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Internal null controllability of a linear Schrödinger–KdV system on a bounded interval ☆

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Abstract

The control of a linear dispersive system coupling a Schrödinger and a linear Korteweg–de Vries equation is studied in this paper. The system can be viewed as three coupled real-valued equations by taking real and imaginary parts in the Schrödinger equation. The internal null controllability is proven by using either one complex-valued control on the Schrödinger equation or two real-valued controls, one on each equation. Notice that the single Schrödinger equation is not known to be controllable with a real-valued control. The standard duality method is used to reduce the controllability property to an observability inequality, which is obtained by means of a Carleman estimates approach.

Keywords: Dispersive system; Schrödinger equation; Korteweg–de Vries equation; Null controllability; Carleman estimates

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1. Introduction

In last years, a lot of works have been devoted to the study of controllability properties for systems of coupled partial differential equations, and new phenomena have appeared. For instance, some linear parabolic systems have been proven to be null controllable only if the time of control is large enough, which never happens when controlling single linear parabolic equations. Most of these works have dealt with the controllability of either parabolic (see [3,4]) or hyperbolic systems (see [1,2,5,10,17]). Different tools, as Carleman estimates, moment problems, energy methods and microlocal techniques, have been applied to obtain internal and boundary controllability results. In particular, efforts have been addressed to study the controllability of a given system with less controls than equations.

Concerning the controllability of dispersive systems, there are few results in the literature. Several Boussinesq systems have been considered in [23], where exact internal controllability results are proven. Other systems coupling Korteweg–de Vries equations have been studied in [14,22], where exact boundary controllability results have been established.

In this paper we are interested in a linear dispersive system posed on the interval $[0, 1]$ and formed by two coupled PDEs: a Schrödinger equation and a linear Korteweg–de Vries (KdV) equation. We consider internal controls supported on a nonempty open subset $\omega \subset (0, 1)$ and homogeneous boundary conditions.

Given $T > 0$, we denote $Q = (0, 1) \times (0, T)$ and $Q_\omega = \omega \times (0, T)$. Moreover, $\mathbf{1}_\omega$ stands for the characteristic function of ω and M, a_1, a_2, a_3, a_4 are given functions. Throughout this work, for a complex number z, we denote by $\overline{z}$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ the conjugate, the real part and the imaginary part of z, respectively.

The control system reads as

$$
\begin{cases}
i w_t + w_{xx} = a_1 w + a_2 y + h\mathbf{1}_\omega & \text{in} \quad Q, \\
y_t + y_{xxx} + (My)_x = Re(a_3 w) + a_4 y & \text{in} \quad Q, \\
w(0,t) = w(1,t) = 0 & \text{in} \quad (0,T), \\
y(0,t) = y(1,t) = y_x(1,t) = 0 & \text{in} \quad (0,T), \\
w(x,0) = w_0(x), \quad y(x,0) = y_0(x) & \text{in} \quad (0,1),
\end{cases}
\tag{1}
$$

where the state is given by the complex-valued function w and the real-valued function y, and the control is given by the complex-valued function h. System (1) is a linearized version of a Schrödinger–Korteweg–de Vries system used in fluid mechanics as well as plasma physics as a model of the interactions between a short-wave $w = w(x,t)$ and a longwave $y = y(x,t)$ (see for instance [19] where capillary-gravity waves are considered). Well posedness studies have been performed when the system is studied on the whole line [8,15] or on the torus [6].

Let us take a look at the controllability properties for each equation in our system. From now on, complex-valued function spaces are denoted using bold letters.

Concerning the Schrödinger equation posed on a domain $\Omega \subset \mathbb{R}^n$, it is well known that controllability is true if the region of control satisfies the geometric condition of the wave equation (see [20]). In the one-dimensional setting, this condition is fulfilled for an arbitrary open set $\omega \subset \Omega$. We refer to [21] and [25], were explicit observability inequalities were obtained, both in the L^2 and the H^1 settings. A Carleman inequality was obtained in [7] in the context of an inverse problem, which implies an observability result in H^1. In [27], an L^2 observability inequality is derived from a H^{-1} Carleman estimate. In [24], observability estimates have been obtained by the control transmutation method. In all the mentioned works, the control is a complex-valued function. As far as we know, the problem of controllability of the Schrödinger equation using a pure real or pure imaginary control is open. In this work, we are able to prove a H^1 Carleman inequality where the observation does not consist in the H^1 local term, but just the L^2 norms of the solution and the real part of its derivative (see Theorem 3.2). This is a first step in addressing that problem.

For the controllability of the KdV equation on an interval $[0, L]$, we refer to the recent work [11] where the internal null controllability is proven in the state space $L^2(0, L)$ with controls in $L^2(0, T; L^2(\omega))$. In [11], the authors prove a Carleman inequality, which has been obtained in an independent way to the one proved in the present paper (see Theorem 3.1). We refer to [13,26] for surveys on the controllability of the KdV equation.

In this article we obtain controllability of system (1) with a single complex-valued control. To our best knowledge, there are no previous control results about system (1). We hope the present paper will be the starting point for further research. Our first main result is the following.

Theorem 1.1. *Let $T > 0$. We suppose $M \in L^2(0, T; H^1(0, 1)) \cap L^\infty(0, T; L^2(0, 1))$, $a_1, a_4 \in L^\infty(0, T; W^{1,\infty}(0, 1))$, and $a_2, a_3 \in L^\infty(0, T; \mathbf{W}^{1,\infty}(0, 1))$. Suppose also that*

$$\mathrm{Im}(a_3) \in C([0, T]; W^{1,\infty}(0, 1)) \ \textit{with} \ |\mathrm{Im}(a_3)| \geq \delta \ \textit{in} \ \omega, \ \textit{for some} \ \delta > 0. \tag{2}$$

Then, for any $(w_0, y_0) \in \boldsymbol{H}^{-1}(0, 1) \times L^2(0, 1)$, there exists a control $h\mathbf{1}_\omega \in L^2(0, T; \boldsymbol{H}^{-1}(0, 1))$ such that the unique solution

$$(w, y) \in C([0, T], \boldsymbol{H}^{-1}(0, 1) \times L^2(0, 1))$$

of (1) satisfies

$$w(T, \cdot) = 0, \quad y(T, \cdot) = 0.$$

Remark 1.2. We actually obtain a control $h \in L^2(0, T; H^1(\omega)')$, where $H^1(\omega)'$ is the dual space of $H^1(\omega)$. In the previous theorem, $h\mathbf{1}_\omega$ denotes the element in $L^2(0, T; H^{-1}(0, 1))$ defined for almost every $t \in (0, T)$ by

$$\langle h(t, \cdot)\mathbf{1}_\omega, \theta\rangle_{H^{-1}(0,1), H_0^1(0,1)} = \langle h(t, \cdot), \theta\mathbf{1}_\omega\rangle_{(H^1(\omega))', H^1(\omega)}, \quad \forall \theta \in H_0^1(0, 1).$$

We recall that system (1) is formed by three real-valued equations: the KdV equation and the real and imaginary parts of the complex-valued Schrödinger equation. Hence, Theorem 1.1 states that this system is null controllable by using two real controls, given by the complex control h.

Moreover, we are also able to prove null-controllability by using two real-valued controls: either a purely real or a purely imaginary control h, and a control ℓ in the KdV equation. More precisely, for the system

$$\begin{cases} i w_t + w_{xx} = a_1 w + a_2 y + h\mathbf{1}_\omega & \text{in} \quad Q, \\ y_t + y_{xxx} + (My)_x = Re(a_3 w) + a_4 y + \ell\mathbf{1}_\omega & \text{in} \quad Q, \\ w(0, t) = w(1, t) = 0 & \text{in} \quad (0, T), \\ y(0, t) = y(1, t) = y_x(1, t) = 0 & \text{in} \quad (0, T), \\ w(x, 0) = w_0(x), \quad y(x, 0) = y_0(x) & \text{in} \quad (0, 1), \end{cases} \tag{3}$$

we have the following result.

Theorem 1.3. *Let $T > 0$. We suppose $M \in L^2(0,T; H^1(0,1)) \cap L^\infty(0,T; L^2(0,1))$, a_1, $a_4 \in L^\infty(0,T; W^{1,\infty}(0,1))$, and $a_2, a_3 \in L^\infty(0,T; \mathbf{W}^{1,\infty}(0,1))$. Suppose also that*

$$\mathrm{Im}(a_2) \in C([0,T]; W^{1,\infty}(0,1)) \text{ with } |\mathrm{Im}(a_2)| \geq \delta \text{ in } \omega, \text{ for some } \delta > 0. \tag{4}$$

Then, for any $(w_0, y_0) \in \boldsymbol{H}^{-1}(0,1) \times L^2(0,1)$, there exists a pair of real-valued controls $(h\mathbf{1}_\omega, \ell\mathbf{1}_\omega) \in L^2(0,T; H^{-1}(0,1) \times L^2(0,1))$, such that the unique solution

$$(w, y) \in C([0,T], \boldsymbol{H}^{-1}(0,1) \times L^2(0,1))$$

of (3) satisfies

$$w(T,\cdot) = 0, \quad y(T,\cdot) = 0.$$

Remark 1.4. Notice that, in Theorem 1.3, the control h acting on the Schrödinger equation is a real-valued function. If we consider the hypothesis $|\mathrm{Re}(a_2)| > 0$ instead of $|\mathrm{Im}(a_2)| > 0$, we still obtain a null-controllability result. In this case, the control of the Schrödinger equation is a pure imaginary-valued function.

Remark 1.5. Hypotheses (2) and (4) imply that some coupling terms in equations (1) and (3), respectively, do not vanish in the control zone. This is crucial, under our approach, to eliminate one of the observations (see Theorems 3.3 and 3.4). Similar conditions appear in related results about parabolic systems: see [4] and the references therein. Recently, null-controllability of hyperbolic and parabolic systems with disjoint control and coupling domains has been proved. For example [2], where both domains have to satisfy some geometric conditions, and [3], where it is needed a minimal time of controllability. It would be very interesting to know if similar results are true for system (1).

In order to prove Theorems 1.1 and 1.3, we follow the standard controllability–observability duality, which reduces a null controllability property to an observability inequality for the solutions of the adjoint system, which in this case is given by

$$\begin{cases} i\phi_t + \phi_{xx} = a_1\phi + \bar{a}_3\psi & \text{in} \quad Q, \\ -\psi_t - \psi_{xxx} - M\psi_x = \mathrm{Re}(\bar{a}_2\phi) + a_4\psi & \text{in} \quad Q, \\ \phi(0,t) = \phi(1,t) = 0 & \text{in} \quad (0,T), \\ \psi(0,t) = \psi(1,t) = \psi_x(0,t) = 0 & \text{in} \quad (0,T), \\ \phi(x,T) = \phi^T(x), \quad \psi(x,T) = \psi^T(x) & \text{in} \quad (0,1). \end{cases} \tag{5}$$

More precisely, we will prove the next result.

Theorem 1.6.

(a) *Under the hypothesis of Theorem 1.1, there exists a constant $C > 0$ such that*

$$\|\phi(\cdot,0)\|^2_{\mathbf{H}^1_0(0,1)} + \|\psi(\cdot,0)\|^2_{L^2(0,1)} \leq C \left(\iint_{Q_\omega} (|\phi|^2 + |\mathrm{Re}(\phi_x)|^2) dx dt \right), \tag{6}$$

for any $(\phi^T, \psi^T) \in \boldsymbol{H}_0^1(0,1) \times L^2(0,1)$, *where* $(\phi, \psi) \in C([0,T]; \boldsymbol{H}_0^1(0,1) \times L^2(0,1))$ *is the solution of system* (5).

(b) *Under the hypothesis of* Theorem 1.3, *there exists a constant* $C > 0$ *such that*

$$\|\phi(\cdot,0)\|^2_{\mathbf{H}_0^1(0,1)} + \|\psi(\cdot,0)\|^2_{L^2(0,1)} \leq C \left(\iint\limits_{Q_\omega} (|\mathrm{Re}(\phi)|^2 + |\mathrm{Re}(\phi_x)|^2 + |\psi|^2) dx dt \right) \tag{7}$$

for any $(\phi^T, \psi^T) \in \boldsymbol{H}_0^1(0,1) \times L^2(0,1)$, *where* $(\phi, \psi) \in C([0,T]; \boldsymbol{H}_0^1(0,1) \times L^2(0,1))$ *is the solution of system* (5).

Remark 1.7. Notice that, in inequality (7), ϕ appears in the observation only by its real part. This allows us to prove that the control h acting on the Schrödinger equation in (3) can be chosen as a real-valued function.

The work is organized as follows. In Section 2 we state well-posedness results we need in this work. In Section 3, we prove the Carleman estimates we will use later. In fact, we prove one-parameter Carleman estimates for the KdV and for the Schrödinger equations, and we use them in order to get appropriate Carleman estimates for the adjoint system (5). Section 4 is devoted to prove the observability inequalities stated in Theorem 1.6, and then to deduce the controllability results of Theorems 1.1 and 1.3.

2. Well-posedness

Let us introduce some functional spaces which will be used along the paper:

$$\begin{gathered}
X_0 := L^2(0,T; H^{-2}(0,1)), \quad X_1 := L^2(0,T; H_0^2(0,1)), \\
\tilde{X}_0 := L^1(0,T; H^{-1}(0,1)), \quad \tilde{X}_1 := L^1(0,T; H^3(0,1) \cap H_0^1(0,1)), \\
Y_0 := L^2(0,T; L^2(0,1)) \cap C([0,T]; H^{-1}(0,1)), \\
Y_1 := L^2(0,T; H^4(0,1)) \cap C([0,T]; H^3(0,1)).
\end{gathered} \tag{8}$$

In addition, we define (see e.g. [9]), for each $\theta \in [0,1]$, the interpolation spaces

$$X_\theta := (X_0, X_1)_{[\theta]}, \quad \tilde{X}_\theta := (\tilde{X}_0, \tilde{X}_1)_{[\theta]} \quad \text{and} \quad Y_\theta := (Y_0, Y_1)_{[\theta]}.$$

In this section we assume the following regularity of the coefficients:

$$\begin{aligned}
&a_1 \in L^\infty(0,T; W^{1,\infty}(0,1)), \qquad a_2 \in \boldsymbol{L}^\infty(Q), \\
&a_3 \in L^\infty(0,T; \boldsymbol{W}^{1,\infty}(0,1)), \qquad a_4 \in L^\infty(Q),\ M \in Y_{\frac{1}{4}}.
\end{aligned} \tag{9}$$

Notice that $Y_{\frac{1}{4}} = L^2(0,T; H_0^1(0,1)) \cap C([0,T]; L^2(0,1))$.

The main goal of this section is to prove the well posedness of system

$$\begin{cases} iw_t + w_{xx} = a_1 w + a_2 y + f_1 & \text{in} \quad Q, \\ y_t + y_{xxx} + (My)_x = \mathrm{Re}(a_3 w) + a_4 y + f_2 & \text{in} \quad Q, \\ w(0,t) = w(1,t) = 0 & \text{in} \quad (0,T), \\ y(0,t) = y(1,t) = y_x(1,t) = 0 & \text{in} \quad (0,T), \\ w(x,0) = w_0(x), \quad y(x,0) = y_0(x) & \text{in} \quad (0,1), \end{cases} \tag{10}$$

and its adjoint system given by

$$\begin{cases} i\phi_t + \phi_{xx} = a_1\phi + \bar{a}_3\psi + g_1 & \text{in} \quad Q, \\ -\psi_t - \psi_{xxx} - M\psi_x = \mathrm{Re}(\bar{a}_2\phi) + a_4\psi + g_2 & \text{in} \quad Q, \\ \phi(0,t) = \phi(1.t) = 0 & \text{in} \quad (0,T), \\ \psi(0,t) = \psi(1,t) = \psi_x(0,t) = 0 & \text{in} \quad (0,T), \\ \phi(x,T) = \phi_T(x), \quad \psi(x,T) = \psi_T(x) & \text{in} \quad (0,1). \end{cases} \tag{11}$$

Proposition 2.1. *Under hypotheses (9), for any* $(g_1, g_2) \in L^1(0,T; \boldsymbol{H}_0^1(0,1) \times L^2(0,1))$ *and* $(\phi_T, \psi_T) \in \boldsymbol{H}_0^1(0,1) \times L^2(0,1)$*, the system (11) has a unique solution*

$$(\phi, \psi) \in C([0,T]; \boldsymbol{H}_0^1(0,1)) \times Y_{\frac{1}{4}}.$$

Concerning system (10), we consider solutions in the sense of transposition.

Definition 2.2. Given $(w_0, y_0) \in \boldsymbol{H}^{-1}(0,1) \times L^2(0,1)$ and $(f_1, f_2) \in L^2(0,T; \boldsymbol{H}^{-1}(0,1) \times L^2(0,1))$, we say that $(w,y) \in L^\infty(0,T; \boldsymbol{H}^{-1}(0,1) \times L^2(0,1))$ is a solution (by transposition) of system (10) if

$$\begin{aligned} \int_0^T \langle w, \overline{g}_1 \rangle_{\boldsymbol{H}^{-1}, \boldsymbol{H}_0^1} dt + \iint_Q y g_2 \, dx dt = \int_0^T \langle f_1, \overline{\phi} \rangle_{\boldsymbol{H}^{-1}, \boldsymbol{H}_0^1} dt \\ + \iint_Q f_2 \psi \, dx dt + i \langle w_0, \bar{\phi}|_{t=0} \rangle_{\boldsymbol{H}^{-1}, \boldsymbol{H}_0^1} \\ + \int_0^1 y_0(x) \psi(x,0) dx, \end{aligned} \tag{12}$$

for all $(g_1, g_2) \in L^1(0,T; \boldsymbol{H}_0^1(0,1) \times L^2(0,1))$, where $(\phi, \psi) \in C([0,T]; \boldsymbol{H}_0^1(0,1)) \times Y_{\frac{1}{4}}$ is the solution of system (11) with $(\phi_T, \psi_T) = (0,0)$.

The following result holds.

Proposition 2.3. *Under hypotheses (9), for any* $(f_1, f_2) \in L^1(0,T; \boldsymbol{H}^{-1}(0,1) \times L^2(0,1))$ *and* $(w_0, y_0) \in \boldsymbol{H}^{-1}(0,1) \times L^2(0,1)$*, the system (10) has a unique solution*

$$(w, y) \in C([0,T]; \boldsymbol{H}^{-1}(0,1) \times L^2(0,1)). \tag{13}$$

Before proving Propositions 2.1 and 2.3, we recall some known results about the well-posedness of each equation appearing in system (10).

2.1. Previous regularity results

Let us consider the linear KdV equation given by

$$\begin{cases} -\psi_t - \psi_{xxx} - M\psi_x = g & \text{in} \quad Q, \\ \psi(0,t) = \psi(1,t) = \psi_x(0,t) = 0 & \text{in} \quad (0,T), \\ \psi(x,T) = \psi_T(x) & \text{in} \quad (0,1). \end{cases} \tag{14}$$

Proposition 2.4. *(See [18], Section 2.2.2.) Let $M \in Y_{\frac{1}{4}}$ be given. If $\psi_T \in L^2(0,1)$ and $g \in G$ with $G = L^2(0,T;H^{-1}(0,1))$ or $G = L^1(0,T;L^2(0,1))$, then system (14) has a unique solution $\psi \in Y_{\frac{1}{4}}$. Moreover, there exists a constant $C > 0$ such that*

$$\|\psi\|_{Y_{\frac{1}{4}}} \le C(\|g\|_G + \|\psi_T\|_{L^2(0,1)}). \tag{15}$$

In the case $M = 0$, we have the following improved regularity results.

Proposition 2.5. *(See [18], Section 2.3.1.) Suppose that $M = 0$. If $\psi_T \in H^3(0,1)$ is such that $\psi_T(0) = \psi_T(1) = \psi_T'(1) = 0$, and $g \in G$ with $G = L^2(0,T;H_0^2(0,1))$ or $G = L^1(0,T;H^3(0,1) \cap H_0^2(0,1))$, then system (14) has a unique solution $\psi \in Y_1$. Moreover, there exists a constant $C > 0$ such that*

$$\|\psi\|_{Y_1} \le C(\|g\|_G + \|\psi_T\|_{H^3(0,1)}). \tag{16}$$

Proposition 2.6. *(See [18], Section 2.3.2.) Let $\theta \in [\frac{1}{4}, 1]$ be given and suppose $M = 0$ and $\psi_T = 0$. If $g \in G$ with $G = X_\theta$ or $G = \tilde{X}_\theta$, then system (14) has a unique solution $\psi \in Y_\theta$. Moreover, there exists a constant $C > 0$ such that*

$$\|\psi\|_{Y_\theta} \le C\|g\|_G. \tag{17}$$

Let us consider now the linear Schrödinger equation

$$\begin{cases} i\phi_t + \phi_{xx} = a_1\phi + g & \text{in} \quad Q, \\ \phi(0,t) = \phi(1,t) = 0 & \text{in} \quad (0,T), \\ \phi(x,T) = \phi_T(x) & \text{in} \quad (0,1). \end{cases} \tag{18}$$

Proposition 2.7. *(See [12].) Suppose $a_1 \in \mathbf{L}^\infty(0,T;\mathbf{W}^{1,\infty}(0,1))$. For any $\phi_T \in \mathbf{X}$ and $g \in L^1(0,T;\mathbf{X})$, with $\mathbf{X} = \mathbf{L}^2(0,1)$ or $\mathbf{X} = \boldsymbol{H}_0^1(0,1)$, there exists a unique solution $\phi \in C([0,T];\mathbf{X})$ of system (18).*

2.2. Proofs of Propositions 2.1 and 2.3

Proof of Proposition 2.1. Let us consider the map

$$\Pi : L^1(0,T;L^2(0,1)) \to [C([0,T];L^2(0,1))]^2$$

defined by $\Pi\tilde{\psi} = (\phi,\psi)$, where

$$\begin{cases} i\phi_t + \phi_{xx} = a_1\phi + \bar{a}_3\tilde{\psi} + g_1 & \text{in} \quad Q, \\ -\psi_t - \psi_{xxx} - M\psi_x = \mathrm{Re}(\bar{a}_2\phi) + a_4\tilde{\psi} + g_2 & \text{in} \quad Q, \\ \phi(0,t) = \phi(1,t) = 0 & \text{in} \quad (0,T), \\ \psi(0,t) = \psi(1,t) = \psi_x(0,t) = 0 & \text{in} \quad (0,T), \\ \phi(x,T) = \phi_T(x), \quad \psi(x,T) = \psi_T(x) & \text{in} \quad (0,1). \end{cases} \tag{19}$$

From Proposition 2.7, we get $\phi \in C([0,T];L^2(0,1))$, and then, Proposition 2.4 gives us $\psi \in Y_{\frac{1}{4}}$ and hence operator Π is well defined. Moreover, the follow identity holds

$$\int_0^T \langle \psi_t, \pi\rangle_{\mathcal{D}',\mathcal{D}} = l(\pi), \tag{20}$$

where

$$l(\pi) = -\iint_Q (\psi_x\pi_{xx} + M\psi_x\pi - \mathrm{Re}(\bar{a}_2\phi)\pi + a_4\tilde{\psi}\pi + g_2\pi)\,dxdt, \quad \forall \pi \in \mathcal{D}(Q). \tag{21}$$

Notice that l is continuous in $L^2(0,T;H_0^2(0,1))$ and, in this way, ψ_t is a distribution in $L^2(0,T;H^{-2}(0,1))$. Now we set

$$\Lambda : L^1(0,T;L^2(0,1)) \to L^1(0,T;L^2(0,1))$$

by $\Lambda\tilde{\psi} = \psi$. Then, we get that the range of Λ is contained in

$$\mathcal{W} = \{\psi \in L^2(0,T;H_0^1(0,1));\, \psi_t \in L^2(0,T;H^{-2}(0,1))\},$$

which, by the Aubin–Lions Lemma, is a compact subset of $L^1(0,T;L^2(0,1))$. Thus, by Schauder's Theorem, Λ has a fixed point $\psi \in L^2(0,T;H_0^1(0,1))$, and then $(\phi,\psi) = \Pi\psi$ solves system (11). Now, since $a_3 \in L^\infty(0,T;\boldsymbol{W}^{1,\infty}(0,1))$, we get $\bar{a}_3\psi \in L^2(0,T;\boldsymbol{H}_0^1(0,1))$ and, from Proposition 2.7, we deduce that $(\phi,\psi) \in C([0,T];\boldsymbol{H}_0^1(0,1)) \times Y_{\frac{1}{4}}$, which ends the proof. □

Remark 2.8. If we suppose $(\phi_T,\psi_T) \in \mathbf{H}_0^1(0,1) \times H_0^1(0,1)$, condition (9) and the additional regularity $a_4 \in L^\infty(0,T;W^{1,\infty}(0,1))$, then we can proceed as in the proof of Proposition 2.1 to obtain a solution $(\phi,\psi) \in C([0,T];\boldsymbol{H}_0^1(0,1)) \times Y_{\frac{1}{2}}$.

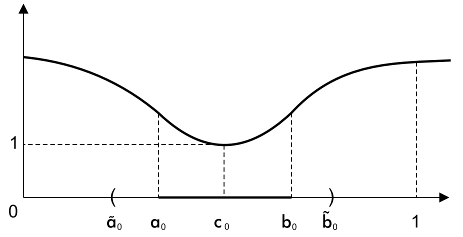

Fig. 1. The weight function ϕ_0.

Proof of Proposition 2.3. The right hand side of (12) defines a linear functional which maps $(g_1, g_2) \in L^1(0, T; \boldsymbol{H}_0^1(0, 1) \times L^2(0, 1))$ to the corresponding value in $\mathbb{R}$. By the regularity stated in Proposition 2.1, this functional is continuous. By Riesz's Theorem, there exists a unique pair $(w, y) \in L^\infty(0, T; \boldsymbol{H}^{-1}(0, 1) \times L^2(0, 1))$ satisfying (12). The regularity (13) follows by a density argument. $\square$

3. Carleman estimates

This section is devoted to the proof of several appropriate Carleman estimates which will be useful in next section in order to prove the observability inequalities and then the null controllability results of our Schrödinger–KdV system. First, we deal with the single equations by separate and then we address the coupled system. In all these cases, we use the same weight functions defined as follows.

Let us suppose that $\omega = (\tilde{a}_0, \tilde{b}_0) \subset (0, 1)$ and let $[a_0, b_0] \subset \omega$. Let $c_0 = (a_0 + b_0)/2$ and consider, for $K_1, K_2 > 0$ to be chosen later, the functions (see Fig. 1)

$$\phi_0(x) = -K_1 \exp(-K_2(x - c_0)^2) + K_1 + 1, \tag{22}$$

$$\xi(t) = \frac{1}{t(T - t)} \quad \text{and} \quad \Phi(x, t) = \phi_0(x)\xi(t). \tag{23}$$

We take $K_2 = \frac{1}{2(c_0 - a_0)^2}$. If $c_0 \geq 1/2$, then the constant K_1 is chosen such that

$$3K_1 < \frac{1}{1 - \exp(-K_2 c_0^2)}.$$

If $c_0 < 1/2$, K_1 is chosen satisfying

$$3K_1 < \frac{1}{1 - \exp(-K_2(1 - c_0)^2)}.$$

In both cases, there exists a positive constant C such that

$$\begin{aligned}
&-\phi_0''(x) \geq C \quad \text{and} \quad |\phi_0'(x)|^2 \geq C \quad \text{in } [0, 1] \setminus \overline{\omega}, \\
&\phi_0'(1) > 0 \quad \text{and} \quad \phi_0'(0) < 0, \\
&8\check{\Phi}(t) - 7\hat{\Phi}(t) > 0 \qquad\qquad \text{in } [0, T],
\end{aligned} \tag{24}$$

where $(\hat{\Phi}(t), \check{\Phi}(t)) = (\max_{x \in [0,1]} \Phi(t, x), \min_{x \in [0,1]} \Phi(t, x))$.

3.1. Carleman estimate for the KdV equation

Following [18], we get a Carleman inequality for the KdV equation. This result is similar than the estimate obtained in [11], and has been independently obtained. However, in order to deal with the system, we will use the same weight function for the Carleman estimates of both equations.

Theorem 3.1. *If $M \in L^2(0,T;H^1(0,1)) \cap L^\infty(0,T;L^2(0,1))$, then there exist $C_0 > 0$ and $s_0 \geq 1$ such that*

$$s^5 \iint_Q e^{-2s\Phi}\xi^5|v|^2\,dxdt + s^3 \iint_Q e^{-2s\Phi}\xi^3|v_x|^2\,dxdt + s\iint_Q e^{-2s\Phi}\xi|v_{xx}|^2\,dxdt$$
$$\leq C_0\left(\iint_Q e^{-2s\Phi}|Lv|^2\,dxdt + s^5\iint_{Q_\omega} e^{-2s\Phi}\xi^5|v|^2\,dxdt + s\iint_{Q_\omega} e^{-2s\Phi}\xi|v_{xx}|^2\,dxdt\right) \quad (25)$$

for all $s > s_0$ and $v \in L^2(0,T;H^2 \cap H_0^1(0,1))$ with $v_x(0,t) = 0$ and such that $Lv := (v_t + v_{xxx} + Mv_x)$ belongs to $L^2(0,T;L^2(0,1))$.

Proof. Let us define

$$w = e^{-s\Phi}v, \quad (26)$$

for each $s > 0$ and $v \in C^\infty(Q)$ with $v(0,t) = v(1,t) = v_x(0,t) = 0$. Then $w(x,0) = w(x,T) = 0$ and

$$\begin{aligned}
v_t &= se^{s\Phi}\Phi_t w + e^{s\Phi}w_t,\\
v_x &= se^{s\Phi}\Phi_x w + e^{s\Phi}w_x,\\
v_{xx} &= s^2e^{s\Phi}(\Phi_x)^2w + se^{s\Phi}\Phi_{xx}w + 2se^{s\Phi}\Phi_x w_x + e^{s\Phi}w_{xx},\\
v_{xxx} &= s^3e^{s\Phi}(\Phi_x)^3w + 3s^2e^{s\Phi}\Phi_x\Phi_{xx}w + 3s^2e^{s\Phi}(\Phi_x)^2w_x,\\
&\quad + se^{s\Phi}\Phi_{xxx}w + 3se^{s\Phi}\Phi_{xx}w_x + 3se^{s\Phi}\Phi_x w_{xx} + e^{s\Phi}w_{xxx}.
\end{aligned}$$

In this way, if we define $L_\Phi w := e^{-s\Phi}L(e^{s\Phi}w)$, then we have the following identity

$$\begin{aligned}
L_\Phi w &= s\Phi_t w + w_t + s^3(\Phi_x)^3w + 3s^2\Phi_x\Phi_{xx}w + 3s^2(\Phi_x)^2w_x + s\Phi_{xxx}w\\
&\quad + 3s\Phi_{xx}w_x + 3s\Phi_x w_{xx} + w_{xxx} + M(s\Phi_x w + w_x).
\end{aligned} \quad (27)$$

If we write

$$\begin{aligned}
L_1 w &= w_t + w_{xxx} + 3s^2(\Phi_x)^2w_x,\\
L_2 w &= 3s\Phi_x w_{xx} + s^3(\Phi_x)^3w + 3s\Phi_{xx}w_x,
\end{aligned} \quad (28)$$

we have that

$$\|L_1 w\|^2_{L^2(Q)} + \|L_2 w\|^2_{L^2(Q)} + 2(L_1 w, L_2 w)_{L^2(Q)} = \|L_\Phi w - Rw\|^2_{L^2(Q)}, \tag{29}$$

where

$$Rw = s\Phi_t w + s\Phi_{xxx} w + 3s^2 \Phi_x \Phi_{xx} w + M(s\Phi_x w + w_x).$$

We now examine each integral term coming from $(L_1 w, L_2 w)_{L^2(Q)}$. Denoting I_{ij} the L^2-product of the i-th term of $L_1 w$ with the j-th term of $L_2 w$, we have:

$$I_{11} = 3s \iint_Q \Phi_x w_t w_{xx} \, dxdt = -3s \iint_Q \Phi_{xx} w_t w_x \, dxdt + \frac{3}{2} s \iint_Q \Phi_{xt} |w_x|^2 \, dxdt, \tag{30}$$

$$I_{12} = \frac{1}{2} s^3 \iint_Q (\Phi_x)^3 \frac{d}{dt} |w|^2 \, dxdt = -\frac{3}{2} s^3 \iint_Q (\Phi_x)^2 \Phi_{xt} |w|^2 \, dxdt, \tag{31}$$

$$I_{13} = 3s \iint_Q \Phi_{xx} w_x w_t \, dxdt = \frac{3}{2} s \iint_Q \Phi_{xt} |w_x|^2 \, dxdt - I_{11}, \tag{32}$$

$$\begin{aligned} I_{21} &= \frac{3}{2} s \iint_Q \Phi_x \partial_x |w_{xx}|^2 \, dxdt \\ &= \frac{3}{2} s \int_0^T \Phi_x(1,t) |w_{xx}(1,t)|^2 \, dt - \frac{3}{2} s \int_0^T \Phi_x(0,t) |w_{xx}(0,t)|^2 \, dt \\ &\quad - \frac{3}{2} s \iint_Q \Phi_{xx} |w_{xx}|^2 \, dxdt, \end{aligned} \tag{33}$$

$$\begin{aligned} I_{22} &= s^3 \iint_Q (\Phi_x)^3 w w_{xxx} \, dxdt = -3s^3 \iint_Q (\Phi_x)^2 \Phi_{xx} w w_{xx} \, dxdt \\ &\quad - \frac{1}{2} s^3 \iint_Q (\Phi_x)^3 \partial_x |w_x|^2 \, dxdt \\ &= -\frac{3}{2} s^3 \iint_Q ((\Phi_x)^2 \Phi_{xx})_{xx} |w|^2 \, dxdt + \frac{9}{2} s^3 \iint_Q (\Phi_x)^2 \Phi_{xx} |w_x|^2 \, dxdt \\ &\quad - \frac{1}{2} s^3 \int_0^T (\Phi_x(1,t))^3 |w_x(1,t)|^2 \, dt, \end{aligned} \tag{34}$$

$$
\begin{aligned}
I_{23} &= 3s \int_0^T \Phi_{xx}(1,t) w_x(1,t) w_{xx}(1,t)\, dt - \frac{3}{2} s \iint_Q \Phi_{xxx} \partial_x |w_x|^2 \, dxdt \\
&\quad - 3s \iint_Q \Phi_{xx} |w_{xx}|^2 \, dxdt \\
&= 3s \int_0^T \Phi_{xx}(1,t) w_x(1,t) w_{xx}(1,t) dt - \frac{3}{2} s \int_0^T \Phi_{xxx}(1,t) |w_x(1,t)|^2 \, dt \\
&\quad + \frac{3}{2} s \iint_Q \Phi_{xxxx} |w_x|^2 \, dxdt - 3s \iint_Q \Phi_{xx} |w_{xx}|^2 \, dxdt,
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
I_{31} &= \frac{9}{2} s^3 \iint_Q \Phi_x^3 \partial_x |w_x|^2 \, dxdt = \frac{9}{2} s^3 \int_0^T \Phi_x(1,t)^3 |w_x(1,t)|^2 \, dt \\
&\quad - \frac{27}{2} s^3 \iint_Q (\Phi_x)^2 \Phi_{xx} |w_x|^2 \, dxdt,
\end{aligned} \tag{36}
$$

$$
I_{32} = \frac{3}{2} s^5 \iint_Q (\Phi_x)^5 \partial_x |w|^2 \, dxdt = -\frac{15}{2} s^5 \iint_Q (\Phi_x)^4 \Phi_{xx} |w|^2 \, dxdt, \tag{37}
$$

and

$$
I_{33} = 9 s^3 \iint_Q (\Phi_x)^2 \Phi_{xx} |w_x|^2 \, dxdt. \tag{38}
$$

Gathering all the computations, we get

$$
\begin{aligned}
(L_1 w, L_2 w)_{L^2(Q)} &= \iint_Q \left(-\frac{3}{2} s^3 (\Phi_x)^2 \Phi_{xt} - \frac{3}{2} s^3 ((\Phi_x)^2 \Phi_{xx})_{xx} - \frac{15}{2} s^5 (\Phi_x)^4 \Phi_{xx} \right) |w|^2 \, dxdt \\
&\quad + \iint_Q \left(\frac{3}{2} s \Phi_{xt} + \frac{3}{2} s \Phi_{xxxx} \right) |w_x|^2 \, dxdt - \frac{9}{2} s \iint_Q \Phi_{xx} |w_{xx}|^2 \, dxdt \\
&\quad + \frac{3}{2} s \int_0^T \left(\Phi_x(1,t) |w_{xx}(1,t)|^2 - \Phi_x(0,t) |w_{xx}(0,t)|^2 \right) dt \\
&\quad + \int_0^T \left(-\frac{3}{2} \Phi_{xxx}(1,t) + 4 s^3 (\Phi_x(1,t))^3 \right) |w_x(1,t)|^2 \, dt \\
&\quad + 3s \int_0^T \Phi_{xx}(1,t) w_x(1,t) w_{xx}(1,t) \, dt.
\end{aligned} \tag{39}
$$

Replacing (39) in (29) we obtain

$$
\begin{aligned}
&\|L_1 w\|^2_{L^2(Q)} + \|L_2 w\|^2_{L^2(Q)} - 15s^5 \iint_Q (\Phi_x)^4 \Phi_{xx} |w|^2\, dxdt \\
&\quad - 9s \iint_Q \Phi_{xx} |w_{xx}|^2\, dxdt + 3s \int_0^T (\Phi_x(1,t)|w_{xx}(1,t)|^2 - \Phi_x(0,t)|w_{xx}(0,t)|^2)\, dt \\
&\quad + 8s^3 \int_0^T (\Phi_x(1,t))^3 |w_x(1,t)|^2\, dt = \|L_\Phi w - Rw\|^2_{L^2(Q)} + \Psi(w),
\end{aligned}
\tag{40}
$$

where

$$
\begin{aligned}
\Psi(w) = &\iint_Q \left(3s^3 (\Phi_x)^2 \Phi_{xt} + 3s^3 ((\Phi_x)^2 \Phi_{xx})_{xx}\right) |w|^2\, dxdt \\
&- \iint_Q (3s\Phi_{xt} + 3s\Phi_{xxxx})\, |w_x|^2\, dxdt \\
&3 \int_0^T \Phi_{xxx}(1,t)|w_x(1,t)|^2 dt - 6s \int_0^T \Phi_{xx}(1,t) w_x(1,t) w_{xx}(1,t)\, dt.
\end{aligned}
$$

Integrating by parts and using Young inequality we also have

$$
s^3 \iint_Q \xi^3 |w_x|^2 dxdt \le s^5 \iint_Q \xi^5 |w_{xx}|^2 dxdt + s \iint_Q \xi |w|^2 dxdt. \tag{41}
$$

Consider $\omega_0 \subset\subset \omega$ such that hypotheses (24) still hold in ω_0. Hence, combining (40) and (41) we have that there exists $C > 0$ such that

$$
\begin{aligned}
&\|L_1 w\|^2_{L^2(Q)} + \|L_2 w\|^2_{L^2(Q)} + \iint_Q \left(s^5 \xi^5 |w|^2 + s^3 \xi^3 |w_x|^2 + s\xi |w_{xx}|^2\right) dxdt \\
&\quad + s \int_0^T \xi (|w_{xx}(1,t)|^2 + |w_{xx}(0,t)|^2)\, dt + s^3 \int_0^T \xi^3 |w_x(1,t)|^2\, dt \\
&\le C \iint_Q |L_\phi w|^2\, dxdt \\
&\quad + C \iint_{Q_{\omega_0}} \left(s^5 \xi^5 |w|^2 + s\xi |w_{xx}|^2\right) dxdt + C\|Rw\|^2_{L^2(Q)} + C\Psi(w).
\end{aligned}
\tag{42}
$$

In order to estimate Ψ, notice that there exists $C > 0$ such that

$$\int\int_Q (3s^3(\Phi_x)^2\Phi_{xt} + 3s^3((\Phi_x)^2\Phi_{xx})_{xx})|w|^2\,dxdt \le Cs^3 \iint_Q \xi^4|w|^2\,dxdt, \tag{43}$$

and

$$\iint_Q (3s\Phi_{xt} + 3s\Phi_{xxxx})|w_x|^2\,dx\,dt \le Cs \iint_Q \xi^2|w_x|^2\,dxdt. \tag{44}$$

We also have that

$$\begin{aligned} &3\int_0^T \Phi_{xxx}(1,t)|w_x(1,t)|^2dt - 12s\int_0^T \Phi_{xx}(1,t)w_x(1,t)w_{xx}(1,t)dt \\ &\le Cs^2\int_0^T \xi^3(t)|w_x(1,t)|^2dt + C\int_0^T \xi(t)|w_{xx}(1,t)|^2dt. \end{aligned} \tag{45}$$

Combining (43), (44) and (45) we obtain

$$\begin{aligned} |\Psi(w)| \le C\Bigg(&s^3 \iint_Q \xi^4|w|^2\,dxdt + s\iint_Q \xi^2|w_x|^2\,dxdt \\ &+ s^2\int_0^T \xi^3(t)|w_x(1,t)|^2\,dt + \int_0^T \xi(t)|w_{xx}(1,t)|^2\,dt\Bigg). \end{aligned} \tag{46}$$

Let $N(w)$ be the left hand side of (42). For any $\varepsilon > 0$ there exists $s_1 > 1$ such that

$$|\Psi(w)| \le \varepsilon N(w), \tag{47}$$

for all $s \ge s_1$.

In order to estimate Rw, we use that $H^{\frac{3}{4}}(0,1)$ embeds in $L^\infty(0,1)$ to get

$$\begin{aligned} \|Mw_x\|_{L^2(Q)} &\le C\|M\|_{L^\infty(0,T;L^2(0,1))}\|w\|_{L^2(0,T;H^{\frac{7}{4}}(0,1))} \\ &\le C\|M\|_{L^\infty(0,T;L^2(0,1))}\left(\|w\|_{L^2(0,T;H^2(0,1))} + \|w\|_{L^2(0,T;H^1(0,1))}\right). \end{aligned}$$

Then, there exists $s_2 \ge 1$ such that

$$\|Rw\|^2_{L^2(Q)} \le C\left(s^4\iint_Q \xi^4|w|^2\,dxdt + \iint_Q |w_x|^2\,dxdt + \iint_Q |w_{xx}|^2\,dxdt\right) \le \varepsilon N(w), \tag{48}$$

for $s \geq s_2$. From (42), (47) and (48) we obtain

$$s^5 \iint\limits_Q \xi^5 |w|^2 \, dxdt + s^3 \iint\limits_Q \xi^3 |w_x|^2 \, dxdt + s \iint\limits_Q \xi |w_{xx}|^2 \, dxdt$$
$$\leq C \iint\limits_Q |L_\phi w|^2 \, dxdt + Cs^5 \iint\limits_{Q_{\omega_0}} \xi^5 |w|^2 \, dxdt + Cs \iint\limits_{Q_{\omega_0}} \xi |w_{xx}|^2 \, dxdt. \tag{49}$$

Now we get an estimate in variable v. Taking into account (26), we have that

$$e^{-2s\Phi}|v_x|^2 \leq C(s^2\xi^2|w|^2 + |w_x|^2) \quad \text{and}$$
$$e^{-2s\Phi}|v_{xx}|^2 \leq C(s^4\xi^4|w|^2 + s^2\xi^2|w_x|^2 + |w_{xx}|^2). \tag{50}$$

Also from (26) we get

$$|w_{xx}|^2 \leq Ce^{-2s\Phi}\left(s^4\xi^4|v|^2 + s^2\xi^2|v_x|^2 + |v_{xx}|^2\right). \tag{51}$$

From (49) to (51) we obtain (25). $\square$

3.2. *Carleman inequality for the Schrödinger equation*

This section is devoted to prove the one parameter Carleman estimate for the Schrödinger equation given in the following theorem.

Theorem 3.2. *There exist constants $C > 0$ and $s_0 \geq 1$ such that*

$$s \iint\limits_Q e^{-2s\Phi}\xi |p_x|^2 \, dxdt + s^3 \iint\limits_Q e^{-2s\Phi}\xi^3 |p|^2 \, dxdt \leq C \iint\limits_Q e^{-2s\Phi}|Bp|^2 \, dxdt$$
$$+ Cs^3 \iint\limits_{Q_\omega} \xi^3 e^{-2s\Phi}|p|^2 \, dxdt + Cs \iint\limits_{Q_\omega} e^{-2s\Phi}\xi |\mathrm{Re}(p_x)|^2 \, dxdt, \tag{52}$$

for all $s > s_0$, and $p \in L^2(0,T;\mathbf{H}_0^1(0,1))$ such that $Bp := (ip_t + p_{xx}) \in L^2(0,T;\mathbf{L}^2(0,1))$.

Proof. Let us define

$$q = e^{-s\Phi} p \tag{53}$$

for each $s > 0$ and $p \in \boldsymbol{C}^\infty(Q)$ such that $p(0,\cdot) = p(1,\cdot) = 0$. Hence we have

$$B_\Phi q := e^{-s\Phi} B(e^{-s\Phi} q) = i\,(s\Phi_t q + q_t) + s^2\Phi_x^2 q + s\Phi_{xx} q + 2s\Phi_x q_x + q_{xx}. \tag{54}$$

If we denote

$$B_1 q = i q_t + q_{xx} + s^2 \Phi_x^2 q \quad \text{and}$$
$$B_2 q = 2s \Phi_x q_x + s \Phi_{xx} q, \tag{55}$$

then we get

$$\|B_1 q\|_{L^2(Q)}^2 + \|B_2 q\|_{L^2(Q)}^2 + 2Re \iint\limits_Q B_1 q \overline{B_2 q} dxdt$$
$$\leq C \left(\iint\limits_Q |B_\Phi q|^2 \, dxdt + s^2 \iint\limits_Q |\Phi_t|^2 |q|^2 \, dxdt \right). \tag{56}$$

In order to analyze the term $(B_1 q, B_2 q)_{L^2(Q)}$, we denote by F_{ij} the L^2-product of the i-th term of $B_1 q$ with the j-th term of $B_2 q$. We have that

$$F_{11} = 2sRe \iint\limits_Q i q_t \Phi_x \bar{q}_x \, dxdt = -2s\text{Im} \iint\limits_Q q_t \Phi_x \bar{q}_x \, dxdt$$
$$= 2s\text{Im} \iint\limits_Q \Phi_x q_{tx} \bar{q} \, dxdt + 2s\text{Im} \iint\limits_Q \Phi_{xx} q_t \bar{q} \, dxdt$$
$$= -2s\text{Im} \iint\limits_Q \Phi_{xt} q_x \bar{q} \, dx, dt - 2s\text{Im} \int\limits_Q \Phi_x q_x \bar{q}_t \, dxdt$$
$$+ 2s\text{Im} \iint\limits_Q \Phi_{xx} q_t \bar{q} \, dxdt$$
$$= -2s\text{Im} \iint\limits_Q \Phi_{xt} q_x \bar{q} \, dxdt - F_{11} + 2s\text{Im} \iint\limits_Q \Phi_{xx} q_t \bar{q} \, dxdt. \tag{57}$$

In this way

$$F_{11} = -s\text{Im} \iint\limits_Q \Phi_{xt} q_x \bar{q} \, dxdt + s\text{Im} \iint\limits_Q \Phi_{xx} q_t \bar{q} \, dxdt \tag{58}$$

and

$$F_{12} = s\text{Re} \iint\limits_Q i q_t \Phi_{xx} \bar{q} \, dxdt. \tag{59}$$

We also have that

$$F_{21} = 2s\mathrm{Re}\iint_Q q_{xx}\Phi_x\bar{q}_x\,dxdt = s\mathrm{Re}\iint_Q \Phi_x\partial_x|q_x|^2\,dxdt$$

$$= s\mathrm{Re}\int_0^T \left(\Phi_x(1,t)|q_x(1,t)|^2 - \Phi_x(0,t)|q_x(0,t)|^2\right)dt - s\mathrm{Re}\iint_Q \Phi_{xx}|q_x|^2\,dxdt, \tag{60}$$

and

$$F_{22} = s\mathrm{Re}\iint_Q q_{xx}\Phi_{xx}\bar{q}\,dxdt = -s\mathrm{Re}\iint_Q \left(q_x\Phi_{xxx}\bar{q} + \Phi_{xx}|q_x|^2\right)dxdt$$

$$= -\frac{s}{2}\mathrm{Re}\iint_Q \Phi_{xxx}\partial_x|q|^2\,dxdt - s\mathrm{Re}\iint_Q \Phi_{xx}|q_x|^2\,dxdt$$

$$= \frac{s}{2}\mathrm{Re}\iint_Q \Phi_{xxxx}|q|^2\,dxdt - s\mathrm{Re}\iint_Q \Phi_{xx}|q_x|^2\,dxdt. \tag{61}$$

To finish we have

$$F_{31} = 2s^3\mathrm{Re}\iint_Q \Phi_x^3 q\bar{q}_x\,dxdt = s^3\mathrm{Re}\iint_Q \Phi_x^3\partial_x|q|^2\,dxdt$$

$$= -3s^3\mathrm{Re}\iint_Q \Phi_x^2\Phi_{xx}|q|^2\,dxdt, \tag{62}$$

and

$$F_{32} = s^3\mathrm{Re}\iint_Q \Phi_x^2\Phi_{xx}|q|^2\,dxdt. \tag{63}$$

Gathering all the previous integral terms we get

$$\mathrm{Re}(B_1q, \overline{B_2q})_{L^2(Q)} = s\mathrm{Re}\int_0^T \left(\Phi_x(1,t)|q_x(1,t)|^2 - \Phi_x(0,t)|q_x(0,t)|^2\right)dt$$

$$- s\mathrm{Im}\iint_Q \Phi_{xt}q_x\bar{q}\,dxdt - s\mathrm{Re}\iint_Q \Phi_{xx}|q_x|^2\,dxdt$$

$$\frac{s}{2}\mathrm{Re}\iint_Q \Phi_{xxxx}|q|^2\,dxdt - s\mathrm{Re}\iint_Q \Phi_{xx}|q_x|^2\,dxdt$$

$$- 2s^3\mathrm{Re}\iint_Q \Phi_x^2\Phi_{xx}|q|^2\,dxdt. \tag{64}$$

We have that there exists $\omega_0 \subset\subset \omega$ such that hypotheses (24) still hold in ω_0. Hence, from (64) we have that there exist constants $C > 0$ and s_1 such that

$$s^3 \iint_Q \xi^3 |q|^2 \, dx dt + s \iint_Q \xi |q_x|^2 \, dx dt$$
$$\leq C \left(\iint_Q |B_\Phi q|^2 \, dx dt + s^3 \iint_{Q_{\omega_0}} \xi^3 |q|^2 \, dx dt + s \iint_{Q_{\omega_0}} \xi |q_x|^2 \, dx dt \right), \tag{65}$$

for all $s \geq s_1$. Taking into account that $p = e^{s\Phi} q$, we have that

$$e^{-2s\Phi} |p_x|^2 \leq C(s^2 \xi^2 |q|^2 + |q_x|^2) \quad \text{and}$$
$$|q_x|^2 \leq C e^{-2s\Phi} (s^2 \xi^2 |p|^2 + |q_x|^2). \tag{66}$$

By (65) and (66) we get the following Carleman estimate

$$s^3 \iint_Q e^{-2s\Phi} \xi^3 |p|^2 \, dx dt + s \iint_Q e^{-2s\Phi} \xi |p_x|^2 \, dx dt$$
$$\leq \iint_Q e^{-2s\Phi} |Bp|^2 \, dx dt + s^3 \iint_{Q_{\omega_0}} e^{-2s\Phi} \xi^3 |p|^2 \, dx dt + s \iint_{Q_{\omega_0}} e^{-2s\Phi} \xi |p_x|^2 \, dx dt. \tag{67}$$

To conclude the proof, it is sufficient to obtain an estimate for the imaginary part of p_x, obtaining in this way (52). In order to do this, we decompose the Schrödinger equation into the real and imaginary parts. We write $p_1 = \mathrm{Re}(p)$ and $p_2 = \mathrm{Im}(p)$. Then Schrödinger equation is equivalent to the system given by

$$\begin{cases} p_{1t} + p_{2xx} = \mathrm{Im}(Bp) & \text{in} \quad Q, \\ -p_{2t} + p_{1xx} = \mathrm{Re}(Bp) & \text{in} \quad Q. \end{cases} \tag{68}$$

Let us take $\rho \in C_0^\infty(\omega)$ such that $\rho = 1$ in ω_0. Multiplying the second equation by $s\xi\rho e^{-2s\Phi} p_1$ and integrating by parts on $\omega \times (0, T)$

$$s \iint_{Q_\omega} (e^{-2s\Phi} \xi)_t \rho p_2 p_1 \, dx dt + s \iint_{Q_\omega} e^{-2s\Phi} \xi \rho p_2 p_{1t} \, dx dt$$
$$- s \iint_{Q_\omega} (e^{-2s\Phi} \xi \rho)_x p_{1x} p_1 \, dx dt - s \iint_{Q_\omega} e^{-2s\Phi} \xi \rho |p_{1x}|^2 \, dx dt$$
$$= s \iint_{Q_\omega} e^{-2s\Phi} \xi \rho \mathrm{Re}(Bp) p_1 \, dx dt. \tag{69}$$

Multiplying the first equation by $s\xi\rho e^{-2s\Phi} p_2$ and integrating by parts on $\omega \times (0, T)$

$$
\begin{aligned}
& s \iint\limits_{Q_\omega} e^{-2s\Phi} \xi \rho p_{1t} p_2 \, dx dt - s \iint\limits_{Q_\omega} (e^{-2s\Phi} \xi \rho)_x p_{2x} p_2 \, dx dt - s \iint\limits_{Q_\omega} e^{-2s\Phi} \xi \rho |p_{2x}|^2 \, dx dt \\
& \quad = s \iint\limits_{Q_\omega} e^{-2s\Phi} \xi \rho \mathrm{Im}(Bp) p_2 \, dx dt. \qquad (70)
\end{aligned}
$$

Subtracting both expressions and using the property of ρ we obtain

$$
\begin{aligned}
s \iint\limits_{Q_{\omega_0}} e^{-2s\Phi} \xi |p_{2x}|^2 \, dx dt \leq & -s \iint\limits_{Q_\omega} (e^{-2s\Phi} \xi \rho)_x p_{2x} p_2 \, dx dt - s \iint\limits_{Q_\omega} e^{-2s\Phi} \xi \rho \mathrm{Im}(Bp) p_2 \, dx dt \\
& - s \iint\limits_{Q_\omega} (e^{-2s\Phi} \xi)_t \rho p_2 p_1 \, dx dt + s \iint\limits_{Q_\omega} (e^{-2s\Phi} \xi \rho)_x p_{1x} p_1 \, dx dt \\
& + s \iint\limits_{Q_\omega} e^{-2s\Phi} \xi \rho |p_{1x}|^2 \, dx dt + s \iint\limits_{Q_\omega} e^{-2s\Phi} \xi \rho \mathrm{Re}(Bp) p_1 \, dx dt.
\end{aligned}
\qquad (71)
$$

The right hand side of (71) can be bounded by local terms of p_1, p_2 and p_{1x}. In accordance with this and (67), we deduce (52). $\square$

3.3. Carleman estimate for the Schrödinger–KdV system. Observations of ψ and $\mathrm{Re}(\phi)$

We state and prove a Carleman estimate for system (5). This inequality will be used in next section to prove the observability estimate (7). The main part of the proof consists in removing, after the Carleman estimates for both equations are combined, one component of the observation. Similar arguments have been applied for systems of parabolic equations (see, for example, [4]).

Theorem 3.3. *Assuming hypotheses of Theorem 1.3, there exist $C > 0$ and $s_0 > 0$ such that for all $s \geq s_0$,*

$$
\begin{aligned}
& s \iint\limits_{Q} e^{-2s\hat{\Phi}} \xi |\phi_x|^2 \, dx dt + s^3 \iint\limits_{Q} e^{-2s\hat{\Phi}} \xi^3 |\phi|^2 \, dx dt + s^5 \iint\limits_{Q} e^{-2s\hat{\Phi}} \xi^5 |\psi|^2 \, dx dt \\
& \quad + s^3 \iint\limits_{Q} e^{-2s\hat{\Phi}} \xi^3 |\psi_x|^2 \, dx dt + s \iint\limits_{Q} e^{-2s\hat{\Phi}} \xi |\psi_{xx}|^2 \, dx dt \leq C s^5 \iint\limits_{Q_\omega} e^{-2\check{\Phi}} \xi^5 |\psi|^2 \, dx dt \\
& \quad + C s^{29} \iint\limits_{Q_\omega} \xi^{47} e^{s(6\hat{\Phi} - 8\check{\Phi})} |\psi|^2 dt + C s^3 \iint\limits_{Q_\omega} \xi^3 e^{-2s\check{\Phi}} |\mathrm{Re}(\phi)|^2 \, dx dt \\
& \quad + C s \iint\limits_{Q_\omega} e^{-2s\check{\Phi}} \xi^3 |\mathrm{Re}(\phi_x)|^2 \, dx dt, \qquad (72)
\end{aligned}
$$

for all $(\phi_T, \psi_T) \in \boldsymbol{H}_0^1(0,1) \times L^2(0,1)$, where (ϕ, ψ) stands for the solution of system (5).

Proof. We start supposing that $(\phi_T, \psi_T) \in \boldsymbol{H}_0^1(0,1) \times H_0^1(0,1)$. The case $(\phi_T, \psi_T) \in \boldsymbol{H}_0^1(0,1) \times L^2(0,1)$ follows by a density argument. We recall that, by Remark 2.8, $(\phi, \psi) \in C([0,T]; \boldsymbol{H}_0^1(0,1)) \times Y_{\frac{1}{2}}$. The rest of the proof is ordered in two steps.

Step 1: We take $\omega_1 \subset\subset \omega$ and apply Carleman inequalities (25) and (52) to each equation of system (5) with observations in ω_1. Adding up both inequalities, we can absorb the zero-order terms of the right hand side, obtaining

$$\begin{aligned}
& s \iint_Q e^{-2s\Phi} \xi |\phi_x|^2 \, dx dt + s^3 \iint_Q e^{-2s\Phi} \xi^3 |\phi|^2 \, dx dt s^5 \iint_Q e^{-2s\Phi} \xi^5 |\psi|^2 \, dx dt \\
& \quad + s^3 \iint_Q e^{-2s\hat{\Phi}} \xi^3 |\psi_x|^2 \, dx dt + s \iint_Q e^{-2s\Phi} \xi |\psi_{xx}|^2 \, dx dt \\
& \leq C s^5 \iint_{Q_{\omega_1}} e^{-2\Phi} \xi^5 |\psi|^2 \, dx dt + C s \iint_{Q_{\omega_1}} e^{-2s\Phi} \xi |\psi_{xx}|^2 \, dx dt \\
& \quad + C s^3 \iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi} |\phi|^2 \, dx dt + C s \iint_{Q_{\omega_1}} e^{-2s\Phi} \xi |\mathrm{Re}(\phi_x)|^2 \, dx dt. \qquad (73)
\end{aligned}$$

In order to remove the imaginary part of the control acting in the Schrödinger equation, we have to remove the weighted integral of $\mathrm{Im}(\phi)$ on the right hand side of (73). Since $|\mathrm{Im}(a_2)| \geq \delta > 0$ in ω, we get

$$\begin{aligned}
s^3 \iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi} |\phi|^2 \, dx dt &= s^3 \iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi} |\mathrm{Re}(\phi)|^2 \, dx dt + s^3 \iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi} |\mathrm{Im}(\phi)|^2 \, dx dt \\
&\leq s^3 \iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi} |\mathrm{Re}(\phi)|^2 \, dx dt \\
&\quad + \frac{s^3}{\delta^2} \iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi} |\mathrm{Im}(a_2)|^2 |\mathrm{Im}(\phi)|^2 \, dx dt. \qquad (74)
\end{aligned}$$

Let $\theta \in C_0^\infty(\omega)$ such that $\theta = 1$ in ω_1 and Sgn the sign function. Multiplying the second equation of system (5) by $s^3 \mathrm{Sgn}(\mathrm{Im}(a_2)) e^{-2s\Phi} \xi \theta \mathrm{Im}(\phi)$ and integrating in $\omega \times (0,T)$, we have

$$\begin{aligned}
s^3 \iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi} |\mathrm{Im}(a_2)||\mathrm{Im}(\phi)|^2 \, dx dt &\leq s^3 \iint_{Q_\omega} \theta \xi^3 e^{-2s\Phi} |\mathrm{Im}(a_2)||\mathrm{Im}(\phi)|^2 \, dx dt \\
&= -s^3 \iint_{Q_\omega} \theta e^{-2s\Phi} \xi^3 \mathrm{Sgn}(\mathrm{Im}(a_2)) \mathrm{Re}(a_2) \mathrm{Re}(\phi) \mathrm{Im}(\phi) \, dx dt \\
&\quad - s^3 \iint_{Q_\omega} \theta e^{-2s\Phi} \xi^3 \mathrm{Sgn}(\mathrm{Im}(a_2)) a_4 \psi \mathrm{Im}(\phi) \, dx dt
\end{aligned}$$

$$
\begin{aligned}
&- s^3 \iint\limits_{Q_\omega} \theta e^{-2s\Phi}\xi^3 \mathrm{Sgn}(\mathrm{Im}(a_2))\psi_t \mathrm{Im}(\phi)\, dxdt \\
&- s^3 \iint\limits_{Q_\omega} \theta e^{-2s\Phi}\xi^3 \mathrm{Sgn}(\mathrm{Im}(a_2))\psi_{xxx} \mathrm{Im}(\phi)\, dxdt \\
&- s^3 \iint\limits_{Q_\omega} \theta e^{-2s\Phi}\xi^3 \mathrm{Sgn}(\mathrm{Im}(a_2))M\psi_x \mathrm{Im}(\phi)\, dxdt. \qquad (75)
\end{aligned}
$$

We denote by J_i the i-th term in the right hand side of (75). Until the end of this proof, we systematically apply inequality $ab \le \varepsilon a^2 + Cb^2$, where $\varepsilon > 0$ is small enough. We have

$$
|J_1| \le \varepsilon s^3 \iint\limits_{Q} e^{-2s\Phi}\xi^3 |\mathrm{Im}(\phi)|^2\, dxdt + Cs^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3 |\mathrm{Re}(\phi)|^2\, dxdt. \qquad (76)
$$

Analogously,

$$
|J_2| \le \varepsilon s^3 \iint\limits_{Q} e^{-2s\Phi}\xi^3 |\mathrm{Im}(\phi)|^2\, dxdt + Cs^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3 |\psi|^2\, dxdt. \qquad (77)
$$

For J_3 we have

$$
\begin{aligned}
J_3 = s^3 &\iint\limits_{Q_\omega} (\mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi^3\theta)_t \psi \mathrm{Im}(\phi)\, dxdt \\
&+ s^3 \iint\limits_{Q_\omega} \mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi^3\theta\psi \mathrm{Im}(\phi_t)\, dxdt, \qquad (78)
\end{aligned}
$$

and using the first equation of (5) we obtain

$$
\begin{aligned}
J_3 = s^3 &\iint\limits_{Q_\omega} (\mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi^3\theta)_t \psi \mathrm{Im}(\phi)\, dxdt \\
&+ s^3 \iint\limits_{Q_\omega} \mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi^3\theta\psi \mathrm{Re}(\phi_{xx})\, dxdt \\
&- s^3 \iint\limits_{Q_\omega} \mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi^3\theta\psi \mathrm{Re}(a_1\phi + \bar{a}_3\psi)\, dxdt. \qquad (79)
\end{aligned}
$$

We remark that it makes sense to calculate the time derivative of $\mathrm{Sgn}(\mathrm{Im}(a_2))$ in (79). This is due to the fact that, in ω, the Sgn of $\mathrm{Im}(a_2)$ is constant and equals to one or minus one.

Denoting by J_3^i the i-th term in the right hand side of (79), and noticing

$$\begin{aligned}|(e^{-2s\Phi}\xi^3\theta\mathrm{Sgn}(\mathrm{Im}(a_2)))_t| &= |-2se^{-2s\Phi}\Phi_t\xi^3\theta\mathrm{Sgn}(\mathrm{Im}(a_2)) + e^{-2s\Phi}(\xi^3)_t\theta\mathrm{Sgn}(\mathrm{Im}(a_2))| \\ &\leq sCe^{-2s\Phi}\xi^5,\end{aligned}$$

we obtain

$$|J_3^1| \leq \varepsilon s^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\,dxdt + Cs^5 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^7|\psi|^2\,dxdt. \tag{80}$$

Integrating by parts we see that

$$\begin{aligned}J_3^2 = &-s^3 \iint\limits_{Q_\omega} (e^{-2s\Phi}\xi^3\theta\mathrm{Sgn}(\mathrm{Im}(a_2)))_x\psi\mathrm{Re}(\phi_x)\,dxdt \\ &- s^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3\theta\mathrm{Sgn}(\mathrm{Im}(a_2))\psi_x\mathrm{Re}(\phi_x)\,dxdt,\end{aligned} \tag{81}$$

and using that

$$\begin{aligned}(e^{-2s\Phi}\xi^3\theta\mathrm{Sgn}(\mathrm{Im}(a_2)))_x &= -2se^{-2s\Phi}\Phi_x\xi^3\theta\mathrm{Im}(a_2) + e^{-2s\Phi}\xi^3(\theta\mathrm{Sgn}(\mathrm{Im}(a_2)))_x \\ &\leq sCe^{-2s\Phi}\xi^4,\end{aligned}$$

we find

$$\begin{aligned}|J_3^2| \leq &\, Cs^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Re}(\phi_x)|^2\,dxdt + Cs^5 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^5|\psi|^2\,dxdt \\ &+ \varepsilon s^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\psi_x|^2\,dxdt.\end{aligned} \tag{82}$$

We see that

$$\begin{aligned}|J_3^3| \leq &\, Cs^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\psi|^2\,dxdt + Cs^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Re}(\phi)|^2\,dxdt \\ &+ \varepsilon s^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\,dxdt.\end{aligned} \tag{83}$$

We have

$$J_4 = s^3 \iint\limits_{Q_\omega} (e^{-2s\Phi}\xi^3\theta \mathrm{Sgn}(\mathrm{Im}(a_2)))_x \psi_{xx} \mathrm{Im}(\phi)\, dxdt$$
$$+ s^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3\theta \mathrm{Sgn}(\mathrm{Im}(a_2)) \psi_{xx} \mathrm{Im}(\phi_x)\, dxdt, \tag{84}$$

and therefore

$$|J_4| \le \varepsilon s^3 \iint\limits_{Q} e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\, dxdt + \varepsilon s \iint\limits_{Q} e^{-2s\Phi}\xi|\mathrm{Im}(\phi_x)|^2\, dxdt$$
$$+ Cs^5 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^5|\psi_{xx}|^2\, dxdt. \tag{85}$$

Finally, we have

$$|J_5| \le Cs^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|M\psi_x|^2\, dxdt + \varepsilon s^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\, dxdt$$
$$\le C\|M\|_{L^\infty(0,T;L^2(0,1))} \left(s^5 \iint\limits_{Q_\omega} e^{-2s\check{\Phi}}\xi^5|\psi|^2\, dxdt + s \iint\limits_{Q_\omega} e^{-2s\check{\Phi}}\xi|\psi_{xx}|^2\, dxdt \right)$$
$$+ \varepsilon s^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\, dxdt. \tag{86}$$

From (75) and the subsequent inequalities, we get

$$s^3 \iint\limits_{Q_{\omega_1}} \xi^3 e^{-2s\Phi}|\mathrm{Im}(a_2)|^2|\mathrm{Im}(\phi)|^2\, dxdt \le \varepsilon s^3 \iint\limits_{Q} e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\, dxdt$$
$$+ Cs^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Re}(\phi)|^2\, dxdt + Cs^5 \iint\limits_{Q_\omega} e^{-2s\check{\Phi}}\xi^7|\psi|^2\, dxdt$$
$$+ Cs^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Re}(\phi_x)|^2\, dxdt + \varepsilon s^3 \iint\limits_{Q_\omega} e^{-2s\Phi}\xi^3|\psi_x|^2\, dxdt$$
$$+ \varepsilon s \iint\limits_{Q} e^{-2s\Phi}\xi|\mathrm{Im}(\phi_x)|^2\, dxdt + Cs^5 \iint\limits_{Q_\omega} e^{-2s\check{\Phi}}\xi^5|\psi_{xx}|^2\, dxdt. \tag{87}$$

From (73), (74) and (87) we obtain the Carleman inequality

$$
\begin{aligned}
& s \iint_Q e^{-2s\hat{\Phi}} \xi |\phi_x|^2 \, dxdt + s^3 \iint_Q e^{-2s\hat{\Phi}} \xi^3 |\phi|^2 \, dxdt + s^5 \iint_Q e^{-2s\hat{\Phi}} \xi^5 |\psi|^2 \, dxdt \\
& \quad + s^3 \iint_Q e^{-2s\hat{\Phi}} \xi^3 |\psi_x|^2 \, dxdt + s \iint_Q e^{-2s\hat{\Phi}} \xi |\psi_{xx}|^2 \, dxdt \le C s^5 \iint_{Q_\omega} e^{-2\check{\Phi}} \xi^7 |\psi|^2 \, dxdt \\
& \quad + C s^5 \iint_{Q_\omega} e^{-2s\check{\Phi}} \xi^5 |\psi_{xx}|^2 \, dxdt + C s^3 \iint_{Q_\omega} \xi^3 e^{-2s\check{\Phi}} |\mathrm{Re}(\phi)|^2 \, dxdt \\
& \quad + C s^3 \iint_{Q_\omega} e^{-2s\check{\Phi}} \xi^3 |\mathrm{Re}(\phi_x)|^2 \, dxdt. \qquad (88)
\end{aligned}
$$

Step 2: In this step we follow [18] in order to eliminate the observation of ψ_{xx} appearing in the right hand side of (88). By an interpolation argument and the Young inequality we have

$$
\begin{aligned}
& s^5 \iint_{Q_\omega} e^{-2s\check{\Phi}} \xi^5 |\psi_{xx}|^2 \, dxdt \\
& \le C s^5 \int_0^T e^{-2s\check{\Phi}} \xi^5 \|\psi\|_{L^2(\omega)}^{\frac{1}{2}} \|\psi\|_{H^{\frac{8}{3}}(\omega)}^{\frac{3}{2}} dt \\
& = C \int_0^T e^{-2s\check{\Phi}} \xi^5 \left[(s^{\frac{29}{4}} \xi^{\frac{21}{2}} e^{\frac{3}{2}s\hat{\Phi}} e^{-\frac{3}{2}s\check{\Phi}}) \|\psi\|_{L^2(\omega)}^{\frac{1}{2}} (s^{\frac{-9}{4}} \xi^{-\frac{21}{2}} e^{-\frac{3}{2}s\hat{\Phi}} e^{\frac{3}{2}s\check{\Phi}}) \|\psi\|_{H^{\frac{8}{3}}(\omega)}^{\frac{3}{2}} \right] dt \\
& \le C \int_0^T e^{-2s\check{\Phi}} \xi^5 \left[C_\varepsilon (s^{29} \xi^{42} e^{6s\hat{\Phi}} e^{-6s\check{\Phi}}) \|\psi\|_{L^2(\omega)}^2 + \varepsilon (s^{-3} \xi^{-14} e^{-2s\hat{\Phi}} e^{2s\check{\Phi}}) \|\psi\|_{H^{\frac{8}{3}}(\omega)}^2 \right] dt \\
& = C s^{29} \int_0^T \xi^{47} e^{s(6\hat{\Phi} - 8\check{\Phi})} \|\psi\|_{L^2(\omega)}^2 dt + \varepsilon s^{-3} \int_0^T \xi^{-9} e^{-2s\hat{\Phi}} \|\psi\|_{H^{\frac{8}{3}}(\omega)}^2 dt, \qquad (89)
\end{aligned}
$$

with $\varepsilon > 0$ taken sufficiently small. Now we prove that the $H^{\frac{8}{3}}$ term in the right hand side of (89) can be estimated by the left hand side of (88), which is denoted by $I(\phi, \psi)$. This will be done by using a bootstrap-kind argument for the KdV equation.

Let $\theta_1 = e^{-s\hat{\Phi}} \xi^{-\frac{1}{2}}$. Given $(\phi, \psi) \in C([0, T]; \boldsymbol{H}_0^1(0, 1)) \times Y_{\frac{1}{4}}$ solution of system (5), we have that $(\phi_1, \psi_1) := (\theta_1 \phi, \theta_1 \psi)$ is solution of

$$
\begin{cases}
i\phi_{1t} + \phi_{1xx} = k & \text{in} \quad Q, \\
\psi_{1t} + \psi_{1xxx} = g & \text{in} \quad Q, \\
\phi_1(0,t) = \phi_1(1,t) = 0 & \text{in} \quad (0,T), \\
\psi_1(0,t) = \psi_1(1,t) = \psi_{1x}(0,t) = 0 & \text{in} \quad (0,T), \\
\phi_1(x,T) = 0, \ \psi_1(x,T) = 0 & \text{in} \quad (0,1),
\end{cases}
\tag{90}
$$

where

$$
\begin{aligned}
&k = i\theta_1'\phi + \theta_1(a_1\phi + \bar{a_3}\psi), \\
&g = \theta_1'\psi - \theta_1(Re(\bar{a_2}\phi) + a_4\psi) + M\theta_1\psi_x.
\end{aligned}
\tag{91}
$$

From the facts that $M \in L^\infty(0,T;L^2(0,1))$ and $|\theta_1'| \le Cs\xi^{\frac{3}{2}}e^{-s\hat{\Phi}}$, we get $k \in L^2(0,T;\boldsymbol{H}_0^1(0,1))$ and $g \in L^2(0,T;L^2(0,1))$. In particular we have

$$
\|k\|^2_{L^2(0,T;\boldsymbol{L}^2(0,1))} + \|g\|^2_{L^2(0,T;L^2(0,1))} \le Cs^{-1}I(\phi,\psi).
\tag{92}
$$

From Proposition 2.6 we get

$$
\|\phi_1\|^2_{L^\infty(0,T;\boldsymbol{L}^2(0,1))} + \|\psi_1\|^2_{L^4(0,T;H^{\frac{3}{2}}(0,1))} \le C\|k\|^2_{L^2(0,T;\boldsymbol{L}^2(0,1))} + C\|g\|^2_{L^2(0,T;\boldsymbol{L}^2(0,1))}.
\tag{93}
$$

Combining (93) and (92) we get

$$
\|\phi_1\|^2_{L^\infty(0,T;\boldsymbol{L}^2(0,1))} + \|\psi_1\|^2_{L^4(0,T;H^{\frac{3}{2}}(0,1))} \le Cs^{-1}I(\phi,\psi).
\tag{94}
$$

Consider now $\theta_2 = e^{-s\hat{\Phi}}\xi^{-\frac{5}{2}}$. Thus $(\phi_2,\psi_2) := (\theta_2\phi, \theta_2\psi)$ satisfies

$$
\begin{cases}
i\phi_{2t} + \phi_{2xx} = k_1 & \text{in} \quad Q, \\
\psi_{2t} + \psi_{2xxx} = g_1 & \text{in} \quad Q, \\
\phi_2(0,t) = \phi_2(1,t) = 0 & \text{in} \quad (0,T), \\
\psi_2(0,t) = \psi_2(1,t) = \psi_{2x}(0,t) = 0 & \text{in} \quad (0,T), \\
\phi_2(x,T) = 0, \psi_2(x,T) = 0 & \text{in} \quad (0,1),
\end{cases}
\tag{95}
$$

where

$$
\begin{aligned}
k_1 &= i\theta_2'\phi + \theta_2(a_1\phi + \bar{a_3}\psi) \\
&= i\theta_2'\theta_1^{-1}\phi_1 + \theta_2\theta_1^{-1}(a_1\phi_1 + \bar{a_3}\psi_1), \\
g_1 &= \theta_2'\psi - \theta_2(Re(\bar{a_2}\phi) + a_4\psi) + M\theta_2\psi_x \\
&= \theta_2'\theta_1^{-1}\psi_1 - \theta_2\theta_1^{-1}(Re(\bar{a_2}\phi_1) + a_4\psi_1) + M\theta_2\theta_1^{-1}\psi_{1x}.
\end{aligned}
\tag{96}
$$

We have that $|\theta_2\theta_1^{-1}| \le C$ and $|\theta_2'\theta_1^{-1}| \le Cs$. Taking into account these inequalities and the fact that $M, \psi_{1x} \in L^4(0,T;H^{\frac{1}{2}}(0,1))$, we have $(k_1, g_1) \in L^2(0,T;\boldsymbol{H}_0^1(0,1) \times H^{\frac{1}{3}}(0,1))$. Here we have used that the product of two functions in $H^{\frac{1}{2}}(0,1)$ belongs to $H^{\frac{1}{3}}(0,1)$. Being $L^2(0,T;H^{\frac{1}{3}}(0,1)) = X_{7/12}$, we use (93), (92) and Proposition 2.6 to obtain

$$\begin{aligned}
&\|\phi_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\psi_2\|^2_{Y_{\frac{7}{12}}} \\
&\quad \le C\|k_1\|^2_{L^2(0,T;\boldsymbol{L}^2(0,1))} + C\|g_1\|^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))} \le CsI(\phi,\psi),
\end{aligned} \tag{97}$$

where

$$Y_{\frac{7}{12}} = L^2(0,T;H^{\frac{7}{3}}(0,1)) \cap L^\infty(0,T;H^{\frac{4}{3}}(0,1)).$$

Finally, consider $\theta_3 = e^{-s\check{\Phi}}\xi^{-\frac{9}{2}}$. Then $(\phi_3,\psi_3) := (\theta_3\phi,\theta_3\psi)$ is solution of

$$\begin{cases}
i\phi_{3t} + \phi_{3xx} = k_2 & \text{in} \quad Q, \\
\psi_{3t} + \psi_{3xxx} = g_2 & \text{in} \quad Q, \\
\phi_3(0,t) = \phi_3(1,t) = 0 & \text{in} \quad (0,T), \\
\psi_3(0,t) = \psi_3(1,t) = \psi_{3x}(0,t) = 0 & \text{in} \quad (0,T), \\
\phi_3(x,T), \psi_3(x,T) = 0 = 0 & \text{in} \quad (0,1),
\end{cases} \tag{98}$$

where

$$\begin{aligned}
k_2 &= i\theta_3'\phi + \theta_3(a_1\phi + \bar{a_3}\psi) \\
&= i\theta_3'\theta_2^{-1}\phi_2 + \theta_3\theta_2^{-1}(a_1\phi_2 + \bar{a_3}\psi_2), \\
g_2 &= \theta_3'\psi - \theta_3(Re(\bar{a_2}\phi) + a_4\psi) + M\theta_3\psi_x \\
&= \theta_3'\theta_2^{-1}\psi_2 - \theta_3\theta_2^{-1}(Re(\bar{a_2}\phi_2) + a_4\psi_2) + M\theta_3\theta_2^{-1}\psi_{2x}.
\end{aligned} \tag{99}$$

Proceeding as before, we see that $|\theta_3\theta_2^{-1}| \le C$ and $|\theta_3'\theta_2^{-1}| \le Cs$. Also, $M \in L^3(0,T;H^{\frac{2}{3}}(0,1))$ and $\psi_{2x} \in L^6(0,T;H^{2/3}(0,1))$. In this way $(k_2,g_2) \in L^2(0,T;\boldsymbol{H}_0^1(0,1) \times H^{\frac{2}{3}}(0,1))$. Here we have used that the product of two functions in $H^{\frac{2}{3}}(0,1)$ belongs to $H^{\frac{2}{3}}(0,1)$. Since $L^2((0,T);H^{\frac{2}{3}}(0,1)) = X_{2/3}$, we have

$$\begin{aligned}
&\|\phi_3\|^2_{L^\infty((0,T);\boldsymbol{L}^2(\Omega))} + \|\psi_3\|^2_{Y_{\frac{2}{3}}} \\
&\quad \le C\|k_2\|^2_{L^2(0,T;\boldsymbol{L}^2(0,1))} + C\|g_2\|^2_{L^2(0,T;H^{\frac{2}{3}}(0,1))} \le Cs^3 I(\phi,\psi),
\end{aligned} \tag{100}$$

where

$$Y_{\frac{2}{3}} = L^2(0,T;H^{\frac{8}{3}}(0,1)) \cap L^\infty(0,T;H^{\frac{5}{3}}(0,1)).$$

By the definition of θ_3, inequality (100) implies that

$$s^{-3}\int_0^T \xi^{-9}e^{-2s\hat{\Phi}}\|\psi\|^2_{H^{\frac{8}{3}}(\omega)}\,dt \le CI(\phi,\psi). \tag{101}$$

Inequality (101), combined with (88) and (89) imply Carleman inequality (72). $\square$

3.4. *Carleman estimate for the Schrödinger–KdV system. Observations of ϕ*

We state and prove a Carleman estimate for system (5), with observations given only by local terms of the solution of the Schrödinger equation. This inequality will be used in next section to prove the observability estimate (6).

Theorem 3.4. *Assuming hypotheses of Theorem 1.1, there exist $C > 0$ and $s_0 > 0$ such that, for all $s \ge s_0$, we have*

$$\begin{aligned} &s\iint_Q e^{-2s\hat{\Phi}}\xi|\phi_x|^2\,dxdt + s^3\iint_Q e^{-2s\hat{\Phi}}\xi^3|\phi|^2\,dxdt + s^5\iint_Q e^{-2s\hat{\Phi}}\xi^5|\psi|^2\,dxdt \\ &\quad + s^3\iint_Q e^{-2s\hat{\Phi}}\xi^3|\psi_x|^2\,dxdt + s\iint_Q e^{-2s\hat{\Phi}}\xi|\psi_{xx}|^2\,dxdt \\ &\le Cs^{12}\iint_{Q_\omega} e^{-2s(8\check{\Phi}-7\hat{\Phi})}\xi^{61}(|\phi|^2 + |\mathrm{Re}(\phi_x)|^2)\,dxdt \end{aligned} \tag{102}$$

for all $(\phi_T,\psi_T) \in \boldsymbol{H}_0^1(0,1)\times L^2(0,1)$, where (ϕ,ψ) stands for the solution of system (5).

Proof. We take $\omega_1 \subset\subset \omega$ and use Carleman inequality (73). Applying a similar argument as in Step 2 of the proof of Theorem 3.3 (see (89)), we obtain the inequality

$$\begin{aligned} s\iint_{Q_{\omega_1}} e^{-2s\check{\Phi}}\xi|\psi_{xx}|^2dxdt &\le Cs^{13}\int_0^T \xi^{31}e^{-s(8\check{\Phi}-6\hat{\Phi})}\|\psi\|^2_{L^2(\omega_1)}dt \\ &\quad + \varepsilon s^{-3}\int_0^T \xi^{-9}e^{-2s\hat{\Phi}}\|\psi\|^2_{H^{\frac{8}{3}}(\omega_1)}\,dt. \end{aligned} \tag{103}$$

Using (103) and (101) in (73) we get

$$\begin{aligned} &s\iint_Q e^{-2s\Phi}\xi|\phi_x|^2\,dxdt + s^3\iint_Q e^{-2s\Phi}\xi^3|\phi|^2\,dxdts^5\iint_Q e^{-2s\Phi}\xi^5|\psi|^2\,dxdt \\ &\quad + s^3\iint_Q e^{-2s\hat{\Phi}}\xi^3|\psi_x|^2\,dxdt + s\iint_Q e^{-2s\Phi}\xi|\psi_{xx}|^2\,dxdt \end{aligned}$$

$$\leq C\left(s^3\iint\limits_{Q_{\omega_1}}\xi^3 e^{-2s\Phi}|\phi|^2\,dxdt + s\iint\limits_{Q_{\omega_1}} e^{-2s\Phi}\xi|\mathrm{Re}(\phi_x)|^2\,dxdt + s^{13}\iint\limits_{Q_{\omega_1}}\xi^{31}e^{-s(8\check{\Phi}-6\hat{\Phi})}|\psi|^2 dt\right). \tag{104}$$

The task now is to eliminate the local term of ψ on the right hand side of (104). Consider $\Gamma(t)=\xi^{31}(t)e^{s(6\hat{\Phi}-8\check{\Phi})(t)}$ and θ a $C^\infty(0,1)$ function, such that $\theta=1$ in ω_1 and Supp $\theta\subset\omega$. Multiplying the imaginary part of the Schrödinger equation in (5) by $\Gamma(t)\theta(x)\mathrm{Im}(a_3)\psi$ and integrating on Q, we have

$$\iint\limits_{Q_{\omega_1}}|\mathrm{Im}(a_3)|^2\xi^{31}(t)e^{s(6\hat{\Phi}-8\check{\Phi})(t)}|\psi|^2\,dxdt \leq \iint\limits_{Q_\omega}\Gamma(t)\theta(x)\left(\mathrm{Im}(a_3)a_1\mathrm{Im}(\phi)-\mathrm{Re}(\phi_t)-\mathrm{Im}(\phi_{xx})\right)\psi\,dxdt. \tag{105}$$

We will now estimate the three terms on the right hand side of (105). The first term can be bounded as follows

$$\begin{aligned}
&\iint\limits_{Q_\omega}\Gamma(t)\theta(x)\mathrm{Im}(a_3)a_1\mathrm{Im}(\phi)\psi\,dxdt\\
&=\iint\limits_{Q_\omega}\mathrm{Im}(a_3)a_1\Gamma(t)\theta(x)(\Gamma(t)\xi^{-\frac{5}{2}}e^{s\hat{\Phi}})\mathrm{Im}(\phi)(\Gamma(t)^{-1}\xi^{\frac{5}{2}}e^{-s\hat{\Phi}})\psi\,dxdt\\
&\leq C_\epsilon s^{-5}\iint\limits_{Q_\omega}e^{-2s(8\check{\Phi}-7\hat{\Phi})}\xi^{57}|\mathrm{Im}(\phi)|^2\,dxdt+\epsilon s^5\iint\limits_{Q_\omega}\xi^5e^{-2s\hat{\Phi}}|\psi|^2\,dxdt.
\end{aligned}\tag{106}$$

The second term is given by

$$\begin{aligned}
-\iint\limits_{Q_\omega}\Gamma(t)\theta(x)\mathrm{Im}(a_3)\mathrm{Re}(\phi_t)\psi\,dxdt &= \iint\limits_{Q_\omega}(\Gamma(t)\mathrm{Im}(a_3))_t\theta(x)\mathrm{Re}(\phi)\psi\,dxdt\\
&\quad+\iint\limits_{Q_\omega}\Gamma(t)\mathrm{Im}(a_3)\theta(x)\mathrm{Re}(\phi)\psi_t\,dxdt\\
&=X_1+X_2.
\end{aligned}\tag{107}$$

Above, we used the fact that Γ decreases exponentially to zero at $t=0$ and $t=T$. Using that

$$\Gamma_t(t)=-(T-2t)e^{s(6\hat{\Phi}-8\check{\Phi})(t)}\left(31\xi^{32}(t)+s\xi^{33}(6\hat{\phi}_0-8\check{\phi}_0)\right)\leq Cs\Gamma(t)\xi^2(t),$$

where $(\hat{\phi_0}, \check{\phi_0}) = (\max_{x\in[0,1]} \phi_0, \min_{x\in[0,1]} \phi_0)$, we get the following bound for X_1

$$
\begin{aligned}
|X_1| &\le Cs \iint_{Q_\omega} \Gamma(t)\xi^2(t)|\mathrm{Re}(\phi)\psi|\,dxdt \\
&= Cs \iint_{Q_\omega} \Gamma(t)\xi^2(t)(\Gamma^{\frac{1}{2}}\xi^{-\frac{3}{2}}e^{s\hat{\Phi}})|\mathrm{Re}(\phi)|(\Gamma^{-\frac{1}{2}}\xi^{\frac{3}{2}}e^{-s\hat{\Phi}})|\psi|\,dxdt \\
&\le C_\epsilon s^{-5} \iint_{Q_\omega} e^{-2s(8\check{\Phi}-7\hat{\Phi})}\xi^{61}|\mathrm{Re}(\phi)|^2\,dxdt + \epsilon s^5 \iint_{Q_\omega} e^{-2s\hat{\Phi}}\xi^5|\psi|^2\,dxdt.
\end{aligned}
\tag{108}
$$

Using the second equation of (5) we have for X_2 that

$$
\begin{aligned}
X_2 &= \iint_{Q_\omega} \Gamma(t)\mathrm{Im}(a_3)\theta(x)\mathrm{Re}(\phi)(-\psi_{xxx} - M\psi_x - \mathrm{Re}(\bar{a}_2\phi) - a_4\psi)\,dxdt \\
&= X_2^1 + X_2^2 + X_2^3 + X_2^4.
\end{aligned}
\tag{109}
$$

The task now is to estimate the variables X_2^i in terms of the Schrödinger variables ϕ and $\mathrm{Re}(\phi_x)$. Indeed,

$$
\begin{aligned}
X_2^1 &= \iint_{Q_\omega} \Gamma(t)(\mathrm{Im}(a_3)\theta(x))_x\mathrm{Re}(\phi)\psi_{xx}\,dxdt + \iint_{Q_\omega} \Gamma(t)\mathrm{Im}(a_3)\theta(x)\mathrm{Re}(\phi)_x\psi_{xx}\,dxdt \\
&\le C_\epsilon s^{-1} \iint_{Q_\omega} \xi^{61}e^{-2s(8\check{\phi}-7\hat{\phi})}(|\mathrm{Re}(\phi)|^2 + |\mathrm{Re}(\phi_x)|)\,dxdt + \epsilon s \iint_{Q_\omega} e^{-2s\hat{\phi}}\xi|\psi_{xx}|^2\,dxdt.
\end{aligned}
\tag{110}
$$

For X_2^2 we have

$$
X_2^2 = C_\epsilon s^{-1} \iint_{Q_\omega} \xi^{61}e^{-2s(8\check{\phi}-7\hat{\phi})}|\mathrm{Re}(\phi)|^2\,dxdt + \epsilon\|M\|_{L^\infty(0,T;L^2(\Omega))}s \iint_{Q} e^{-2s\hat{\phi}}\xi|\psi_{xx}|^2\,dx\,dt.
\tag{111}
$$

The computations of X_2^3 and X_2^4 are simpler and results on

$$
\begin{aligned}
X_2^3 &\le C \iint_{Q_\omega} \xi^{31}e^{s(6\hat{\Phi}-8\check{\Phi})}|\mathrm{Re}(\phi)|^2\,dxdt, \\
X_2^4 &\le C_\epsilon s^{-5} \iint_{Q_\omega} e^{2s(7\hat{\Phi}-8\check{\Phi})}\xi^{57}|\mathrm{Re}(\phi)|^2\,dxdt + \epsilon s^5 \iint_{Q_\omega} \xi^5e^{-2s\hat{\Phi}}|\psi|^2\,dxdt.
\end{aligned}
\tag{112}
$$

To finish, we must bound the third term on the right hand side of (105). In fact,

$$
\begin{aligned}
-\iint\limits_{Q_\omega} \Gamma(t)\theta(x)\mathrm{Im}(\phi_{xx})\psi\,dxdt &= -\iint\limits_{Q_\omega} \Gamma(t)\theta_{xx}(x)\mathrm{Im}(\phi)\psi\,dxdt \\
&\quad -2\iint\limits_{Q_\omega} \Gamma(t)\theta_x(x)\mathrm{Im}(\phi)\psi_x\,dxdt - \iint\limits_{Q_\omega} \Gamma(t)\theta(x)\mathrm{Im}(\phi)\psi_{xx}\,dxdt \\
&\le C_\epsilon \iint\limits_{Q_\omega} e^{-2s(8\check{\Phi}-7\hat{\Phi})}\xi^{57}(s^{-5}+s^{-3}\xi^2+s^{-1}\xi^4)|\mathrm{Im}(\phi)|^2\,dxdt \\
&\quad +\epsilon\left(\iint\limits_{Q_\omega} e^{-2s\hat{\Phi}}(s^5\xi^5|\psi|^2+s^3\xi^3|\psi_x|^2+s\xi|\psi_{xx}|^2)\,dxdt\right).
\end{aligned}
\tag{113}
$$

Combining (105)–(113), we get

$$
\begin{aligned}
\iint\limits_{Q_{\omega_1}} \xi^{31}(t)e^{-s(8\check{\Phi}-6\hat{\Phi})(t)}|\psi|^2\,dxdt &\le Cs^{-1}\iint\limits_{Q_\omega} e^{-2s(8\check{\Phi}-7\hat{\Phi})}\xi^{61}(|\phi|^2+|\mathrm{Re}(\phi_x)|^2)\,dxdt \\
&\qquad \epsilon\left(\iint\limits_{Q_\omega} e^{-2s\hat{\Phi}}(\xi^5|\psi|^2+\xi^3|\psi_x|^2+\xi|\psi_{xx}|^2)\,dxdt\right).
\end{aligned}
\tag{114}
$$

Replacing (114) in the Carleman estimate (104) we obtain (102). □

4. Observability and control

The observability inequalities stated in Theorem 1.6 are proved in this section. From these inequalities we deduce the null controllability results stated in Theorems 1.1 and 1.3.

4.1. Observability inequalities

In order to prove observability inequalities (6) and (7), let us assume the hypotheses of Theorem 1.6. Given $(\phi^T, \psi^T) \in \boldsymbol{H}_0^1(0,1)\times L^2(0,1)$, we define, for each $t\in[0,T]$,

$$
E(t) = \int_0^1 \left(|\phi(x,t)|^2 + |\phi_x(x,t)|^2 + |\psi(x,t)|^2\right)dx, \tag{115}
$$

where (ϕ, ψ) is the solution of system (5). We have the following property of $E(t)$.

Lemma 4.1. *There exists a constant $C > 0$ such that for every $(\phi^T, \psi^T) \in \boldsymbol{H}_0^1(0,1) \times L^2(0,1)$ we have*

$$E(0) \leq C \int_{T/4}^{3T/4} E(t)dt. \tag{116}$$

Proof. Multiplying the first equation of system (5) by $\bar{\phi}$ and integrating in $(0,1)$ we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |\phi|^2\,dx = \mathrm{Im}\int_0^1 (a_1\phi + \bar{a_3}\psi)\bar{\phi}\,dx = \mathrm{Im}\int_0^1 \bar{a_3}\psi\bar{\phi}\,dx. \tag{117}$$

Denoting $f = a_1\phi + \bar{a_3}\psi$, multiplying the same equation by $\bar{\phi}_t$ and integrating in $(0,1)$ we get

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\int_0^1 |\phi_x|^2\,dx &= \mathrm{Re}\int_0^1 f\bar{\phi}_t\,dx = \mathrm{Re}\int_0^1 f(-i\bar{\phi}_{xx} + i\bar{f})\,dx \\ &= \mathrm{Re}\int_0^1 (-i\bar{\phi}_{xx} f)\,dx \end{aligned}$$

and integrating by parts in x we get that there exists a constant $C > 0$ depending on a_1 and a_3 such that

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 |\phi_x|^2\,dx \leq C\int_0^1 \left(|\phi|^2 + |\phi_x|^2 + |\psi|^2\right)dx + \frac{1}{2}\int_0^1 |\psi_x|^2. \tag{118}$$

Multiplying the second equation of system (5) by ψ, and denoting $g = \mathrm{Re}(\bar{a_2}\phi) + a_4\psi$, we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 |\psi|^2\,dx + \frac{1}{2}|\psi_x(1,t)|^2 \leq \int_0^1 |g\psi|dx + \frac{1}{2}\|M\|^2_{L^\infty(0,1)}\int_0^1 |\psi|^2\,dx + \frac{1}{2}\int_0^1 |\psi_x|^2\,dx. \tag{119}$$

Multiplying the same equation, this time by $(1-x)\psi$, we get

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}\int_0^1 (1-x)|\psi|^2\,dx + \frac{3}{2}\int_0^1 |\psi_x|^2\,dx \\ &\leq \int_0^1 |g\psi|dx + \frac{1}{2}\|M\|^2_{L^\infty(0,1)}\int_0^1 |\psi|^2\,dx + \frac{1}{2}\int_0^1 |\psi_x|^2\,dx. \end{aligned} \tag{120}$$

From (120) and (119), there exists a constant $C > 0$ depending on a_2 and a_4 such that

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 (2-x)|\psi|^2\,dx + \frac{1}{2}\int_0^1 |\psi_x|^2\,dx \le C(1+\|M\|^2_{L^\infty(0,1)})\int_0^1 \left(|\phi|^2+|\psi|^2\right)dx. \tag{121}$$

From (117), (118) and (121), we have

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_0^1 \left((2-x)|\psi|^2+|\phi|^2+|\phi_x|^2\right)dx\\
&\le C(1+\|M\|^2_{L^\infty(0,1)})\int_0^1 \left(|\phi_x|^2+|\phi|^2+|\psi|^2\right)dx,
\end{aligned} \tag{122}$$

where the constant $C > 0$ depends on a_1, a_2, a_3 and a_4. Therefore, denoting

$$\tilde{E}(t) := \frac{1}{2}\int_0^1 \left((2-x)|\psi|^2+|\phi|^2+|\phi_x|^2\right)dx, \tag{123}$$

we get

$$\frac{d}{dt}\tilde{E}(t) \ge -C(1+\|M(t)\|^2_{L^\infty(0,1)})\tilde{E}(t), \quad \forall t \in (0,T). \tag{124}$$

From (124) we obtain that

$$\frac{d}{dt}\left(e^{C\int_0^t (1+\|M(s)\|^2_{L^\infty(0,1)})ds}\,\tilde{E}(t)\right) \ge 0, \quad \forall t \in (0,T). \tag{125}$$

Integrating (125) on the time interval $(0,t)$ we get

$$\tilde{E}(0) \le e^{C(T+\|M\|^2_{L^2(0,T;H^1(0,1))})}\,\tilde{E}(t), \quad \forall t \in (0,T). \tag{126}$$

Integrating (126) on the interval $[T/4, 3T/4]$ and taking into account that $1 \le 2-x \le 2$, for each $x \in [0,1]$, we obtain (116) and Lemma 4.1 is proved. $\square$

From definition (23) we have that there exists $\delta > 0$ such that

$$e^{-2s\hat{\Phi}}\xi^k \ge \delta, \tag{127}$$

for all $t \in [T/4, 3T/4]$, $x \in [0,1]$, and $k = 1, 3, 5$. Hence

$$\delta \int_{T/4}^{3T/4} E(t)dt \le \iint_Q e^{-2s\hat{\Phi}}\xi^3|\phi|^2\,dxdt + \iint_Q e^{-2s\hat{\Phi}}\xi|\phi_x|^2\,dxdt + \iint_Q e^{-2s\hat{\Phi}}\xi^5|\psi|^2\,dxdt. \tag{128}$$

From (128), Lemma 4.1, and Carleman estimate (102), we deduce the observability inequality (6). Analogously, but using Carleman estimate (72), we deduce the observability inequality (7). This concludes the proof of Theorem 1.6.

4.2. Null controllability

The duality between observability and controllability is well known in the literature. See for instance Theorems 2.42, 2.43 and 2.44 in [16]. In the sake of completeness, we prove Theorem 1.1 as a consequence of the observability (6). The proof of Theorem 1.3 by using (7) is very similar and then is omitted here.

We start by the following characterization of a control driving system (1) to the rest. This kind of result is already classic for parabolic systems.

Lemma 4.2. *A control $h \in L^2(0,T;\boldsymbol{H}^1(\omega)')$ drives system (1) from $w(0,\cdot)=w_0$ and $y(0,\cdot)=y_0$ to $w(T,\cdot)=0$ and $y(T,\cdot)=0$ if and only if*

$$-i\langle w_0,\bar{\phi}|_{t=0}\rangle_{\boldsymbol{H}^{-1},\boldsymbol{H}_0^1} - \int_0^1 y_0(x)\psi(x,0)dx = \int_0^T \langle h,\bar{\phi}\mathbf{1}_\omega\rangle_{\boldsymbol{H}^1(\omega)',\boldsymbol{H}^1(\omega)}dt,$$

for all $(\phi^T,\psi^T)\in \boldsymbol{H}_0^1(0,1)\times L^2(0,1)$, where $(\phi,\psi)\in C([0,T];\boldsymbol{H}_0^1(0,1)\times L^2(0,1))$ is the solution of system (5).

Proof. This can be obtained, for regular solutions, by simple integration by parts after multiplying system (1) by the solutions of (5). The less regular framework can be proved using density arguments. □

In order to prove Theorem 1.1, we define the set

$$B = \{\phi\mathbf{1}_\omega;\ (\phi^T,\psi^T)\in \boldsymbol{H}_0^1(0,1)\times L^2(0,1)\} \subset L^2(0,T;\boldsymbol{H}^1(\omega))$$

and H its closure with respect to the $L^2(0,T;\boldsymbol{H}^1(\omega))$ norm. In addition, we define the map

$$\Lambda : \phi\mathbf{1}_\omega \in B \longmapsto (\phi(0,x),\psi(0,x)) \in \boldsymbol{H}_0^1(0,1)\times L^2(0,1)$$

which is well-defined thanks to (6). Moreover, Λ is linear and continuous due to (6). Now, we define

$$N : \phi\mathbf{1}_\omega \in B \longmapsto -i\langle w_0,\bar{\phi}|_{t=0}\rangle_{\boldsymbol{H}^{-1},\boldsymbol{H}_0^1} - \int_0^1 y_0(x)\psi(x,0)dx \in \mathbf{C}$$

which is linear and continuous and can be extended in a continuous way, still thanks to (6), to H, the closure of B. We now extend the operator N to the whole space $L^2(0,T;\boldsymbol{H}^1(\omega))$ by requiring N vanishes on $H^\perp$. Thus, N is linear and continuous and consequently, it belongs to $L^2(0,T;\boldsymbol{H}^1(\omega)')$. In conclusion, we have proved the existence of an element $N \in L^2(0,T;\boldsymbol{H}^1(\omega)')$ such that, when evaluated in B, is such that

$$-i\langle w_0, \bar{\phi}|_{t=0}\rangle_{\boldsymbol{H}^{-1},\boldsymbol{H}_0^1} - \int_0^1 y_0(x)\psi(x,0)dx = N(\phi\mathbf{1}_\omega), \forall \phi\mathbf{1}_\omega \in B.$$

Using Lemma 4.2 and the definition of the space $L^2(0,T;\boldsymbol{H}^1(\omega)')$, we see that the control we look for is given by N. Hence Theorem 1.1 is proved.

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