PDE Control Methods: Stabilization Methods for KdV equation

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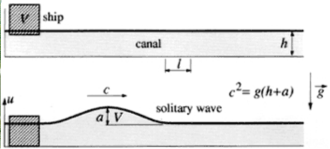
Korteweg-de Vries equation 1895



Function $u = u(t,x)$ models for a time t the amplitude of the water wave at position x. The nonlinear dispersive partial differential equation, named Korteweg-de Vries equation and abbreviated as KdV, describes approximately long waves in water of relatively shallow depth

$$\boxed{u_t + u_{xxx} + uu_x = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}}$$

Korteweg-de Vries equation on a bounded domain

On a bounded interval, the extra term u_x should be incorporated in the equation in order to obtain an appropriate model for water waves in a uniform channel when coordinates x is taken with respect to a fixed frame. Thus, for $L > 0$ the equation considered here is

$$\boxed{u_t + u_x + u_{xxx} + uu_x = 0, \quad x \in [0, L], t \geq 0}$$

+ Boundary conditions, for instance posed on

$$\boxed{u(t, 0) = u(t, L) = u_x(t, L) = 0, \quad t \geq 0}$$

+ Initial data

$$\boxed{u(0, x) = u_0 \in L^2(0, L)}$$

Asymptotic behaviour

We are interested in the long-time behavior of the energy

$$E(t) = \int_0^L |u(t,x)|^2 \, dx.$$

More precisely we want to prove the exponential decay of $E(t)$ as t goes to infinity.

$$E(t) \leq Ce^{-\omega t} E(0), \quad \forall t \in [0, \infty)$$

Let us start considering the linear equation

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0 \end{array}}$$

Asymptotic behaviour

By performing integration by parts in the equation

$$\int_0^L (u_t + u_x + u_{xxx}) u \, dx = 0$$

we get

$$\frac{d}{dt} \int_0^L |u(t,x)|^2 \, dx = -|u_x(t,0)|^2 \leq 0.$$

The energy is non-increasing, but is it strictly decreasing?

Remember we are looking for an exponential decay.

Solutions with constant energy

The energy is not decreasing. In fact there are solutions with constant energy!

For instance, if $L = 2\pi$ and

$$u_0 = (1 - \cos(x)),$$

the solution of the linear KdV $u_t + u_x + u_{xxx} = 0$ is stationary

$$u(t,x) = (1 - \cos(x))$$

which satisfies $u_x(t,0) = 0$ for any $t \geq 0$ and then

$$\boxed{\dot{E}(t) = \frac{d}{dt}\int_0^L |u(t,x)|^2 dx = 0}$$

Critical domains

For the linear KdV equation there exist constant energy solutions if and only if

$$L \in \mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2 + k\ell + \ell^2}{3}};\ k, \ell \in \mathbb{N}^* \right\}.$$

This phenomena is linked to the controllability of a linear KdV from the boundary.

(Controllability)
Take a look at the linear control system

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0 \\ u(t,0) = u(t,L) = 0, \quad u_x(t,L) = \kappa(t), \\ u(0,\cdot) = 0 \end{array}}$$

(Controllability)

- Linear KdV is controllable $\Leftrightarrow$ the following map is onto

$$B : \kappa \in L^2(0,T) \mapsto u(T,\cdot) \in L^2(0,L)\,.$$

- The map B is onto $\Leftrightarrow$ the following inequality holds

$$\text{(Obs)} \qquad \|B^*(\phi_T)\|_{L^2(0,T)} \geq C\|\phi_T\|_{L^2(0,L)}$$

- The map B is onto $\Leftrightarrow$ its adjoint system is observable, i.e.

$$\text{(Obs)} \qquad \|\phi_x(t,L)\|_{L^2(0,T)} \geq C\|\phi_T\|_{L^2(0,L)}$$

where $\phi = \phi(t,x)$ satisfies,

$$\text{(Adj)} \quad \begin{cases} \phi_t + \phi_x + \phi_{xxx} = 0, \\ \phi(t,0) = \phi(t,L) = \phi_x(t,0) = 0, \\ \phi(T,\cdot) = \phi_T. \end{cases}$$

(Controllability)

Theorem (Rosier 97)

- *The linear KdV system is controllable iff* $L \notin \mathcal{N}$.
- *If* $L \notin \mathcal{N}$, *the nonlinear system (KdV) is locally exactly controllable.*

Theorem (Coron-Crépeau 04, EC 07, EC-Crépeau 09)

Let $L \in \mathcal{N}$, *there exists* $T_L \geq 0$ *such that (KdV) is locally exactly controllable in* $L^2(0,L)$ *if* $T \geq T_L$.

Back to stabilization

We will design some feedback control laws in order to get

$$E(t) \leq Ce^{-\omega t}E(0), \quad \forall t \geq 0.$$

Internal control:

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + uu_x = F(u), \;\; u(0,\cdot) = u_0, \\ u(t,0) = 0, \quad u(t,L) = 0, \quad u_x(t,L) = 0, \end{array}}$$

Boundary control from the right:

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + uu_x = 0, \;\; u(0,\cdot) = u_0, \\ u(t,0) = 0, \quad u(t,L) = 0, \quad u_x(t,L) = F_\omega(u), \end{array}}$$

Boundary control from the left:

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + uu_x = 0, \;\; u(0,\cdot) = u_0, \\ u(t,0) = K_\omega(u), \quad u(t,L) = 0, \quad u_x(t,L) = 0, \end{array}}$$

Internal Control

Internal Control

Equation with internal control

$$\boxed{u_t + u_x + u_{xxx} + uu_x = F}$$

We consider a feedback law in the form

$$\boxed{F(u) = -au}$$

where $a \in L^\infty(0, L; \mathbb{R}^+)$ satisfies

$$\begin{cases} a(x) \geq a_0 > 0, \quad \forall x \in \mathcal{O}, \\ \text{where } \mathcal{O} \text{ is nonempty open subset of } (0, L). \end{cases}$$

Closed-loop system

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + a(x)u + uu_x = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot). \end{array}}$$

Internal Control - Linear

A natural strategy is to consider first the linearized equation around the origin

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot), \end{array}} \tag{1}$$

and prove the exponential decay of its solutions.

Theorem (Perla-Vasconcellos-Zuazua 02)

Let $L > 0$ and $a = a(x)$ as before. There exist $C, \omega > 0$:

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of (1) with $u_0 \in L^2(0,L)$.

Internal Control - Nonlinear

Nonlinear system

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au + uu_x = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot) \end{array}} \tag{2}$$

Using a perturbative argument, a local version of this theorem is proven by adding a smallness condition on the initial data.

Theorem (Perla-Vasconcellos-Zuazua 02)

Let $L > 0$ and $a = a(x)$ as before. There exist $C, r > 0$ and $\omega > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of (2) with $\|u_0\|_{L^2(0,L)} \leq r$.

Internal Control - Semiglobal

Nonlinear system

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au + uu_x = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot). \end{array}} \tag{3}$$

Theorem (Pazoto 05)

Let $L > 0$, $a = a(x)$ as before and $R > 0$. There exist $C = C(R) > 0$ and $\omega = \omega(R) > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of (3) *with* $\|u_0\|_{L^2(0,L)} \leq R$.

This result was proved in [P-V-Z 02] by assuming

$$\exists \delta > 0, \quad (0,\delta) \cup (L-\delta, L) \subset \mathcal{O}$$

which has been removed by Pazoto.

Linear System on a noncritical case

No damping ($a(x) = 0$) and $L \notin \mathcal{N}$.
We have the observability inequality for $T = 1$

$$\forall u_0 \in L^2(0,L), \quad C\|u_x(\cdot,0)\|_{L^2(0,T)} \geq \|u_0\|_{L^2(0,L)}$$

Integrating with respect to time

$$\frac{d}{dt}\int_0^L |u(t,x)|^2\,dx = -|u_x(t,0)|^2$$

from $t = 0$ to $t = 1$ we get

$$\int_0^L |u(1,x)|^2\,dx - \int_0^L |u_0(x)|^2\,dx$$

$$= -\int_0^1 |u_x(s,0)|^2\,ds \leq -\frac{1}{C^2}\int_0^L |u_0(x)|^2\,dx,$$

that implies

$$\int_0^L |u(1,x)|^2\,dx \leq \frac{C^2-1}{C^2}\int_0^L |u_0(x)|^2\,dx.$$

Linear System on a noncritical case

Of course we also have

$$\int_0^L |u(t+1,x)|^2\,dx \le \frac{C^2-1}{C^2}\int_0^L |u(t,x)|^2\,dx,$$

that implies the exponential decay.

Indeed, let $k \le t \le k+1$. Denoting $\gamma := \frac{C^2-1}{C^2} < 1$, we have

$$\begin{aligned} E(t) \le E(k) \le \gamma E(k-1) \le \gamma^2 E(k-2) \le \dots \\ \le \gamma^k E(0) = \frac{\gamma^{k+1}}{\gamma}E(0) = \frac{1}{\gamma}e^{(k+1)\ln(\gamma)}E(0) \\ \le \frac{1}{\gamma}e^{-t|\ln(\gamma)|}E(0) \end{aligned}$$

Linear System on a critical case

With damping $a(x)u$ active in $\mathcal{O}$ and $L \in \mathcal{N}$. From

$$\int_0^L (u_t + u_x + u_{xxx} + au)u\,dx = 0$$

we get

$$\frac{d}{ds}\int_0^L |u(s,x)|^2\,dx = -|u_x(s,0)|^2 - \int_0^L a(x)|u(s,x)|^2\,dx \leq 0$$

and then by integrating on $(0,1)$ we obtain

$$\int_0^L |u(1,x)|^2\,dx - \int_0^L |u_0(x)|^2\,dx$$
$$= -\int_0^1 |u_x(s,0)|^2\,ds - \int_0^1\int_0^L a(x)|u(s,x)|^2\,dxds.$$

Linear System on a critical case

$$\int_0^L |u(1,x)|^2\,dx - \int_0^L |u_0(x)|^2\,dx$$
$$= -\int_0^1 |u_x(s,0)|^2\,ds - \int_0^1\int_0^L a(x)|u(s,x)|^2\,dxds.$$

$\Big($same proof as before runs if we are able to prove $\exists C > 0$:

$$\leq -C^2 \int_0^L |u_0(x)|^2\,dx\Big)$$

Let us prove that for any $T, L > 0$, there exists $C > 0$:

$$\forall u_0 \in L^2(0,L), \quad \|u_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|u(t,x)|^2\,dxdt$$
$$\geq C^2\|u_0\|^2_{L^2(0,L)}$$

Linear System on a critical case

By integrating by parts

$$\int_0^L (u_t + u_x + u_{xxx} + au)(T-t)u\,dx = 0$$

we obtain

$$\|u_0\|^2_{L^2(0,L)} \leq \frac{1}{T}\|u\|^2_{L^2(0,T;L^2(0,L))} + \|u_x(\cdot,0)\|^2_{L^2(0,T)} + 2\int_0^T\int_0^L a(x)|u(t,x)|^2\,dxdt$$

and therefore we will be done if we prove that there exists a constant $K > 0$ such that

$$K\|u\|^2_{L^2(0,T;L^2(0,L))} \leq \|u_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|u(t,x)|^2\,dxdt$$

Linear System on a critical case

We proceed by contradiction by supposing that

$$\boxed{\begin{array}{l} \forall\, K>0, \exists\, u=u(t,x), \text{ such that} \\ K\|u\|^2_{L^2(0,T;L^2(0,L))} > \|u_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T \int_0^L a(x)|u(t,x)|^2\,dxdt \end{array}}$$

By using this successively with $K=1/n$, we obtain a sequence $\{u^n\}_{n\in\mathbb{N}}$ of solutions such that $\|u^n\|_{L^2(0,T;L^2(0,L))}=1$ (if not, we normalize. This is due to the linearity of the equation) and

$$\frac{1}{n} > \|u^n_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T \int_0^L a(x)|u^n(t,x)|^2\,dxdt$$

Then, as n goes to ∞

$$u^n_x(t,0)\to 0, \text{ in } L^2(0,T), \quad au^n(t,x)\to 0, \text{ in } L^2(0,T,L^2(0,L))$$

Linear System on a critical case

We pass to the limit (see the notes) in the equation

$$u^n_t + u^n_x + u^n_{xxx} + au^n = 0.$$

and get a solution u of

$$u_t + u_x + u_{xxx} = 0.$$

with

$$a(x)u(t,x) = 0 \quad \forall x \in [0,L], \forall t \in (0,T)$$

From the properties of the damping (active in $\mathcal{O}$), we get

$$u(t,x) = 0, \quad \forall x \in \mathcal{O}, \forall t \in (0,T).$$

A unique continuation principle (Holmgrem's Theorem) implies that $u = 0$, which contradicts the fact that

$$\|u\|_{L^2(0,T;L^2(0,L))} = 1$$

Stabilization of the Linear System

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot), \end{array}}$$

Theorem (Perla-Vasconcellos-Zuazua 02)

Let $L > 0$ and $a = a(x)$ as before. There exist $C, \omega > 0$:

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of linear KdV with $u_0 \in L^2(0,L)$.

Nonlinear System

The solution u of

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au + uu_x = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot), \end{array}}$$

can be written as $u = u^1 + u^2$ where u^1 is the solution of

$$\boxed{\begin{array}{l} u^1_t + u^1_x + u^1_{xxx} + au^1 = 0, \\ u^1(t,0) = u^1(t,L) = u^1_x(t,L) = 0, \\ u^1(0,x) = u_0 \end{array}}$$

and u^2 is the solution of

$$\boxed{\begin{array}{l} u^2_t + u^2_x + u^2_{xxx} + au^2 = -uu_x, \\ u^2(t,0) = u^2(t,L) = u^2_x(t,L) = 0, \\ u^2(0,x) = 0 \end{array}}$$

Nonlinear System

- From some linear estimates of the system

$$
\begin{aligned}
\|u(t,\cdot)\|_{L^2(0,L)} &\leq \|u^1(t,\cdot)\|_{L^2(0,L)} + \|u^2(t,\cdot)\|_{L^2(0,L)} \\
&\leq \gamma \|u_0\|_{L^2(0,L)} + C\|uu_x\|_{L^1(0,T;L^2(0,L))} \\
&\leq \gamma \|u_0\|_{L^2(0,L)} + C\|u\|^2_{L^2(0,T;H^1(0,L))}
\end{aligned}
$$

  where $\gamma < 1$.
- Here we need a nonlinear estimate

$$
\int_0^L (u_t + u_x + u_{xxx} + au + uu_x) xu \, dx = 0
$$

  we get

$$
\begin{aligned}
3\int_0^T \int_0^L |u_x|^2 dxdt + \int_0^L x|u(T,\cdot)|^2 dx + 2\int_0^T \int_0^L xa|u|^2 dxdt \\
= \int_0^T \int_0^L |u|^2 dxdt + \int_0^L x|u_0|^2 dx + \frac{2}{3}\int_0^T \int_0^L |u|^3 dxdt
\end{aligned}
$$

Nonlinear System

We obtain

$$\|u\|^2_{L^2(0,T;H^1(0,L))} \leq \frac{(3T+L)}{3}\|u_0\|^2_{L^2(0,L)} + \frac{2}{9}\int_0^T\int_0^L |u|^3 dxdt$$

As $u \in L^2(0,T;H^1(0,L))$ and $H^1(0,L)$ embeds into $C([0,L])$:

$$\begin{aligned}
\int_0^T\int_0^L |u|^3 dxdt &\leq \int_0^T \|u\|_{L^\infty(0,L)} \int_0^L |u|^2 dxdt \\
&\leq C\int_0^T \|u\|_{H^1(0,L)} \int_0^L |u|^2 dxdt \\
&\leq C\|u_0\|^2_{L^2(0,L)} \int_0^T \|u\|_{H^1(0,L)} dt \\
&\leq CT^{1/2}\|u_0\|^2_{L^2(0,L)} \|u\|_{L^2(0,T;H^1(0,L))}
\end{aligned}$$

Nonlinear System

We obtain

$$\|u\|^2_{L^2(0,T;H^1(0,L))} \leq \frac{(8T+2L)}{3}\|u_0\|^2_{L^2(0,L)} + \frac{TC}{27}\|u_0\|^4_{L^2(0,L)}$$

which gives the existence of $C > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq \|u_0\|_{L^2(0,L)} \left\{\gamma + C\|u_0\|_{L^2(0,L)} + C\|u_0\|^3_{L^2(0,L)}\right\}$$

Given $\epsilon > 0$ small enough such that $(\gamma+\epsilon) < 1$, we can take r small enough so that $r + r^3 < \frac{\epsilon}{C}$, in order to have

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq (\gamma+\epsilon)\|u_0\|_{L^2(0,L)}$$

The rest of the proof runs as before thanks to the fact that $(\gamma+\epsilon) < 1$.

Stabilization of the Nonlinear System

We have introduced an internal damping mechanism in order to be sure the energy of the system decreases to zero in an exponential way. We have proved a local result for the KdV equation.

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au + uu_x = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot), \end{array}}$$

Theorem (Perla-Vasconcellos-Zuazua 02)

Let $L > 0$ and $a = a(x)$ as before. There exist $C, r, \omega > 0$:

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of KdV with $\|u_0\|_{L^2(0,L)} \leq r$.

Remark

- Similar results have been proven recently for coupled systems of KdV equations. See [Capistrano-Fihlo, Komornik, and Pazoto. 2014], [Pazoto, Souza, 2014 and 2013], Massarolo, Perla-Mezala, and Pazoto, 2011], [Nina, Pazoto, and Rosier, 2011], [Pazoto, Rosier, 2010].
- That could seem strange, but as mentioned before, a similar phenomena appears when studying the controllability of the system from the right Neumann boundary condition. The linear system is controllable if and only if L is not critical but in despite of that the nonlinear system is always controllable.

Saturated inputs

What is saturation for a function? Different choices

$$\mathtt{sat}_{loc}(f)(x) = \begin{cases} -u_0 & \text{if } f(x) \leq -u_0, \\ f(x) & \text{if } -u_0 \leq f(x) \leq u_0, \\ u_0 & \text{if } f(x) \geq u_0, \end{cases}$$

$$\mathtt{sat}_2(f)(x) = \begin{cases} f(x) & \text{if } \|f(x)\|_{L^2(0,L)} \leq u_0, \\ \dfrac{f(x)u_0}{\|f(x)\|_{L^2(0,L)}} & \text{if } \|f(x)\|_{L^2(0,L)} \geq u_0. \end{cases}$$

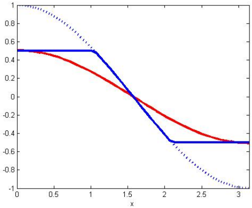


Figure: $x \in [0, \pi]$. Red: $\mathtt{sat}_2(\cos)(x)$ and $u_0 = 0.5$, Blue: $\mathtt{sat}_{loc}(\cos)(x)$ and $u_0 = 0.5$, Dotted lines: $\cos(x)$.

Saturated inputs

Let us consider the KdV equation controlled by a saturated distributed control as follows

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x + \mathtt{sat}(ay) = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = y_0(x), \end{cases}$$

where `sat` is any of previous saturations, and a is a localized function as in previous sections.

Theorem (Marx, EC, Prieur, Andrieu, under review)

There exist a positive value $\mu^\star$ and a class $\mathcal{K}$ function $\alpha_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for any $y_0 \in L^2(0,L)$, the mild solution y of saturated-KdV satisfies

$$\|y(t,.)\|_{L^2(0,L)} \leq \alpha_0(\|y_0\|_{L^2(0,L)}) e^{-\mu^\star t}, \quad \forall t \geq 0. \tag{4}$$

Simulations: $\texttt{sat}_2$, $u_0 = 0.5$, $\omega = [0, L]$, $a = 1$

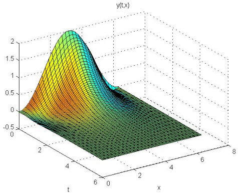


Figure: Solution with no sat.

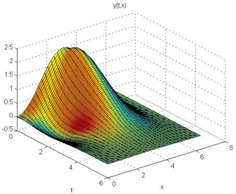


Figure: Saturated solution

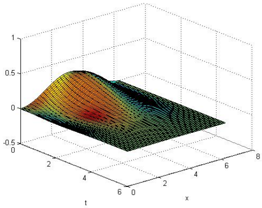


Figure: Saturated control

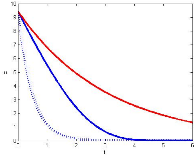


Figure: Blue: Saturated energy. Red: Theoretical energy. Dotted line: Energy with no sat

Simulations: $\texttt{sat}_{loc}$, $u_0 = 0.5$, $\omega = [\frac{L}{3}, \frac{2L}{3}]$, $a = 1$

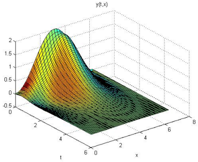


Figure: Solution with no sat.

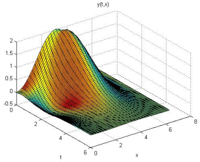


Figure: Saturated solution

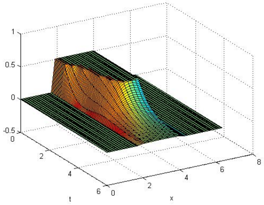


Figure: Saturated control

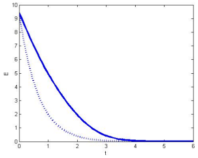


Figure: Blue: Saturated energy. Dotted line: Energy with no sat

Boundary Control from the right

Boundary Control from the right

In all this part the equation is linear.

Let $L > 0$ be fixed. Let us consider the following linear control system for the KdV equation with homogeneous Dirichlet boundary conditions

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0, \\ u(t,0) = u(t,L) = 0, \\ u_x(t,L) = F_\omega(t) \end{array}}$$

State is $u(t,\cdot) : [0,L] \to \mathbb{R}$. Control is $F_\omega(t) \in \mathbb{R}$.

We want to design a feedback control law

$$\boxed{F_\omega = F_\omega(u)}$$

Result

We will use a Gramian-based approach in order to build a feedback law to show the following.

Theorem (EC-Crépeau 09)

Let $\omega > 0$ and $L \notin \mathcal{N}$. The closed-loop system

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0,\ \ u(0,\cdot) = u_0, \\ u(t,0) = u(t,L) = 0,\ \ u_x(t,L) = F_\omega(u(t)), \end{array}}$$

is globally well posed in $H_0^1(0,L)$. Moreover, the solutions decay to zero with an exponential rate of 2ω, i.e.,

$$\exists C > 0, \forall u_0 \in H_0^1(0,L), \quad \|u(t,\cdot)\|_{H_0^1} \leq Ce^{-2\omega t}\|u_0\|_{H_0^1}.$$

Finite Dimensional Control

$$\boxed{\dot{x} = Ax + Bu, \quad x(0) = x_0,}$$

with $n, m \in \mathbb{N}$, $A \in M_{n\times n}(\mathbb{R})$, $B \in M_{n\times m}(\mathbb{R})$. The state is $x(t) \in \mathbb{R}^n$ and the control is $u(t) \in \mathbb{R}^m$. The state x_0 is the initial data. The solution is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{(t-s)A}Bu(s)ds$$

The system is controllable in time T if and only if the Gramian matrix

$$\mathcal{C} = \int_0^T e^{(T-t)A}BB^*e^{(T-t)A^*}dt$$

is invertible. For instance, if $\mathcal{C}$ is invertible, then the system is driven from x_0 to x_1 in time T (for any $x_0, x_1 \in \mathbb{R}^n$) by applying the control

$$u(s) = B^*e^{(T-s)A^*}\mathcal{C}^{-1}(x_1 - e^{TA}x_0), \quad \forall s \in [0, T].$$

Gramian-based stabilization

Let us see how the Gramian matrix can also be used to stabilize the system. Let us suppose the system is controlable. Thus,

$$C_T = e^{-TA}\mathcal{C}e^{-TA^*} = \int_0^T e^{-tA}BB^*e^{-tA^*}dt$$

is invertible and we can define the feedback control

$$u(t) = -B^*C_T^{-1}x(t).$$

By applying a Lyapunov method, it can be easily proven the following (see the notes).

Theorem

$\exists\, M, \mu > 0$ such that solutions of $\dot{x}(t) = (A - BB^*C_T^{-1})x(t)$, satisfies

$$|x(t)| \leq Me^{-\mu t}|x(0)|, \quad \forall t \geq 0.$$

Rapid Stabilization

Now, as we want to impose an exponential decay rate equals to ω, we make the change $y = e^{\omega t}x$. The system becomes

$$\boxed{\dot{y} = (A + \omega I_d)y + Bv}$$

(I_d identity matrix) and the control is given by $v = e^{\omega t}u$. The controllability of this system is equivalent to the controllability of $\dot{x} = Ax + Bu$. Then, the feedback control

$$v(t) = -B^* \left(\int_0^T e^{-t(A+\omega I_d)} BB^* e^{-t(A^*+\omega I_d)} dt \right)^{-1} y(t).$$

gives the exponential decay of y. However, we do not know exactly the rate μ. By coming back to x, we get

$$\boxed{|x(t)| \leq Me^{-\omega t}|x(0)|, \quad \forall t \geq 0}$$

which is what we were looking for.

Rapid stabilization

An improvement of this method: let us consider the matrix

$$C_{\omega,\infty} = \int_0^\infty e^{-t(A+\omega I_d)} BB^* e^{-t(A^*+\omega I_d)} dt$$

We obtain

$$(A+\omega I_d)C_{\omega,\infty} + C_{\omega,\infty}(A+\omega I_d)^* = BB^*$$

and then if we use the control

$$\boxed{u(t) = -B^* C_{\omega,\infty}^{-1} x(t)}$$

in $\dot{x} = Ax + Bu$, then we obtain

$$\boxed{(A - BB^* C_{\omega,\infty}^{-1}) = C_{\omega,\infty}(-A^* - 2\omega I_d) C_{\omega,\infty}^{-1}}$$

In particular, if $A^* = -A$, then the eigenvalues of system $\dot{x} = (A - BB^* C_{\omega,\infty}^{-1})x$ are exactly the eigenvalues of A shifted 2ω units to the left in the complex plane:

$$|x(t)| \leq Me^{-2\omega t}|x(0)|, \quad \forall t \geq 0.$$

Infinite Dimensional Case

$$\boxed{\begin{array}{l} \dot{y}(t) = Ay(t) + B\kappa(t), \\ y(0) = y_0. \end{array}}$$

State $y(t)$ in a Hilbert space Y; Control $\kappa(t)$ in a Hilbert space U; A is a skew-adjoint operator (i.e. $A^* = -A$) in Y; B is an unbounded operator from U to Y; B^* is called observation operator.

We want to define an invertible operator $\Lambda_\omega : Y \to Y$

$$\Lambda_\omega \approx \int_0^\infty e^{-t(A+\omega I_d)} BB^* e^{-t(A^*+\omega I_d)} dt$$

To do so, we use the cuadratic expression: $\forall x, z \in Y$,

$$(\Lambda_\omega x, z)_Y = \int_0^\infty \left(B^* e^{-\tau(A+\omega I)^*} x, B^* e^{-\tau(A+\omega I)^*} z \right)_U d\tau$$

Infinite Dimensional Case

State $y(t)$ in a Hilbert space Y; Control $\kappa(t)$ in a Hilbert space U; A is a skew-adjoint operator (i.e. $A^* = -A$) in Y and B is an unbounded operator from U to Y.

(H1) A is the infinitesimal generator of a strongly continuous group on Y.

(H2) The operator $B : U \to D(A)'$ is linear continuous.

(H3) *Regularity property.* $\forall\, T > 0$, $\exists\, C_T > 0$:

$$\int_0^T \|B^* e^{-tA^*} y\|_U^2 dt \leq C_T \|y\|_Y^2, \quad \forall\, y \in D(A^*).$$

(H4) *Observability inequality.* $\exists\, T, c_T > 0$:

$$\int_0^T \|B^* e^{-tA^*} y\|_U^2 dt \geq c_T \|y\|_Y^2, \quad \forall\, y \in D(A^*).$$

Infinite Dimensional Case

Theorem (Urquiza 05)

Consider A and B such that (H1)-(H4) hold. For any $\omega > 0$:

- (i) *The symmetric positive operator Λ_ω defined above is coercive and an isomorphism on Y.*
- (ii) *Let $F_\omega := -B^*\Lambda_\omega^{-1}$. The operator $A + BF_\omega$ is the infinitesimal generator of a strongly continuous semigroup on Y.*
- (iii) *The closed-loop system with feedback law $F_\omega(y(t))$ is exponentially stable with a decay rate 2ω:*

$$\exists C > 0, \forall y_0 \in Y, \quad \|e^{t(A+BF_\omega)}y_0\|_Y \leq Ce^{-2\omega t}\|y_0\|_Y.$$

Application to our Problem

(H1): Operator A is the infinitesimal generator of a strongly continuous group on Y, $A^* = -A$.

It holds if we take as control, the function v defined by

$$v(t) = F(t) - y_x(t, 0).$$

Hence our system becomes symmetric with respect to the space variable

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0, \\ u(t, 0) = u(t, L) = 0, \\ u_x(t, L) - u_x(t, 0) = v(t). \end{array}}$$

Application to our Problem

We can rewrite latter system in the abstract form by defining $U := L^2(0,T)$, $Y := L^2(0,L)$ and

$$D(A) := \left\{ w \in H^3(0,L); w(0) = w(L) = 0, w'(0) = w'(L) \right\},$$

$$Aw := -w' - w''',$$

$$B : s \in \mathbb{R} \longmapsto L_s \in D(A^*)',$$

$$L_s : z \in D(A^*) \longmapsto s z_x(L) \in \mathbb{R}.$$

It is not difficult to see that $A^* = -A$ and that

$$(Aw, w)_{L^2(0,L)} = 0, \quad \forall w \in D(A).$$

Hence, from classical semigroup results, one sees that the operator A satisfies (H1). We also see that (H2) holds for the operator B.

Application to our Problem

Hypothesis (H3) and (H4) are more delicate to show. As our operator B stands for a boundary control, we will see that assumption (H3) is a sharp trace regularity. Concerning (H4), it is an observability inequality.

Observation operator

$$B^* : w \in D(A^*) \longmapsto w'(L) \in \mathbb{R}$$

and then we have to show $\exists\, c_T, C_T > 0, \forall\, z_0 \in L^2(0,T)$,

$$\boxed{c_T \|z_0\|^2_{L^2(0,T)} \leq \int_0^T |z_x(t,L)|^2 dt \leq C_T \|z_0\|^2_{L^2(0,T)}}$$

where z is the solution of

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \quad z(0,\cdot) = z_0, \\ z(t,0) = z(t,L) = 0, \quad z_x(t,L) - z_x(t,0) = 0, \end{cases}$$

Testing Hypothesis

$$\boxed{c_T\|z_0\|^2_{L^2(0,T)} \leq \int_0^T |z_x(t,L)|^2 dt \leq C_T\|z_0\|^2_{L^2(0,T)}}$$

where

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \quad z(0,\cdot) = z_0, \\ z(t,0) = z(t,L) = 0, \quad z_x(t,L) - z_x(t,0) = 0. \end{cases}$$

We know that $\{\phi_k\}_{k\in\mathbb{Z}}$ where

$$\begin{cases} -\phi' - \phi''' = i\lambda\phi, \\ \phi(0) = 0, \quad \phi(L) = 0, \quad \phi'(0) = \phi'(L). \end{cases}$$

form a basis of $L^2(0,L)$. Thus, for any $f \in L^2(0,L)$ there exists a unique sequence $\{f_k\}_{k\in\mathbb{Z}}$ with $\sum_{k\in\mathbb{Z}} |f_k|^2 < \infty$ such that

$$f = \sum_{k\in\mathbb{Z}} f_k\phi_k \quad \text{and} \quad \|f\|_{L^2(0,L)} = \Big(\sum_{k\in\mathbb{Z}} |f_k|^2\Big)^{1/2}.$$

Testing Hypothesis

$$\boxed{c_T\|z_0\|^2_{L^2(0,T)} \leq \int_0^T |z_x(t,L)|^2 dt \leq C_T\|z_0\|^2_{L^2(0,T)}}$$

If $z_0 = \sum_{k\in\mathbb{Z}} z_0^k \phi_k(x)$, then the solution of

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \quad z(0,\cdot) = z_0, \\ z(t,0) = z(t,L) = 0, \quad z_x(t,L) - z_x(t,0) = 0, \end{cases}$$

is given by

$$\boxed{z(t,x) = \sum_{k\in\mathbb{Z}} e^{i\lambda_k t} z_0^k \phi_k(x)}$$

Testing Hypothesis

As

$$\boxed{z(t,x) = \sum_{k\in\mathbb{Z}} e^{i\lambda_k t} z_0^k \phi_k(x)}$$

one has at least formally,

$$\boxed{z_x(t,L) = \sum_{k\in\mathbb{Z}} e^{i\lambda_k t} \underbrace{z_0^k \phi_k'(L)}_{\gamma_k}}$$

It can be proven that

$$\phi_k'(L) \approx k, \qquad \text{as } |k| >> 1.$$

If $z_0 \in L^2(0,L)$, then $\sum_{k\in\mathbb{Z}} |z_0^k|^2 < \infty$.

If $z_0 \in H^1(0,L)$, then $\sum_{k\in\mathbb{Z}} (1+|k|)^2 |z_0^k|^2 < \infty$.

Testing Hypothesis

Lemma (Ingham's inequality)

Let $T > 0$. Let $\{\beta_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence of pairwise distinct real numbers such that

$$\lim_{|k|\to+\infty} \beta_{k+1} - \beta_k = +\infty.$$

Then there exist two strictly positive constants C_1 and C_2 such that for any sequence $\{\gamma_k\}_{k\in\mathbb{Z}}$ satisfying $\sum_{k\in\mathbb{Z}} \gamma_k^2 < \infty$, the series $f(t) = \sum_{k\in\mathbb{Z}} \gamma_k e^{i\beta_k t}$ converges in $L^2(0,T)$ and satisfies

$$C_1 \sum_{k\in\mathbb{Z}} \gamma_k^2 \le \int_0^T |f(t)|^2 dt \le C_2 \sum_{k\in\mathbb{Z}} \gamma_k^2.$$

In our case we take

$$\beta_k := \lambda_k, \quad \gamma_k := z_0^k \phi_k'(L), \quad f(t) := \sum_{k\in\mathbb{Z}} e^{i\lambda_k t} z_0^k \phi_k'(L)$$

Testing Hypothesis

$$\boxed{\beta_k := \lambda_k, \quad \gamma_k := z_0^k \phi_k'(L), \quad f(t) := \sum_{k\in\mathbb{Z}} e^{i\lambda_k t} z_0^k \phi_k'(L)}$$

Applying Ingham's inequality

$$c_T \sum_{k\in\mathbb{Z}} |z_0^k \phi_k'(L)|^2 \leq \int_0^T |z_x(t,L)|^2 dt \leq C_T \sum_{k\in\mathbb{Z}} |z_0^k \phi_k'(L)|^2$$

In order to put the term $\sum_{k\in\mathbb{Z}}(1+|k|)^2|z_0^k|^2$ by above and below, we have to ask the condition

$$\phi_k'(L) \neq 0, \quad \forall k \in \mathbb{Z}$$

Theorem

This condition holds if and only if $L \notin \mathcal{N}$.

Result

$$\left((\Lambda_\omega x, z)_Y = \int_0^\infty \left(B^* e^{-\tau(A+\omega I)^*} x, B^* e^{-\tau(A+\omega I)^*} z \right)_U d\tau \right)$$

We first define, for any q_0 and $\psi_0 \in H_0^1(0,L)$, the bilinear form

$$a_\omega(q_0,\psi_0) := \int_0^\infty e^{-2\omega\tau} q_x(\tau,L)\psi_x(\tau,L)d\tau,$$

where q and ψ are the respective solutions of

$$\begin{cases} q_\tau + q_x + q_{xxx} = 0, \ \ q(0,.) = q_0, \\ q(\tau,0) = q(\tau,L) = 0, \ \ q_x(\tau,L) - q_x(\tau,0) = 0 \end{cases}$$

and

$$\begin{cases} \psi_\tau + \psi_x + \psi_{xxx} = 0, \ \ \psi(0,.) = \psi_0, \\ \psi(\tau,0) = \psi(\tau,L) = 0, \ \ \psi_x(\tau,L) - \psi_x(\tau,0) = 0. \end{cases}$$

Result

$$\left((\Lambda_\omega x, z)_Y = \int_0^\infty \left(B^* e^{-\tau(A+\omega I)^*} x, B^* e^{-\tau(A+\omega I)^*} z\right)_U d\tau\right)$$

We then define the operator $\Lambda_\omega : H_0^1(0,L) \longrightarrow H^{-1}(0,L)$ assumed to satisfy

$$< \Lambda_\omega q_0, \psi_0 >_{H^{-1},H_0^1} = a_\omega(q_0, \psi_0), \quad \forall q_0, \psi_0 \in H_0^1.$$

Therefore we define $\Lambda_\omega^{-1} z$ as q_0 solution of

$$\boxed{\Lambda_\omega q_0 = z}$$

that is equivalent to

$$\boxed{< \Lambda_\omega q_0, \psi_0 >_{H^{-1},H_0^1} = < z, \psi_0 >_{H^{-1},H_0^1} \quad \forall \psi_0 \in H_0^1}$$

or to the following Lax-Milgram problem

$$\boxed{a_\omega(q_0, \psi_0) = < z, \psi_0 >_{H^{-1},H_0^1} \quad \forall \psi_0 \in H_0^1}$$

Feedback law

Finally, we define $F_\omega(z) = -B^* \Lambda_\omega^{-1} z$

$$\begin{array}{rcl} F_\omega : H_0^1(0,L) & \longrightarrow & \mathbb{R} \\ z & \longrightarrow & F_\omega(z) := -q_0'(L), \end{array}$$

where q_0 is the solution

$$a_\omega(q_0, \psi_0) = < z, \psi_0 >_{H^{-1}, H_0^1}, \quad \forall \psi_0 \in H_0^1.$$

Notice that $q_0 \in H_0^1(0,L)$ is characterized as the minimum of

$$J(q_0) := \frac{1}{2} a_\omega(q_0, q_0) - < z, q_0 >_{H_{-1}, H_1}$$

in $H_0^1(0,L)$.

Result

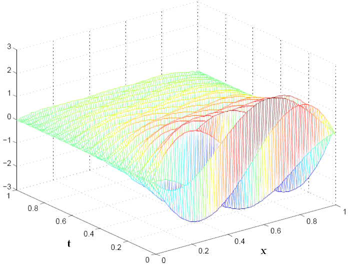
As hypothesis (H1)-(H4) are satisfied under the condition $L \notin \mathcal{N}$, the method can be applied to get

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \ \ u(0,.) = y_0, \\ u(t,0) = u(t,L) = 0, \ \ u_x(t,L) - u_x(t,0) = F_\omega(u(t)), \end{cases}$$

is globally well posed in $H_0^1(0,L)$. Moreover, the solutions decay to zero with an exponential rate of 2ω, i.e.,

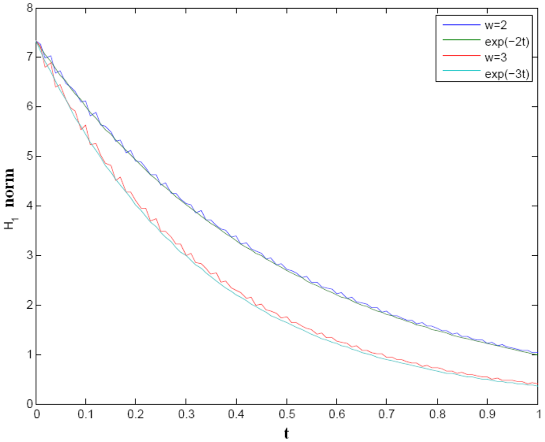
$$\exists C > 0, \forall u_0 \in H_0^1, \quad \|u(t,\cdot)\|_{H_0^1} \leq Ce^{-2\omega t}\|u_0\|_{H_0^1}.$$

Numerical Simulations



Evolution of the solution when $\omega = 2$ (left) and $\omega = 3$ (right).

Numerical Simulations



Time-evolution of the norm $\|u\|_{H1}$ compared with $e^{-\omega t}\|u_0\|_{H^1}$ for $\omega = 2$ and $\omega = 3$.

Remarks

- By using a finite-dimensional method based on the Gramian matrix we have design some feedback controls which make the linear KdV equation stable with an exponential decay rate as large as desired.
- This method can not be applied if the underlying spatial operator is not skew-adjoint.
- For that reason, we consider a first order boundary condition on $(u_x(t,L) - u_x(t,0))$ instead of $u_x(t,L)$.

Remarks

A major difficulty in order to consider the nonlinear KdV equation is to deal with the technical point of well-posedness of the equation with the convenient boundary conditions. Is the system

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \\ u(t,0) = u(t,L) = 0, \\ u_x(t,L) - u_x(t,0) = 0, \end{cases}$$

well-posed in $L^2(0,L)$ or $H^1(0,L)$?

With this boundary conditions there is no Kato smoothing effect allowing us to deal with the nonlinearity uu_x in the well-posedness framework.

Remarks

In [Coron and Lu, 2014] the authors apply a new design strategy (similar to Backstepping method) in order to define a control law acting on the right-hand side of the interval. They do not need to work with a skew-adjoint operator and therefore they obtain stabilization results for the nonlinear KdV equation

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \\ u(t,0) = u(t,L) = 0, \\ u_x(t,L) = K(u(t,\cdot)). \end{cases}$$

Of course, they have to avoid the critical cases because their method is based on a linearization procedure.

Boundary Control from the left

Boundary Control from the left

Given $L > 0$, the linear control system is

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0, \ \ u(0,\cdot) = u_0, \\ u(t,0) = K_\omega, \quad u(t,L) = 0, \quad u_x(t,L) = 0, \end{array}}$$

and the nonlinear one is

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + uu_x = 0, \ \ u(0,\cdot) = u_0, \\ u(t,0) = K_\omega, \quad u(t,L) = 0, \quad u_x(t,L) = 0. \end{array}}$$

Boundary Control from the left

We use the Backstepping method to get

Theorem (EC-Coron 13)

For any $\omega > 0$*, there exist a feedback control law* $K_\omega = K_\omega(u(t,\cdot))$ *and* $D > 0$ *such that*

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq De^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0,$$

for any solution of linear KdV.

Theorem (EC-Coron 13)

For any $\omega > 0$*, there exist a feedback control law* $K_\omega = K_\omega(u(t,\cdot))$*,* $r > 0$ *and* $D > 0$ *such that*

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq De^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0,$$

for any solution of nonlinear KdV satisfying $\|u_0\|_{L^2(0,L)} \leq r$*.*

Boundary Control from the left

In both cases the feedback law K_ω is explicitly defined as follows

$$K_\omega(u(t,\cdot)) = \int_0^L k(0,y)u(t,y)dy,$$

where the function $k = k(x,y)$ will be characterized as the solution of a given partial differential equation depending on ω.

Unlike the cases of the wave and the heat equation, we have not found a closed formula for the gain $k = k(x,y)$.

Control Design

Let us consider the linearized system around the origin

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0, \\ u(t,0) = K_\omega, \quad u(t,L) = 0, \quad u_x(t,L) = 0. \end{array}} \tag{5}$$

Given a positive parameter ω, we look for a transformation $\Pi : L^2(0,L) \to L^2(0,L)$ defined by

$$v(x) = \Pi(u(x)) := u(x) - \int_x^L k(x,y)u(y)dy,$$

such that a trajectory $u = u(t,x)$, solution of (5) with

$$K_\omega(t) = \int_0^L k(0,y)u(t,y)dy,$$

is map into a trajectory $v = v(t,x)$, solution of the linear system

$$\boxed{\begin{array}{l} v_t + v_x + v_{xxx} + \omega v = 0, \\ v(t,0) = 0, \quad v(t,L) = 0 \quad v_x(t,L) = 0. \end{array}}$$

Target System

Take a look at the target system

$$\boxed{\begin{array}{l} v_t + v_x + v_{xxx} + \omega v = 0, \\ v(t,0) = 0, \quad v(t,L) = 0 \quad v_x(t,L) = 0. \end{array}}$$

We have for any $t \geq 0$

$$\begin{aligned} \frac{d}{dt}\int_0^L |v(t,x)|^2 dx = -|v_x(t,0)|^2 - 2\omega \int_0^L |v(t,x)|^2 dx \\ \leq -2\omega \int_0^L |v(t,x)|^2 dx \end{aligned}$$

and therefore we easily obtain for $v = v(t,x)$ the exponential decay at rate ω

$$\|v(t,\cdot)\|_{L^2(0,L)} \leq e^{-\omega t}\|v(0,\cdot)\|_{L^2(0,L)}, \quad \forall t \geq 0.$$

Target System

Is this decay rate sharp? Let us notice that the eigenvalues of target system

$$\boxed{\begin{array}{l} v_t + v_x + v_{xxx} + \omega v = 0, \\ v(t,0) = 0, \quad v(t,L) = 0 \quad v_x(t,L) = 0. \end{array}}$$

are the eigenvalues of

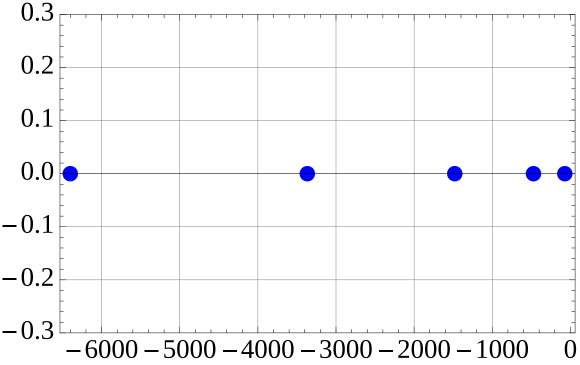
$$\boxed{\begin{array}{l} v_t + v_x + v_{xxx} = 0, \\ v(t,0) = 0, \quad v(t,L) = 0, \quad v_x(t,L) = 0, \end{array}}$$

shifted to the left ω units. Thus, we are lead to study the eigenvalues σ of

$$\begin{cases} -\phi'(x) - \phi'''(x) = \sigma\phi(x), \\ \phi(0) = 0, \quad \phi(L) = 0, \quad \phi'(L) = 0. \end{cases}$$

Eigenvalues

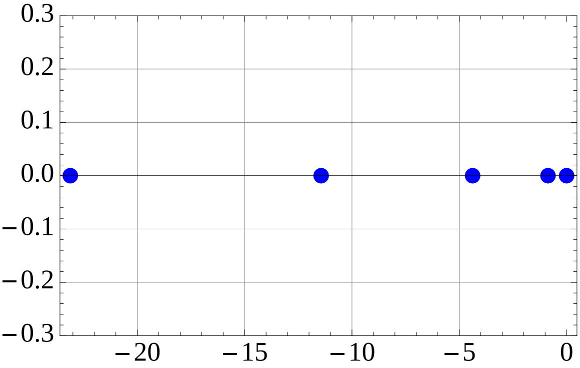
Surprisingly, the eigenvalues behavior depends on the length of the interval.



(a)

In case (a), $L = 1$ (non-critical) and the first eigenvalue σ_1 is approximately -72. The system behaves like a dissipative one.

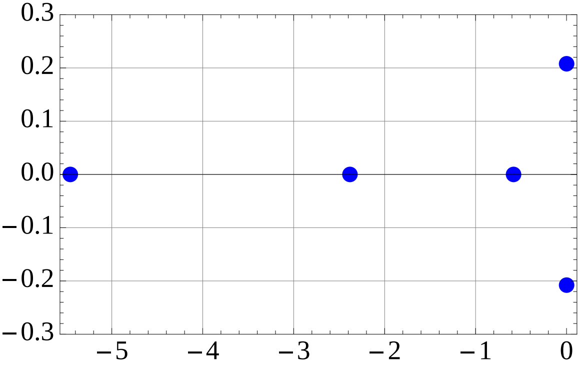
Eigenvalues



(b)

In (b), $L = 2\pi$ (critical) and we have $\sigma_1 = 0$. The system has one conservative component given by the eigenfunction $\phi(x) = 1 - \cos(x)$.

Eigenvalues



(c)

In (c), $L = 2\pi\sqrt{7/3}$ and the first two eigenvalues are imaginary numbers $\sigma_1 = 0.2i$ and $\sigma_2 = -0.2i$.

This examples show the different behaviors that the target system can have and the important role played by the parameter ω in our design.

Kernel function

Let us find the kernel $k = k(x, y)$ such that

$$\boxed{v(x) = u(x) - \int_x^L k(x,y)u(y)dy}$$

is sent into the target. For instance,

$$\begin{aligned}
v_t(t,x) &= u_t(t,x) - \int_x^L u_t(t,y)k(x,y)dy \\
&= u_t(t,x) + \int_x^L (u_y(t,y) + u_{yyy}(t,y))k(x,y)dy \\
&= u_t(t,x) - \int_x^L u(t,y)\left(k_y(x,y) + k_{yyy}(x,y)\right)dy \\
&\quad - k(x,x)(u(t,x) + u_{xx}(t,x)) \\
&\quad + k_y(x,x)u_x(t,x) - k_{yy}(x,x)u(t,x) + k(x,L)u(t,L) \\
&\quad + k(x,L)u_{xx}(t,L) - k_y(x,L)u_x(t,L) + k_{yy}(x,L)u(t,L)
\end{aligned}$$

Kernel function

$$\boxed{v(x) = u(x) - \int_x^L k(x,y)u(y)dy,}$$

$$v_x(t,x) = u_x(t,x) + k(x,x)u(t,x) - \int_x^L k_x(x,y)u(t,y)dy,$$

$$\begin{aligned} v_{xx}(t,x) &= u_{xx}(t,x) + u(t,x)\frac{d}{dx}k(x,x) + k(x,x)u_x(t,x) \\ &\quad + k_x(x,x)u(t,x) - \textstyle\int_x^L k_{xx}(x,y)u(t,y)dy, \end{aligned}$$

and

$$\begin{aligned} v_{xxx}(t,x) &= u_{xxx}(t,x) + u(t,x)\frac{d^2}{dx^2}k(x,x) + 2u_x(t,x)\frac{d}{dx}k(x,x) \\ &\quad + k(x,x)u_{xx}(t,x) + u(t,x)\frac{d}{dx}k_x(x,x) + k_x(x,x)u_x(t,x) \\ &\quad + k_{xx}(x,x)u(t,x) - \int_x^L k_{xxx}(x,y)u(t,y)dy. \end{aligned}$$

Kernel function

Thus, given $\omega \in \mathbb{R}$ we have

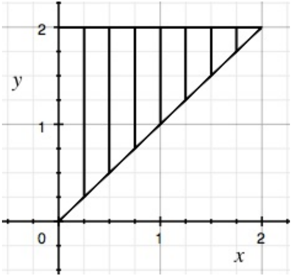
$$v_t(t,x) + v_x(t,x) + v_{xxx}(t,x) + \omega v(t,x) =$$
$$-\int_x^L u(t,y)\Big(k_{xxx}(x,y) + k_x(x,y) + k_{yyy}(x,y) + k_y(x,y) + \omega k(x,y)\Big)dy$$
$$+ k(x,L)u_{xx}(t,L) + u_x(t,x)\left(3\frac{d}{dx}k(x,x)\right)$$
$$+ u(t,x)\left(\omega + k_{xx}(x,x) - k_{yy}(x,x) + \frac{d}{dx}k_x(x,x) + \frac{d^2}{dx^2}k(x,x)\right).$$

Kernel function

Thus, we obtain that the kernel $k = k(x,y)$ defined in the triangle

$$\mathcal{T} = \{(x,y) \,/\, x \in [0,L],\ y \in [x,L]\}$$

must satisfy one third-order PDE with 3 boundary conditions



$$\boxed{\begin{array}{rcl} k_{xxx}(x,y) + k_{yyy}(x,y) + k_x(x,y) + k_y(x,y) & = & -\omega k(x,y) \\ k(x,L) & = & 0 \\ k(x,x) & = & 0 \\ k_x(x,x) & = & \dfrac{\omega}{3}(L-x) \end{array}}$$

Kernel function

Let us make the following change of variable

$$\boxed{t = y - x, \quad s = x + y,}$$

and define

$$G(s,t) := k(x,y)$$

We have

$$k(x,y) = G(x+y, y-x)$$

and therefore

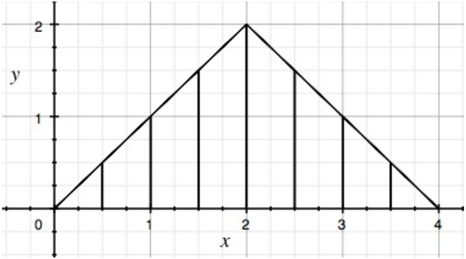
$$\begin{aligned} k_x = G_s - G_t, \quad k_y = G_s + G_t, \\ k_{xx} = G_{ss} - 2G_{st} + G_{tt}, \quad k_{yy} = G_{ss} + 2G_{st} + G_{tt}, \\ k_{xxx} = G_{sss} - 3G_{sst} + 3G_{stt} - G_{ttt}, \\ k_{yyy} = G_{sss} + 3G_{sst} + 3G_{stt} + G_{ttt} \end{aligned}$$

Kernel function

Now, the function $G = G(s,t)$, defined in

$$\mathcal{T}_0 = \{(s,t) \,/\, t \in [0,L], s \in [t, 2L{-}t]\}$$

satisfies



$$\boxed{\begin{array}{rcll} 6G_{tts}(s,t) + 2G_{sss}(s,t) + 2G_s(s,t) & = & -\omega G(s,t), & \text{in } \mathcal{T}_0, \\ G(s, 2L-s) & = & 0, & \text{in } [L, 2L], \\ G(s,0) & = & 0, & \text{in } [0, 2L], \\ G_t(s,0) & = & \dfrac{\omega}{6}(s-2L), & \text{in } [0, 2L]. \end{array}}$$

Kernel function

Let us transform this system into an integral one.

- We write the equation in variables (η, ξ), integrate ξ in $(0, \tau)$ and use that $6G_{ts}(\eta, 0) = \omega$.
- We integrate τ in $(0, t)$ and use that $G_s(\eta, 0) = 0$.
- We integrate η in $(s, 2L - t)$ and use that $G(2L - t, t) = 0$.

Thus, we can write the following integral form for $G = G(s, t)$

$$\begin{aligned} G(s,t) &= -\frac{\omega t}{6}(2L - t - s) \\ &\quad + \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2G_{sss}(\eta,\xi) + 2G_s(\eta,\xi) + \omega G(\eta,\xi)\Big) d\xi d\tau d\eta. \end{aligned}$$

Kernel function

To prove that such a function $G = G(s,t)$ exists, we use the method of successive approximations. We take as an initial guess

$$G^1(s,t) = -\frac{\omega t}{6}(2L - t - s)$$

and define the recursive formula as follows,

$$G^{n+1}(s,t) = \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2G^n_{sss}(\eta,\xi) + 2G^n_s(\eta,\xi) + \omega G^n(\eta,\xi)\Big)d\xi d\tau d\eta.$$

Performing some computations, we get for instance

$$G^2(s,t) = \frac{1}{108}\Big\{t^3\big(\omega - \omega^2 L + \frac{\omega^2 t}{4}\big)(2L - t - s) + \frac{t^3\omega^2}{4}\big[(2L-t)^2 - s^2\big]\Big\},$$

Kernel function

... and more generally the following formula

$$G^k(s,t) = \sum_{i=1}^{k} \left(a_k^i t^{2k-1} + b_k^i t^{2k}\right)\left[(2L-t)^i - s^i\right],$$

where the coefficients satisfy $b_k^k = 0$ and more importantly, there exist positive constants M, B such that, for any $k \geq 1$ and any $(s,t) \in \mathcal{T}_0$

$$\left|G^k(s,t)\right| \leq M \frac{B^k}{(2k)!}(t^{2k-1} + t^{2k}).$$

This implies that the series $\sum_{n=1}^{\infty} G^n(s,t)$ is uniformly convergent in $\mathcal{T}_0$.

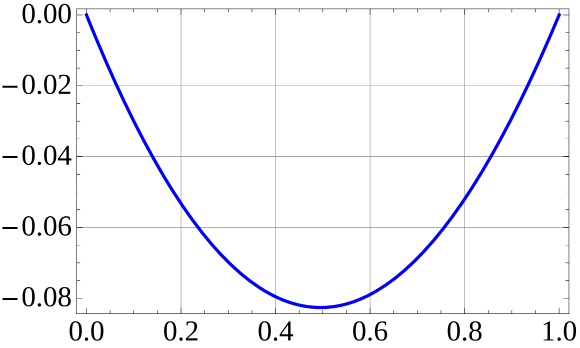
Kernel function

We get a solution of our integral equation. Indeed,

$$
\begin{aligned}
G = G^1 + \sum_{n=1}^{\infty} G^{n+1} \\
&= G^1 + \frac{1}{6} \sum_{n=1}^{\infty} \int_s^{2L-t} \int_0^t \int_0^\tau \Big(2G^n_{sss}(\eta,\xi) \\
&\qquad + 2G^n_s(\eta,\xi) + \omega G^n(\eta,\xi) \Big) d\xi d\tau d\eta \\
&= G^1 + \frac{1}{6} \int_s^{2L-t} \int_0^t \int_0^\tau \Big(2 \sum_{n=1}^{\infty} G^n_{sss}(\eta,\xi) \\
&\qquad + 2 \sum_{n=1}^{\infty} G^n_s(\eta,\xi) + \omega \sum_{n=1}^{\infty} G^n(\eta,\xi) \Big) d\xi d\tau d\eta \\
&= G^1 + \frac{1}{6} \int_s^{2L-t} \int_0^t \int_0^\tau \Big(2G_{sss}(\eta,\xi) + 2G_s(\eta,\xi) + \omega G(\eta,\xi) \Big) d\xi d\tau d\eta.
\end{aligned}
$$

Kernel function

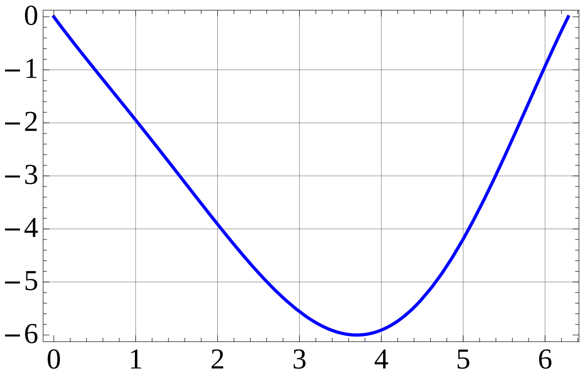
We plot the gain kernel $k(0, y)$ as a function of $y \in [0, L]$ for the length (a) $L = 1$ (non-critical). $\omega = 1$.



(a)

Kernel function

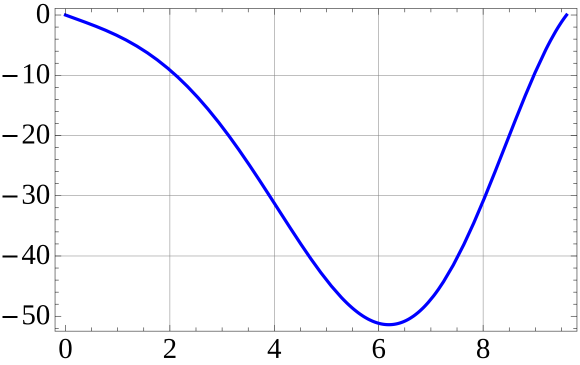
We plot the gain kernel $k(0, y)$ as a function of $y \in [0, L]$ for the length (b) $L = 2\pi$ (critical). $\omega = 1$.



(b)

Kernel function

We plot the gain kernel $k(0,y)$ as a function of $y \in [0,L]$ for the length (c) $L = 2\pi\sqrt{7/3}$ (critical). $\omega = 1$.



(c)

Stability Linear System

We know that the target system is exponentially stable. In order to get the same conclusion for the original linear system the method we are applying uses the inverse transformation Π^{-1}. For that, we introduce a kernel function $\ell(x,y)$ which satisfies

$$\boxed{\begin{array}{rcl} \ell_{xxx}(x,y)+\ell_{yyy}(x,y)+\ell_x(x,y)+\ell_y(x,y) & = & \omega\ell(x,y), \\ \ell(x,L) & = & 0, \\ \ell(x,x) & = & 0, \\ \ell_x(x,x) & = & \frac{\omega}{3}(L-x) \end{array}}$$

The existence and uniqueness of such a kernel $\ell=\ell(x,y)$ are proven in the same way than for the kernel $k=k(x,y)$ previously. Once we have defined $\ell=\ell(x,y)$, it is easy to see that the transformation Π^{-1} is characterized by

$$\boxed{u(x)=\Pi^{-1}(v(x)) := v(x)+\int_x^L \ell(x,y)v(y)dy.}$$

Stability Linear System

The operator $\Pi : L^2(0,L) \to L^2(0,L)$, is continuous and consequently we have the existence of a positive constant D_κ such that

$$\|\Pi(f)\|_{L^2(0,L)} \leq D_\kappa \|f\|_{L^2(0,L)}, \quad \forall f \in L^2(0,L).$$

The map $\Pi^{-1} : L^2(0,L) \to L^2(0,L)$ is also continuous and therefore we get the existence of a positive constant D_ℓ such that

$$\|\Pi^{-1}(f)\|_{L^2(0,L)} \leq D_\ell \|f\|_{L^2(0,L)}, \quad \forall f \in L^2(0,L).$$

Stability Linear System

Given $u_0 \in L^2(0,L)$, we define

$$v_0(x) = \Pi(u_0(x)) := u_0(x) - \int_x^L k(x,y)u_0(y)dy.$$

The solution of target system with initial condition $v(0,x) = v_0(x)$ satisfies

$$\|v(t,\cdot)\|_{L^2(0,L)} \leq e^{-\omega t}\|v_0(\cdot)\|_{L^2(0,L)}, \quad \forall t \geq 0.$$

Moreover, the solution of linear KdV is given by $u(t,x) = \Pi^{-1}(v(t,x))$. Thus,

$$\begin{aligned}\|u(t,\cdot)\|_{L^2(0,L)} \leq D_\ell\|v(t,\cdot)\|_{L^2(0,L)} \leq D_\ell e^{-\omega t}\|v_0(\cdot)\|_{L^2(0,L)} \\ \leq D_\ell D_k e^{-\omega t}\|u_0(\cdot)\|_{L^2(0,L)}\end{aligned}$$

Nonlinear System

Let $u = u(t,x)$ be a solution of the nonlinear KdV equation with the control given by

$$K(t) = \int_0^L k(0,y)u(t,y)dy,$$

Then, $v = \Pi(u(t,x))$ satisfies

$$v_t(t,x) + v_x(t,x) + v_{xxx}(t,x) + \omega v(t,x) = \\ - \left(v(t,x) + \int_x^L \ell(x,y)v(t,y)dy \right) \left(v_x(t,x) + \int_x^L \ell_x(x,y)v(t,y)dy \right)$$

with homogeneous boundary conditions

$$v(t,0) = 0, \quad v(t,L) = 0, \quad v_x(t,L) = 0.$$

Nonlinear System

We multiply by v and integrate in $(0, L)$ to obtain

$$\frac{d}{dt}\int_0^L |v(t,x)|^2 dx = -|v_x(t,0)|^2 \\ - 2\omega\int_0^L |v(t,x)|^2 dx - 2\int_0^L v(t,x)F(t,x)dx$$

where the term $F = F(t,x)$ is given by

$$F(t,x) = v(t,x)\int_x^L \ell_x(x,y)v(t,y)dy + v_x(t,x)\int_x^L \ell(x,y)v(t,y)dy \\ + \left(\int_x^L \ell(x,y)v(t,y)dy\right)\left(\int_x^L \ell_x(x,y)v(t,y)dy\right)$$

Nonlinear System

We can prove that there exists a positive constant $C = C(\|\ell\|_{C^1(\mathcal{T})})$ such that

$$\left|2\int_0^L v(t,x)F(t,x)dx\right| \leq C\left(\int_0^L |v(t,x)|^2\right)^{3/2}$$

and therefore, if there exists $t_0 \geq 0$ such that

$$\|v(t_0,\cdot)\|_{L^2(0,L)} \leq \frac{\omega}{C},$$

then we obtain

$$\frac{d}{dt}\int_0^L |v(t,x)|^2 dx \leq -\omega \int_0^L |v(t,x)|^2 dx, \quad \forall t \geq t_0.$$

Nonlinear System

Thus, we get

Theorem (EC-Coron 2013)

For any $\omega > 0$, there exist a feedback control law $K_\omega = K_\omega(u(t,\cdot))$, $r > 0$ and $D > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq De^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0,$$

for any solution of nonlinear KdV satisfiying $\|u_0\|_{L^2(0,L)} \leq r$.

Remarks

- The backstepping method has been applied to build some boundary feedback laws, which locally stabilize the Korteweg-de Vries equation posed on a finite interval.
- Our control acts on the Dirichlet boundary condition at the left hand side of the interval where the system evolves.
- The closed-loop system is proven to be locally exponentially stable with a decay rate that can be chosen to be as large as we want.

Remarks

Let us consider one or two control inputs at the right hand side

$$u(t,0) = 0, \quad u(t,L) = K_1(t), \quad u_x(t,L) = K_2(t)$$

To impose $v_t + v_x + v_{xxx} + \omega v = 0$, we have to vanish

$$\boxed{k(x,L)u_{xx}(t,L)} + k(x,L)u(t,L) + k_{yy}(x,L)u(t,L) - k_y(x,L)u_x(t,L)$$

As we do not have to our disposal $u_{xx}(t,L)$, the first term above arises the condition $k(x,L) = 0$.

Moreover, to keep $w(t,0) = u(t,0) = 0$, we have to impose $k(0,y) = 0$ for any $y \in (0,L)$. We get four boundary restrictions (the other two are on $k(x,x)$), the third order kernel equation satisfied by $k = k(x,y)$ may become overdetermined.

Remarks

A natural idea to deal with controls at $x = L$ is to use

$$v(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy,$$

If we do so, we deal now with the extra condition $k_y(x,0) = 0$ for any $x \in (0,L)$. This is due to the fact that when imposing $v_t + v_x + v_{xxx} + \omega v = 0$ on the target system, we get the extra term $u_x(t,0)k_y(x,0)$ to be cancelled. As previously, this fourth restriction may give an overdetermined kernel equation for $k = k(x,y)$.

Moreover, the existence of critical lengths when only one control is considered at the right end-point suggests that either the existence of the kernel or the invertibility of the corresponding map Π should fail for some spatial domains.

As mentioned before, [Coron, Liu, 2014] solve this problem changing the structure of the transformation.

Output feedback control

GOAL: To design a controller $u = K(y(t))$ depending on some partial measurements $y(t)$ of the solution and not on the full state $u = u(t,x)$.

What measurements?

The natural choice for the KdV equation should be $y(t) = u_x(t,0)$.

Unfortunately, the system is not observable with this choice. (Critical values)

In this paper we consider the output given by

$$\boxed{y(t) = u_{xx}(t,L).}$$

By using this measurement, we build an observer and apply the backstepping method to design an output feedback control which exponentially stabilizes the closed-loop system.

Output feedback control

Lemma

Let us consider system

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \\ u(t,0) = \kappa(t),\ u(t,L) = 0,\ u_x(t,L) = 0, \\ u(0,x) = u_0(x), \end{cases}$$

where $u_0 \in H^3(0,L)$ *and* $\kappa(t) \in H^1(0,T)$. *Then*
$u \in C([0,T], H^3(0,L)) \cap L^2(0,T; H^4(0,L))$ *and* $u_{xx}(\cdot, L) \in C([0,T])$.

Definition

Let us introduce the new transformation Π_o defined by:

$$u(t,x) = \Pi_o(w(t,x)) = w(t,x) - \int_x^L p(x,y)w(t,y)dy$$

where an appropriate kernel function $p = p(x,y)$.

Output feedback control

By following a classical approach, we construct the following observer:

$$\begin{cases} \hat{u}_t + \hat{u}_x + \hat{u}_{xxx} + p(x,L)[y(t) - \hat{u}_{xx}(t,L)] = 0, \\ \hat{u}(t,0) = \kappa(t),\ \hat{u}(t,L) = \hat{u}_x(t,L) = 0, \\ \hat{u}(0,x) = 0, \end{cases} \tag{6}$$

$$y(t) = u_{xx}(t,L).$$

Theorem (Marx-EC, 2014)

For any $\omega > 0$, there exist a feedback law $\kappa(t) := \kappa(\hat{u}(t,x))$, a function $p = p(x,y)$, and a constant $C > 0$ such that the coupled system (LKdV)-(6) is globally exponentially stable with a decay rate equals to ω, i.e., for any $u_0 \in H^3(0,L)$ we have

$$\|u(t,\cdot)\|_{H^3(0,L)} + \|\hat{u}(t,\cdot)\|_{L^2(0,L)} \le Ce^{-\omega t}\|u_0\|_{H^3(0,L)}$$

Output feedback control

By using the output feedback control

$$\kappa(t) = \int_0^L k(0,y)\hat{u}(t,y)dy,$$

the transformations Π and Π_o, we can see that $(\tilde{u} = u - \hat{u}, \hat{u})$ are mapped into $(\tilde{w}, \hat{w}) = (\Pi_o^{-1}(\tilde{u}), \Pi(\hat{u}))$ solutions of the target system

$$\begin{cases} \hat{w}_t + \hat{w}_x + \hat{w}_{xxx} + \omega \hat{w} = \\ -\left\{ p(x,L) - \displaystyle\int_x^L k(x,y) p(y,L) dy \right\} \tilde{w}_{xx}(t,L), \\ \hat{w}(0) = \hat{w}(L) = \hat{w}_x(L) = 0, \\ \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \omega \tilde{w} = 0, \\ \tilde{w}(0) = \tilde{w}(L) = \tilde{w}_x(L) = 0. \end{cases}$$

Output feedback control

To prove the exponential stability of $(\tilde{w}, \hat{w})$, we use a Lyapunov argument

$$V(t) = \frac{A}{2}\int_0^L |\hat{w}(t,x)|^2\,dx + \frac{B}{2}\int_0^L |\tilde{w}(t,x)|^2\,dx + \frac{B}{2}\int_0^L |\tilde{w}_{xxx}(t,x)|^2\,dx,$$

with A, B to be chosen later.

In this way, by tuning A, B large enough, we get for any $\epsilon > 0$ that

$$\dot{V}(t) \leq 2\Big(-\omega + \epsilon\Big)V(t),$$

which gives an exponential stability with decay rate as close to ω as we want.

Output feedback control

Theorem (Marx-EC, 2014)

Let $\omega > 0$ *given.* $\exists C > 0$, $\forall u_0 \in H^3(0,L)$, *the solution* $(u, \hat{u})$ *of*

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \\ u(t,0) = \int_0^L k(0,y)\hat{u}(t,y)dy, \ u(t,L) = 0, \ u_x(t,L) = 0, \\ u(0,x) = u_0(x), \end{cases}$$

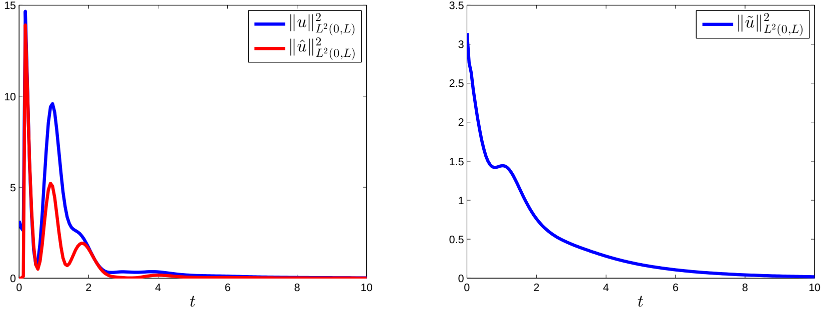
$$\begin{cases} \hat{u}_t + \hat{u}_x + \hat{u}_{xxx} + p(x,L)[u_{xx}(t,L) - \hat{u}_{xx}(t,L)] = 0, \\ \hat{u}(t,0) = \int_0^L k(0,y)\hat{u}(t,y)dy, \ \hat{u}(t,L) = \hat{u}_x(t,L) = 0, \\ \hat{u}(0,x) = 0, \end{cases}$$

satisfies

$$\|u(t,\cdot)\|_{H^3(0,L)} + \|\hat{u}(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{H^3(0,L)}$$

Simulations work fine, even for the nonlinear system

Good behavior of the observer:



Left: Evolution of the L^2-norm for the state (blue line) and the observer (red line). Right: Time evolution of the L^2-norm for the observation error $u - \hat{u}$.

Remarks

- Not able to deal with the nonlinear system because regularity issues.
- We are working with Swann Marx on other configurations inputs-outputs to overcome this mathematical difficulty.
- Other related works by [Tang and Krstic, 2013 and 2015], [Hassan, 2016].