(Q)}^2 = \|P_1w\|_{L^2(Q)}^2 + 2\langle P_1w, P_2w\rangle + \|P_2w\|_{L^2(Q)}^2,$$

where $\langle\cdot,\cdot\rangle$ is the $L^2(Q)$ scalar product.

For any $v \in \mathcal{V}$ we obtain $v_t \in L^2(0, T; L^2(0,1))$ and then $v \in C([0,T]; L^2(0,1))$. From the construction of ϕ (see (27)), we obtain $w \in C([0,T]; L^2(0,1))$ and $w(x,0) = w(x,T) = 0$ for any $w \in \mathcal{W}_\lambda$.

Let us define the notations

$$I(w) = -6\lambda^7\int_0^T\int_0^1 \phi_x^6\phi_{xx}\sigma^2|w|^2\,dx\,dt,$$

$$I(w_x) = -\lambda^5\int_0^T\int_0^1 \phi_x^4\sigma(30\phi_{xx}\sigma + 12\phi_x\sigma_x)|w_x|^2\,dx\,dt,$$

$$I(w_{2x}) = -\lambda^3\int_0^T\int_0^1 \phi_x^2\sigma(58\phi_{xx}\sigma + 40\phi_x\sigma_x)|w_{xx}|^2\,dx\,dt,$$

$$I(w_{3x}) = -\lambda\int_0^T\int_0^1 \sigma(2\phi_{xx}\sigma - 4\phi_x\sigma_x)|w_{xxx}|^2\,dx\,dt$$

and

$$I_x = \int_0^T (10\lambda^3\phi_x^3\sigma^2|w_{xx}|^2 + 2\lambda\phi_x\sigma\sigma_{xx}|w_{xx}|^2 + 2\lambda\phi_x\sigma^2|w_{xxx}|^2)\Big|_{x=0}^{1} \,\mathrm{d}t.$$

The following weighted norm is defined, for any $w\in\mathcal{W}_\lambda$, as

$$\|w\|^2_{\lambda,\phi} = \int_0^T\int_0^1 \left(\lambda^7\phi^7|w|^2 + \lambda^5\phi^5|w_x|^2 + \lambda^3\phi^3|w_{xx}|^2 + \lambda\phi|w_{xxx}|^2\right)\mathrm{d}x\,\mathrm{d}t.$$

We first require the following lemma.

Lemma 3.2 *Under the hypothesis of Theorem* 3.1, *there exists a* $\delta > 0$ *such that*

$$\langle P_1 w, P_2 w\rangle_{L^2(Q)} \geq \delta\|w\|^2_{\lambda,\phi} + I_x \tag{34}$$

for λ *large enough and for all* $w\in\mathcal{W}_\lambda$.

Proof It is sufficient to prove that

$$\langle P_1 w, P_2 w\rangle_{L^2} = \sum_{k=0}^{3} I(w_{kx}) + R_0(w) + I_x \tag{35}$$

for a large enough λ, for all $w\in\mathcal{W}_\lambda$, where $|R_0(w)| \leq \lambda^{-1}\|w\|^2_{\lambda,\phi}$.

Indeed, let us first assume that we have (35). From the hypothesis in (26) we easily check that there exists an $\varepsilon > 0$ such that ϕ satisfies for all $x\in(0, 1)$,

$$\begin{aligned} &\phi_{xx}(x) \leq -\varepsilon\phi < 0, \\ &30\phi_{xx}(x)\sigma(x) + 12\phi_x(x)\sigma_x(x) \leq -\varepsilon\phi < 0, \\ &58\phi_{xx}(x)\sigma(x) + 40\phi_x(x)\sigma_x(x) \leq -\varepsilon\phi < 0 \text{ and} \\ &2\phi_{xx}(x)\sigma(x) - 4\phi_x(x)\sigma_x(x) \leq -\varepsilon\phi < 0. \end{aligned} \tag{36}$$

Then from (29) and assuming (35), we obtain, for λ large enough,

$$\begin{aligned} \langle P_1 w, P_2 w\rangle_{L^2} &= \sum_{k=0}^{3} I(w_{kx}) + R_0(w) + I_x \\ &\geq 2\delta\|w\|^2_{\lambda,\phi} - |R_0(w)| + I_x \\ &\geq \delta\|w\|^2_{\lambda,\phi} + I_x. \end{aligned} \tag{37}$$

Let us now prove (35): we write $\langle P_1 w, P_2 w\rangle_{L^2(Q)} = \sum_{i,j=1}^4 I_{i,j}$, where $I_{i,j}$ denotes the L^2-product between the i-th term of $P_1 w$ in (31) and the j-th term of $P_2 w$ in (32).

Integrations by parts in time or space are performed on each expression $I_{i,j}$. Each resulting expression will be included in one of the terms of the right-hand side of (35). The results are listed below, and we indicate for each term where it will be included.

- $I_{1,1} = -I_{4,1} + \underbrace{3\lambda^2 \iint_Q (\phi_x^2\sigma)_t |w_x|^2}_{R_0(w)}.$
- $I_{1,2} = \underbrace{-12\lambda^5 \iint_Q (\phi_x^5\sigma^2)_x |w_x|^2}_{I(w_x)}.$

- $I_{1,3} = \underbrace{-12\lambda^3 \iint_Q (\phi_x^3\sigma^2)_x |w_{xx}|^2}_{I(w_{2x})} + \underbrace{12\lambda^3 \int_0^T \left[\phi_x^3\sigma^2|w_{xx}|^2\right]_0^1 \mathrm{d}t}_{I_x}$

- $I_{1,4} = \underbrace{12\lambda^5 \iint_Q [\phi_x^3\sigma(\phi_x^2\sigma)_x]_{xx}|w|^2}_{R_0(w)} - \underbrace{24\lambda^5 \iint_Q \phi_x^3\sigma(\phi_x^2\sigma)_x|w_x|^2}_{I(w_x)}.$

- $I_{2,1} = \underbrace{-\frac{\lambda^4}{2} \iint_Q (\phi_x^4\sigma)_t|w|^2}_{R_0(w)}.$

- $I_{2,2} = \underbrace{-2\lambda^7 \iint_Q (\phi_x^7\sigma^2)_x|w|^2}_{I(w)}.$

- $I_{2,3} = \underbrace{-2\lambda^5 \iint_Q (\phi_x^5\sigma^2)_{xxx}|w|^2}_{R_0(w)} + \underbrace{6\lambda^5 \iint_Q (\phi_x^5\sigma^2)_x|w_x|^2}_{I(w_x)}.$

- $I_{2,4} = \underbrace{4\lambda^7 \iint_Q \phi_x^5\sigma(\phi_x^2\sigma)_x|w|^2}_{I(w)}.$

- $I_{3,1} = \frac{1}{2}\int_0^1 \left[\sigma|w_{xx}|^2\right]_0^T \mathrm{d}x = 0.$

- $I_{3,2} = \underbrace{-2\lambda^3 \iint_Q [(\phi_x^3\sigma)_{xx}\sigma]_x|w_x|^2}_{R_0(w)} + \underbrace{4\lambda^3 \iint_Q (\phi_x^3\sigma)_x\sigma|w_{xx}|^2}_{I(w_{2x})}.$

$$+\underbrace{2\lambda^3 \iint_Q (\phi_x^3)_x\sigma^2|w_{xx}|^2}_{I(w_{2x})} - \underbrace{2\lambda^3 \int_0^T \left[\phi_x^3\sigma^2|w_{xx}|^2\right]_0^1 \mathrm{d}t}_{I_x}$$

- $I_{3,3} = \underbrace{2\lambda \int_0^T \left[\phi_x\sigma\sigma_{xx}|w_{xx}|^2\right]_0^1 \mathrm{d}t}_{I_x} - \underbrace{2\lambda \iint_Q (\phi_x\sigma\sigma_{xx})_x|w_{xx}|^2}_{R_0(w)}$

$$+\underbrace{8\lambda \iint_Q \phi_x\sigma\sigma_x|w_{3x}|^2}_{I(w_{xxx})} + \underbrace{2\lambda \int_0^T \left[\phi_x\sigma^2|w_{xxx}|^2\right]_0^1 \mathrm{d}t}_{I_x} - \underbrace{2\lambda \iint_Q (\phi_x\sigma^2)_x|w_{xxx}|^2}_{I(w_{xxx})}$$

- $I_{3,4} = \underbrace{4\lambda^3 \iint_Q (\phi_x(\phi_x^2\sigma)_x)_{xx}\sigma w w_{xx}}_{R_0(w)} - \underbrace{4\lambda^3 \iint_Q (\phi_x(\phi_x^2\sigma)_x)_x\sigma|w_x|^2}_{R_0(w)}.$

$$+\underbrace{4\lambda^3 \iint_Q \phi_x(\phi_x^2\sigma)_x\sigma|w_{2x}|^2}_{I(w_{2x})}$$

- $I_{4,1} = 6\lambda^2 \iint_Q (\phi_x^2\sigma)_x w_x w_t$, which is cancelled when adding with $I_{1,1}$.

- $I_{4,2} = \underbrace{24\lambda^5 \iint_Q (\phi_x^2\sigma)_x \phi_x^3 \sigma |w_x|^2}_{I(w_x)}.$
- $I_{4,3} = \underbrace{12\lambda^3 \iint_Q [(\phi_x^2\sigma)_x \phi_x \sigma]_{xx} |w_x|^2}_{R_0(w)} - \underbrace{24\lambda^3 \iint_Q (\phi_x^2\sigma)_x \phi_x \sigma |w_{xx}|^2}_{I(w_{2x})}.$
- $I_{4,4} = \underbrace{-12\lambda^5 \iint_Q (\phi_x^2\sigma)_x (\phi_x^3\sigma)_x |w|^2}_{R_0(w)}.$

Summing up all the terms, we obtain (35). ■

Then, we will prove a Carleman inequality for the conjugated operator P.

LEMMA 3.3 *There exists a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ we have, for all $w \in \mathcal{W}_\lambda$,*

$$\int_0^T \int_0^1 \left(\lambda^7\phi^7|w|^2 + \lambda^5\phi^5|w_x|^2 + \lambda^3\phi^3|w_{xx}|^2 + \lambda\phi|w_{xxx}|^2\right) \mathrm{d}x\,\mathrm{d}t$$
$$+ \|P_1w\|^2_{L^2(Q)} + \|P_2w\|^2_{L^2(Q)} \leq C\|Pw\|^2_{L^2(Q)} - I_x.$$

Proof From hypothesis (26) and the inequalities listed in (36), we know that there exists a $\delta > 0$ such that

$$\sum_{k=0}^{3} I(w_{kx}) \geq \delta \|w\|^2_{\lambda,\phi} \tag{38}$$

for a parameter λ large enough.

Besides, from the definition (33), the fact $\|q_i\|_{L^\infty((0,T)\times(0,1))} \leq m$ for $i = 0$, 1, 2, and (29), we obtain

$$\begin{aligned}\|Rw\|^2_{L^2((0,T)\times(0,1))} &\leq C\left(\lambda^6 \iint_Q \phi^6|w|^2 + \lambda^2 \iint_Q \phi^2|w_x|^2 + \lambda^2 \iint_Q \phi^2|w_{xx}|^2\right)\\ &\leq C\lambda^{-1}\|w\|^2_{\lambda,\phi}.\end{aligned} \tag{39}$$

Thus, for λ large enough, we have

$$\begin{aligned}\|P_1w\|^2_{L^2} + 2\langle P_1w, P_2w\rangle + \|P_2w\|^2_{L^2} &= \|Pw - Rw\|^2_{L^2}\\ &\leq 2\|Pw\|^2_{L^2} + 2\|Rw\|^2_{L^2}\\ &\leq 2\|Pw\|^2_{L^2} + C\lambda^{-1}\|w\|^2_{\lambda,\phi}.\end{aligned} \tag{40}$$

From Lemma 3.2 and estimates (40) and (38), we conclude the proof of Lemma 3.3. ■

To complete the proof of Theorem 3.1, we have to deal with the norms fo P_1w and P_2w appearing in Lemma 3.3. From the definition of P_2w, and because (26) holds, we have

$$\frac{1}{\lambda\phi}|w_t|^2 \leq \frac{2}{\lambda\phi}|P_2w|^2 + C\left(\lambda^5\phi^5|w|^2 + \lambda^5\phi^5|w_x|^2 + \lambda\phi|w_{xxx}|^2\right)$$

and

$$\iint_Q \frac{1}{\lambda\phi}|w_t|^2 \le C\iint_Q |P_2 w|^2 + C\|w\|^2_{\lambda,\phi}$$

for λ large enough. A similar result is proven for $(\sigma w_{xx})_{xx}$ and $P_1 w$, and we then have

$$\iint_Q \frac{1}{\lambda\phi}\left(|w_t|^2 + |(\sigma w_{xx})_{xx}|^2\right) \le C\iint_Q \left(|P_1 w|^2 + |P_2 w|^2\right) + C\|w\|^2_{\lambda,\phi}. \tag{41}$$

From (41) and Lemma 3.3 we obtain

$$\iint_Q \frac{1}{\lambda\phi}(|w_t|^2 + |(\sigma w_{xx})_{xx}|^2) + \lambda^7\phi^7|w|^2 + \lambda^5\phi^5|w_x|^2 + \lambda^3\phi^3|w_{xx}|^2 + \lambda\phi|w_{xxx}|^2$$
$$\le C\iint_Q |Pw|^2 - CI_x. \tag{42}$$

To handle the terms in I_x, we note that for any $x \in (0, 1)$ and λ large enough,

$$-C\lambda \int_0^T \phi_x(x,t)\sigma(x)\sigma_{xx}(x)|w_{xx}(x,t)|^2 \mathrm{d}t \le C\lambda^3 \int_0^T \phi_x(x,t)^3\sigma(x)^2|w_{xx}(x,t)|^2\mathrm{d}t.$$

Then

$$-CI_x \le C\lambda^3 \int_0^T \phi_x(0,t)^3\sigma(0)^2|w_{xx}(0,t)|^2\mathrm{d}t + C\lambda\int_0^T \phi_x(0,t)\sigma(0)^2|w_{xxx}(0,t)|^2\mathrm{d}t, \tag{43}$$

and from (42) and (43) we obtain

$$\iint_Q \frac{1}{\lambda\phi}\left(|w_t|^2 + |(\sigma w_{xx})_{xx}|^2\right) + \|w\|^2_{\lambda,\phi} \le C\iint_Q |Pw|^2$$
$$+ C\lambda^3 \int_0^T \phi_x(0,t)^3\sigma(0)^2|w_{xx}(0,t)|^2\mathrm{d}t + C\lambda\int_0^T \phi_x(0,t)\sigma(0)^2|w_{xxx}(0,t)|^2\mathrm{d}t. \tag{44}$$

Computing the derivatives of $e^{\lambda\phi}w$, it is trivial to prove that

$$\left|\partial_x^k v\right|^2 = \left|\partial_x^k(e^{\lambda\phi}w)\right|^2 \le C\sum_{j=0}^{k}\left|\lambda^{k-j}\phi^{k-j}\partial_x^j w\right|^2$$

for each $k = 0, \ldots, 3$. Therefore

$$\int_0^T\int_0^1 e^{-2\lambda\phi}\left(\lambda^7\phi^7|e^{\lambda\phi}w|^2 + \lambda^5\phi^5|(e^{\lambda\phi}w)_x|^2 + \lambda^3\phi^3|(e^{\lambda\phi}w)_{xx}|^2 + \lambda\phi|(e^{\lambda\phi}w)_{xxx}|^2\right)\mathrm{d}x\,\mathrm{d}t$$
$$\le C\|w\|_{\lambda,\phi}.$$

Finally considering that $Pw = e^{-\lambda\varphi}Lv$, we obtain the Carleman estimate (30). ■

Remark 3.4 We considered the function β to be increasing. This allows the Carleman inequality to be obtained with boundary terms at $x = 0$. If a decreasing function β was used instead, then an inequality with boundary terms at $x = 1$ would have been obtained. As discussed in the following section, the boundary terms in the Carleman inequality are related to the location of the observations in the inverse problem.

4. Inverse problem

In this section, the local stability of the nonlinear inverse problem stated in Theorem 1.2 will be proved following the ideas of [1,4]. The proof is split into several steps.

Step 1 Local study of the inverse problem

Let γ, $\tilde{\gamma}$, y and $\tilde{y}$ be defined as in Theorem 1.2. If we set $u = y - \tilde{y}$ and $f = \tilde{\gamma} - \gamma$, then u solves the following KS equation:

$$\begin{cases} u_t + (\sigma(x)u_{xx})_{xx} + \gamma u_{xx} + \tilde{y}u_x + \tilde{y}_x u + uu_x = f(x)\tilde{y}_{xx}(x,t), & \forall\,(t,x)\in Q, \\ u(t,0) = u(t,1) = 0, & \forall t\in(0,T), \\ u_x(t,0) = u_x(t,1) = 0, & \forall t\in(0,T), \\ u(0,x) = 0, & \forall x\in(0,1). \end{cases} \tag{45}$$

Then, in order to prove the stability of the inverse problem mentioned in Section 1, it is sufficient to obtain an estimate of f in terms of $u_{xx}(\cdot,0)$, $u_{xxx}(\cdot,0)$ and $u(T_0,\cdot)$, where $\tilde{\gamma}$ and $\tilde{y}$ are given, $\gamma\in H^4(0,1)$ and u is the solution of Equation (45).

We begin by deriving Equation (45) with respect to time. Thus, $v = u_t$ satisfies the following equation:

$$\begin{cases} v_t + (\sigma v_{xx})_{xx} + \gamma v_{xx} + \tilde{y}v_x + \tilde{y}_x v = f\tilde{y}_{xxt} - g, & \forall\,(t,x)\in Q, \\ v(t,0) = v(t,1) = 0, & \forall t\in(0,T), \\ v_x(t,0) = v_x(t,1) = 0, & \forall t\in(0,T), \\ v(0,x) = fR(x,0), & \forall x\in(0,1), \end{cases} \tag{46}$$

where $g(x,t) = u(x,t)y_{xt}(x,t) + u_x(x,t)y_t(x,t)$.

The proof of Theorem 1.2 relies on the use of the Carleman estimate given in Theorem 3.1. This result will be used twice. First, Equation (46) allows to estimate v in terms of f, $\tilde{y}_{xx}$ and g. Then, Equation (45) will be used to handle the terms u and u_x, which appear in the expression of the source term g. The details are given in the next step below.

Step 2 First use of the Carleman estimate

Similar to the proof of the Carleman estimate, we set $w = e^{-\lambda\phi}v$. Then, we work on the term

$$I = 2\int_0^1\int_0^{T_0} w(t,x)w_t(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

On the one hand, we can calculate I and bound it from below. Indeed, using $w(0,x) = e^{-\lambda\phi(0,x)}v(0,x) = 0$ for all $x\in(0,1)$ and Equation (45), we can easily obtain

$$\begin{aligned} I &= \int_0^1 \left|w(T_0,x)\right|^2 \mathrm{d}x \\ &= \int_0^1 e^{-2\lambda\phi(T_0,x)}\left|\left(f\tilde{y}_{xx} - (\sigma u_{xx})_{xx} - \gamma u_{xx} - \tilde{y}u_x - \tilde{y}_x u - uu_x\right)(T_0,x)\right|^2 \mathrm{d}x \\ &\geq \int_0^1 e^{-2\lambda\phi(T_0,x)}|f(x)|^2\left|\tilde{y}_{xx}(T_0,x)\right|^2 \mathrm{d}x - C\left\|u(T_0)\right\|^2_{H^4(0,1)} - C\left\|u(T_0)\right\|^4_{H^1(0,1)}, \end{aligned}$$

where C depends on $\|\gamma\|_{L^\infty(0,1)}$, $\|\tilde{y}(T_0)\|_{W^{1\infty}(0,1)}$ and $\|\sigma\|_{W^{2\infty}(0,1)}$.

On the other hand, in order to bound I from above we apply the Carleman estimate (44) to Equation (46) using $q_0 = \tilde{y}_x$ and $q_1 = \tilde{y}$, which are uniformly bounded in $L^\infty((0, T) \times (0, 1))$ by the hypothesis in Theorem 1.2. We obtain that

$$\begin{aligned}
I &= 2\int_0^1 \int_0^{T_0} w(t,x) w_t(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\le \left(\int_0^1 \int_0^{T_0} \lambda\phi(t,x)|w(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_0^1 \int_0^{T_0} \frac{|w_t(t,x)|^2}{\lambda\phi(t,x)}\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C\lambda^{-3} \int_0^1 \int_0^T e^{-2\lambda\phi} |f(x)\tilde{y}_{xxt}(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + C\lambda^{-3} \int_0^1 \int_0^T e^{-2\lambda\phi} |g(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\lambda^{-3} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t)\sigma^2(0)|v_{xx}(0,t)|^2 + \lambda\phi_x(0,t)\sigma^2(0)|v_{xxx}(0,t)|^2)\,\mathrm{d}t.
\end{aligned}$$

Step 3 Second use of the Carleman estimate

Considering that $g = uy_{xt} + u_x\, y_t$, we will now use a Carleman estimate for the solution of Equation (45) in order to manage the term in g of the previous inequality. The unknown trajectory y is nevertheless such that y_{xt} and y_t belong to $L^\infty(0, T; L^\infty(0, 1))$ since $y \in \mathcal{Z}$. Thus, we have

$$\begin{aligned}
\iint_Q e^{-2\lambda\phi}|g|^2 &\le 2\iint_Q e^{-2\lambda\phi}|u|^2|y_{xt}|^2 + 2\iint_Q e^{-2\lambda\phi}|u_x|^2|y_t|^2 \\
&\le C\iint_Q e^{-2\lambda\phi}\big(|u|^2 + |u_x|^2\big).
\end{aligned}$$

Then we can apply the Carleman estimate (44) to Equation (45), using the identity $\tilde{y}_x u + uu_x = uy_x$, and taking $q_0 = y_x$ and $q_1 = \tilde{y}$, which are bounded in $L^\infty((0, T) \times (0, 1))$. We can choose λ_0 as large as possible in Theorem 3.1: we then obtain

$$\begin{aligned}
\iint_Q e^{-2\lambda\phi}|g|^2 &\le C\lambda^{-5}\iint_Q e^{-2\lambda\phi}\big(\lambda^7|u|^2 + \lambda^5|u_x|^2\big) \\
&\le C\lambda^{-5}\iint_Q e^{-2\lambda\phi}|f\tilde{y}_{xx}|^2 \\
&\quad + C\lambda^{-5}\int_0^T e^{-2\lambda\phi(0,t)}(\lambda^3\phi_x^3(0,t)\sigma^2(0)|u_{xx}(0,t)|^2 + \lambda\phi_x(0,t)\sigma^2(0)|u_{xxx}(0,t)|^2)\,\mathrm{d}t.
\end{aligned}$$

Gathering all the estimates of I and g that were obtained above, we have

$$\begin{aligned}
&\int_0^1 e^{-2\lambda\phi(T_0,x)}|f(x)|^2\big|\tilde{y}_{xx}(T_0,x)\big|^2\,\mathrm{d}x - C\big\|u(T_0)\big\|^2_{H^4(0,1)} - C\big\|u(T_0)\big\|^4_{H^1(0,1)} \\
&\le C\lambda^{-3}\iint_Q e^{-2\lambda\phi}|f\tilde{y}_{xxt}|^2 + C\lambda^{-8}\iint_Q e^{-2\lambda\phi}|f\tilde{y}_{xx}|^2 \\
&\quad + C\lambda^{-8}\int_0^T e^{-2\lambda\phi(0,t)}(\lambda^3\phi_x^3(0,t)\sigma^2(0)|u_{xx}(0,t)|^2 + \lambda\phi_x(0,t)\sigma^2(0)|u_{xxx}(0,t)|^2)\mathrm{d}t \\
&\quad + C\lambda^{-3}\int_0^T e^{-2\lambda\phi(0,t)}(\lambda^3\phi_x^3(0,t)\sigma^2(0)|v_{xx}(0,t)|^2 + \lambda\phi_x(0,t)\sigma^2(0)|v_{xxx}(0,t)|^2)\mathrm{d}t.
\end{aligned}$$

From the hypothesis of the theorem, we have $\tilde{y} \in C([0,T]; H^6(0,1))$ $\tilde{y}_t \in C([0,T]; H^2(0,1))$ and $|\tilde{y}_{xx}(T_0,\cdot)| > \eta > 0$ in $(0, 1)$. Also using that the Carleman weight function satisfies (27), thus $e^{-2\lambda\phi(t,x)} \leq e^{-2\lambda\phi(T_0,x)}$ in $(0,T)\times(0,1)$, we obtain

$$\begin{aligned}
&\int_0^1 e^{-2\lambda\phi(T_0,x)}|f(x)|^2\,\mathrm{d}x \\
&\leq C\Big(\lambda^{-3}\int_0^1 e^{-2\lambda\phi(T_0,x)}|f(x)|^2\,\mathrm{d}x + \|u(T_0)\|^2_{H^4(0,1)} + \|u(T_0)\|^4_{H^1(0,1)} \\
&\quad + \lambda^{-8}\int_0^T e^{-2\lambda\phi(0,t)}(\lambda^3\phi_x^3(0,t)\sigma^2(0)|u_{xx}(0,t)|^2 + \lambda\phi_x(0,t)\sigma^2(0)|u_{xxx}(0,t)|^2)\,\mathrm{d}t \\
&\quad + \lambda^{-3}\int_0^T e^{-2\lambda\phi(0,t)}(\lambda^3\phi_x^3(0,t)\sigma^2(0)|v_{xx}(0,t)|^2 + \lambda\phi_x(0,t)\sigma^2(0)|v_{xxx}(0,t)|^2)\,\mathrm{d}t\Big).
\end{aligned}$$

Therefore, the regularity of ϕ (that come from the assumptions on β and ϕ_0) allows to prove that choosing λ_0 large enough, we obtain the existence of a constant C that depends on r, K, T, λ_0, m such that $\forall\lambda > \lambda_0$,

$$\begin{aligned}
\|f(x)\|^2_{L^2(0,1)} \leq C\Big(&\|u(T_0,\cdot)\|^2_{H^4(0,1)} + \|u(T_0,\cdot)\|^4_{H^1(0,1)} \\
&+ \|u_{xx}(\cdot,0)\|^2_{H^1(0,T)} + \|u_{xxx}(\cdot,0)\|^2_{H^1(0,T)}\Big).
\end{aligned}$$

This estimate leads to the local stability of the initial inverse problem since $f = \tilde{\gamma} - \gamma$ and $u = y - \tilde{y}$, and we have proved Theorem 1.2.

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