Control of some reaction-diffusion equations

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What is the plan?

1. Control of ordinary differential equations
   - Linear ODE
   - Nonlinear ODE
2. What is a reaction-diffusion equation?
3. Control of the heat equation
4. Control of the Kuramoto-Sivashinsky equation

Controllability of ODE

Controllability of ODE

Let us consider $x : [0, T] \to \mathbb{R}^n$ the solution of

$$\dot{x}(t) = f(x(t), u(t))$$

$$x(0) = x_0$$

where $u : [0, T] \to \mathbb{R}^m$ is a given function.

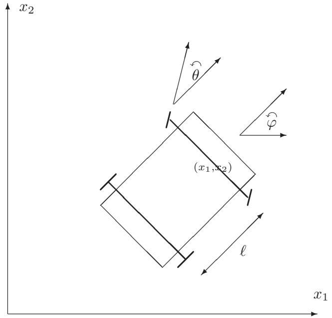
For any $t \in [0, T]$:

- We call $x(t) \in \mathbb{R}^n$ the state of the system
- We call $u(t) \in \mathbb{R}^m$ the control of the system

Goal:
Choosing the control u such that the trajectory x behaves as we desire.

Example: Look for a control u driving the system to the rest $x(T) = 0$.

Example: Nelson's car



$$
\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \varphi \\ \theta \end{pmatrix} = u_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + u_2 \begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin(\theta) \\ 0 \end{pmatrix}
$$

Controllability of ODE

Given $x_0, x_1 \in \mathbb{R}^n$, there exists $u : [0, T] \to \mathbb{R}^m$ such that

$$\dot{x}(t) = f(x(t), u(t))$$

$$x(0) = x_0$$

$$x(T) = x_1 \,?$$

If the answer is YES for any states x_0, x_1, we say the system is controllable.

If the answer is YES when x_0, x_1 are close to some equilibrium point x_e, we say the system is locally controllable around x_e.

Controllability of linear ODE

Given $A \in M_{n\times n}(\mathbb{R})$ y $B \in M_{n\times m}(\mathbb{R})$,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

$$\Longrightarrow \quad x(T) = e^{AT}x_0 + \int_0^T e^{(T-t)A}Bu(t)dt$$

If the symmetric matrix

$$\mathcal{C} = \int_0^T e^{(T-t)A}BB^* e^{(T-t)A^*}dt$$

is invertible, then we can go from x_0 to x_1 in time T by applying the control

$$u(t) = B^* e^{(T-t)A^*}\mathcal{C}^{-1}(x_1 - e^{TA}x_0), \quad \forall s \in [0,T].$$

Theorem

System $\dot{x} = Ax + Bu$ is controllable if and only if $\mathcal{C}$ is invertible.

Kalman condition, 1963 (see LaSalle 1960)

The controllability of

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

is equivalent to the condition that the matrix

$$\underbrace{\left[B \,|\, \underbrace{AB}_{m \text{ columns}} \,|\, A^2 B \,|\, \cdots \,|\, A^{n-1} B \right]}_{n \cdot m \text{ columns}}$$

has full rank n.

Remark

The controllability does not depend on T if A and B are stationary.

Controllability of nonlinear ODE

First idea: To linearize!

It is not a bad idea... in fact we have

Theorem

The control system $\dot{x} = f(x,u)$ *is locally controllable around* (x_e, u_e) *if the system linearized around* (x_e, u_e)*, given by*

$$\dot{x}(t) = \frac{\partial f}{\partial x}(x_e, u_e)x(t) + \frac{\partial f}{\partial u}(x_e, u_e)u(t),$$

is controllable.

Example of Nelson's car

We linearize around $\vec{0}$ the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \varphi \\ \theta \end{pmatrix} = u_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + u_2 \begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin(\theta) \\ 0 \end{pmatrix}$$

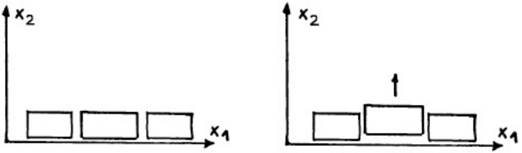
and we get

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \varphi \\ \theta \end{pmatrix} = u_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + u_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We see that $\dot{x}_2(t) = 0$ y $\dot{\varphi}(t) = 0$, for any control (u_1, u_2). Therefore, we cannot impose anything on the second and third components.

Example of Nelson's car

How do we do to get out from a parking spot?
(Without hitting other cars!!)



The answer is given by the experience. We have to iterate the sequence:

1. Turn wheels to the left
2. Go forward.
3. Turn wheels to the right
4. Go backward.

Example of Nelson's car

In terms of our controls, the sequence is:

$$(u_1, u_2)(t) = \begin{cases} (+1,0), & t \in (0,h) \\ (0,+1), & t \in [h,2h) \\ (-1,0), & t \in [2h,3h) \\ (0,-1), & t \in [3h,4h) \end{cases}$$

After Taylor developments, the result is:

$$\vec{x}(4h) = \vec{x}_0 + h^2 \underbrace{(f_2' f_1 - f_1' f_2)(\vec{x}_0)}_{[f_1, f_2](\vec{x}_0)} + o(h^2)$$

where

$$f_1(\vec{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad f_2(\vec{x}) = \begin{pmatrix} \cos(\varphi + \theta) \\ \sin(\varphi + \theta) \\ \sin(\theta) \\ 0 \end{pmatrix}$$

Example of Nelson's car

Notice that $[f_1, f_2](\vec{0}) = (0, 1, 1, 0)^*$.

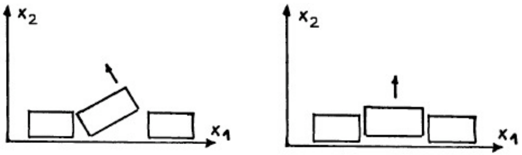


Figure: Directions $(0, 1, 1, 0)^*$ (left) and $(0, 1, 0, 0)^*$ (right)

Combining the previous sequence with u_2 (go forward-backward) we can get a movement in the direction

$$\Big[[f_1, f_2], f_2\Big](\vec{x}_0)$$

In this case, this is the desired lateral translation

$$\Big[[f_1, f_2], f_2\Big](\vec{0}) = (0, 1, 0, 0)^*$$

Lie brackets condition. Rashevski 1938, Chow 1939

Theorem

If any vector in $\mathbb{R}^n$ is generated by

$$span\left(\begin{array}{c}\{f_1(x_0), f_2(x_0), \ldots, f_n(x_0)\}\\ \bigcup \\ \{\textit{Every Lie bracket involving } f_i|_{x_0}\}\end{array}\right),$$

then the system

$$\dot{x} = \sum_{i=1}^{m} u_i(t) f_i(x)$$

is locally controllable around x_0.

What is a reaction-diffusion equation?

Diffusion phenomena

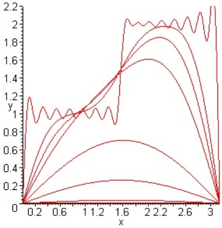
- We call $u(t,x)$ the concentration of some substance
- We call $\vec{Q}(t,x)$ the vector flux of the substance
- Let us consider a small volume V with surface boundary ∂V
- We have

$$\frac{d}{dt}\int_V u\,dV = -\int_{\partial V} \vec{Q}\,d\vec{S}$$

Diffusion phenomena

- We have
$$\frac{d}{dt}\int_V u\,dV = -\int_{\partial V} \vec{Q}\,d\vec{S}$$
- From divergence theorem
$$\frac{d}{dt}\int_V u\,dV = -\int_V div(\vec{Q})\,dV$$
- We assume Fourier's law $\vec{Q} = -k\nabla u$ to get
$$\frac{d}{dt}\int_V u\,dV = k\int_V div(\nabla u)\,dV = k\int_V \Delta u\,dV$$
- As V is arbitrary, we get the diffusion equation
$$\frac{\partial u}{\partial t} = k\Delta u$$
- Some linear or nolinear reaction terms can be added
$$\frac{\partial u}{\partial t} = k\Delta u - u + u^2$$

Heat equation on a bounded domain of $\mathbb{R}^n$



$$\boxed{\begin{array}{l} u_t(t,x) - \Delta u(t,x) = 0,\ x \in \Omega \\ u(t,x) = 0,\ x \in \partial\Omega \\ u(0,x) = u_0(x),\ x \in \Omega \end{array}}$$

- The heat equation has a regularizing effect. Even if u_0 is irregular, then the solution u becomes regular inmediately ($\forall t > 0$).
- In a domain $[0, L]$, with $u_0 = \sum_{n\geq 1} C_n \sin\left(\frac{n\pi x}{L}\right)$, the solution is

$$u(t,x) = \sum_{n\geq 1} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}t}$$

The system is asymptotically stable: $u(t,x) \to 0$ as $t \to \infty$

Control of the heat equation

Control of the heat equation

We desire to control the temperature distribution in a room Ω whose boundaries $\partial\Omega$ are kept at constant temperature 0. We use a heater F localized on a subdomain ω.

Temperature: $u = u(t,x), x \in \Omega, t \in [0,T]$

$$\boxed{\begin{array}{l} u_t(t,x) - \Delta u(t,x) = F(t,x)1_\omega, \quad \text{si } x \in \Omega \\ u(t,x) = 0, \quad \text{si } x \in \partial\Omega \\ u(0,x) = u_0(x), \quad \text{si } x \in \Omega \end{array}}$$

Internal controllability problem:

How do we choose the control F such that $u(T,x)$ is nice for all of us?

Control of the heat equation

By linearity, we can restrict to the case $u_0 = 0$. We define the linear operator

$$B : F \in L^2(0,T;L^2(\omega)) \mapsto u(T,\cdot) \in L^2(\Omega)$$

Definition

- *Heat equation is controllable if B is onto.*
- *Heat equation is approximately controllable if B has a dense image in $L^2(\Omega)$*

Characterization (Functional Analysis)

- *B has dense image in $L^2(\Omega)$ $\Leftrightarrow$ the adjoint operator B^* is one-to-one.*
- *B is onto $\Leftrightarrow$ There exists $C > 0$ such that*

$$\|B^*(\phi_T)\|_{L^2(0,T;L^2(\omega))} \geq C\|\phi_T\|_{L^2(\Omega)}, \quad \forall \phi_T \in L^2(\Omega).$$

REM: Operator B cannot be onto because the regularizing effect.

Control of the heat equation

The adjoint operator of

$$B : F \in L^2(0,T;L^2(\omega)) \mapsto u(T,\cdot) \in L^2(\Omega)$$

is given by

$$B^* : \phi_T \in L^2(\Omega) \mapsto \phi 1_\omega \in L^2(0,T;L^2(\omega))$$

where $\phi = \phi(t,x)$ satisfies

$$\boxed{(\mathrm{Adj}) \begin{array}{ll} -\phi_t(t,x) - \Delta\phi(t,x) = 0, & \text{si } x \in \Omega \\ \phi(t,x) = 0, & \text{si } x \in \partial\Omega \\ \phi(T,x) = \phi_T(x), & \text{si } x \in \Omega \end{array}}$$

The heat equation is approximately controllable $\Leftrightarrow$

$$\boxed{(PCU) \quad \phi(t,x) = 0,\ \forall t,\ \forall x \in \omega \implies \phi(t,x) = 0,\ \forall t,\ \forall x \in \Omega}$$

THIS HOLDS (Tools: Holmgrem's theorem, Carleman inequalities)

Control of the Kuramoto-Sivashinsky equation

The Kuramoto-Sivashinsky equation

The Kuramoto-Sivashinsky equation is a fourth order nonlinear parabolic equation:

$$y_t + y_{xxxx} + \gamma y_{xx} + yy_x = 0.$$

Here, $\gamma > 0$ is called the anti-diffusion coefficient.

It models front propagation in reaction-diffusion systems and describes incipient instabilities in a variety of physical and chemical problems.

To study this equation on a bounded space-interval $[0, 1]$, we have to impose four boundary conditions. For instance,

$$y(t, 0) = 0, \quad y_x(t, 0) = 0, \quad y(t, 1) = 0, \quad y_x(t, 1) = 0.$$

The Kuramoto-Sivashinsky equation

Let us consider the linear equation (drop the term yy_x).

$$A : w \in D(A) \subset L^2(0,1) \longmapsto -w'''' - \gamma w'' \in L^2(0,1),$$
$$D(A) := H^4(0,1) \cap H_0^2(0,1).$$

A is self-adjoint with compact resolvent. The spectrum of A is a discrete subset $\{\sigma_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}$ satisfying $\lim_{k\to\infty} \sigma_k = -\infty$

The eigenfunctions $\{\phi_k\}_{k\in\mathbb{N}}$ form a basis of $L^2(0,1)$.

$$\begin{cases} -\gamma\phi_k'' - \phi_k'''' = \sigma_k\phi_k, \\ \phi_k(0) = \phi_k(1) = \phi_k'(0) = \phi_k'(1) = 0. \end{cases}$$

The operator A has some positive eigenvalues IFF $\gamma > 4\pi^2$.

If $\gamma < 4\pi^2$, then all the eigenvalues are negative and therefore the linear system is asymptotically stable. The nonlinear KS equation is also asymptotically stable [Liu-Krstic, 2001].

The Kuramoto-Sivashinsky equation

Let $\gamma > 0$.

$$\begin{aligned} y_t + y_{xxxx} + \gamma y_{xx} + y y_x &= f, \\ y(t,0) = 0, \quad y(t,1) &= 0, \\ y_x(t,0) = 0, \quad y_x(t,1) &= 0, \\ y(0,x) &= y_0(x). \end{aligned}$$

Well-posedness I:

$$\begin{array}{l} y_0 \in L^2(0,1) \\ f \in L^1(0,T;L^2(0,L)) \end{array} \Longrightarrow \left\{ \begin{array}{l} \exists! \text{ solution} \\ y \in C([0,T],L^2(0,1)) \cap L^2(0,T;H^2(0,1)) \end{array} \right.$$

Well-posedness II:

$$\begin{array}{l} y_0 \in H^{-2}(0,1) \\ f \in L^1(0,T;W^{-1,1}(0,L)) \end{array} \Longrightarrow \left\{ \begin{array}{l} \exists! \text{ solution} \\ y \in C([0,T],H^{-2}(0,1)) \cap L^2(0,T;L^2(0,1 \end{array} \right.$$

Control - Linear Kuramoto-Sivashinsky equation

Question: [Null controllability]

Let $T > 0$ and $\omega \subset (0, L)$. Given a state $y_0 \in H^{-2}(0,1)$. Does there exist some control $h \in L^2(0,T;L^2(\omega))$ *driving the solution of*

$$\begin{aligned} y_t + y_{xxxx} + \gamma y_{xx} &= h\mathbf{1}_\omega, \\ y(t,0) = 0, \quad y(t,1) &= 0, \\ y_x(t,0) = 0, \quad y_x(t,1) &= 0, \\ y(0,x) &= y_0(x). \end{aligned}$$

from y_0 *to* 0*?*

Control - Linear Kuramoto-Sivashinsky equation

Let $z_T \in H_0^2(0,1)$. Let z be the solution of the adjoint equation

$$\begin{cases} -z_t + z_{xxxx} + \gamma z_{xx} = 0, \\ z(t,0) = 0, \quad z(t,1) = 0, \\ z_x(t,0) = 0, \quad z_x(t,1) = 0, \quad z(T,\cdot) = z_T. \end{cases}$$

We get

$$\langle y_0, z(0) \rangle_{-2,2} = \langle y(T), z_T \rangle_{-2,2} + \int_0^T \int_\omega h(t,x) z(t,x)\, dxdt.$$

From this, we obtain the following

Lemma

The function $h = h(t,x)$ is a control driving the system from y_0 to zero if and only if

$$\forall z_T \in H_0^2(0,1), \quad \langle y_0, z(0) \rangle_{-2,2} = \int_0^T \int_\omega h(t,x) z(t,x)\, dxdt.$$

Control - Linear Kuramoto-Sivashinsky equation

There exists a control $h = h(t,x)$ such that

$$\forall z_T \in H_0^2(0,1), \quad \langle y_0, z(0)\rangle_{-2,2} = \int_0^T \int_\omega h(t,x) z(t,x)\, dxdt \quad ??$$

Lemma

Such a control exists if we prove

$$(OBS) \quad \exists C > 0, \quad \forall z_T, \quad \|z(0,x)\|_{H_0^2(0,1)} \leq C \|z\|_{L^2(0,T;L^2(\omega))}$$

where z is the solution of the adjoint equation.

If (OBS) holds, we can define this continuous linear map

$$\Lambda : z1_\omega \in \bar{H} \subset L^2(0,T;L^2(\omega)) \longmapsto z(0,x) \in H_0^2(0,1),$$

where $H = \{z1_\omega ; z_T \in H_0^2(0,1)\}$. Thus, the map
$z1_\omega \in \bar{H} \subset L^2(0,T;L^2(\omega)) \longmapsto \langle y_0, z(0)\rangle_{-2,2} \in \mathbb{R}$
is linear and continuous. By Riesz, there exists a control $h = h(t,x)$.

Control - Kuramoto-Sivashinsky equation

To prove the previous inequality. Important tool: Carleman estimates.

Proposition (EC-Mercado, 2011, Zhou, 2012, Gao, 2015)

There exist $k_1, k_2, C > 0$ *such that* $\forall z_T \in H_0^2(0,L)$, $\forall f \in L^2_{xt}(e^{\frac{-2k_1}{(T-t)}})$

$$\|ze^{-\frac{k_2}{T-t}}\|_{L^\infty(W^{1,\infty})} + \int_0^1 |z_{xx}(0,x)|^2 \leq$$

$$C\left\{ \iint |f|^2 e^{-\frac{2k_1}{T-t}}\, dxdt + \int_0^T \int_\omega z e^{-\frac{2k_1}{T-t}}\, dxdt \right\}$$

$$\begin{cases} -z_t + z_{xxxx} + \gamma z_{xx} + a(x)z = f, \\ z(t,0) = 0, \quad z(t,1) = 0, \\ z_x(t,0) = 0, \quad z_x(t,1) = 0, \\ z(T,\cdot) = z_T. \end{cases}$$

$L^2(\rho) := \{h \in L^2; \int |h(y)|^2 \rho(y)\, dy < \infty\}$

Control - Kuramoto-Sivashinsky equation

By applying an inverse function argument, we obtain

Theorem

The KS equation is locally null controllable, i.e. for any $y_0 \in H^{-2}(0,1)$ *small enough (in norm), there exists some controls* $h \in L^2(0,T;L^2(\omega))$ *such that the solution of*

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} + y y_x = h\mathbf{1}_\omega, \\ y(t,0) = 0, \quad y(t,1) = 0, \\ y_x(t,0) = 0, \quad y_x(t,1) = 0, \\ y(0,\cdot) = y_0. \end{cases}$$

satisfies $y(T) = 0$.

Other Results

Boundary control for KS

$$\boxed{\begin{array}{ll} y_t + y_{xxxx} + \gamma y_{xx} + y y_x = 0, & x \in (0,1),\ t > 0, \\ y(t,0) = h_1(t), \quad y(t,1) = 0, & t > 0, \\ y_x(t,0) = h_2(t), \quad y_x(t,1) = 0, & t > 0, \end{array}}$$

- Control and stabilization for linear KS. Roles of h_1 y h_2. (EC 10, EC+Guzmán+Mercado 15)
- Control for nonlinear KS (EC+Mercado 11)
- Control with neumann boundary conditions (EC+Guzmán+Mercado 15)

$$y_{xx}(t,0) = h_1(t), \quad y_{xxx}(t,0) = h_2(t), \quad y_{xx}(t,1) = y_{xxx}(t,1) = 0$$

Internal control for the Cahn-Hilliard (fourth-order with nonlinearity $(y^3)_{xx}$) with neumann boundary condition (Guzmán 15)

$$y_x(t,0) = 0, \quad y_{xxx}(t,0) = 0, \quad y_x(t,1) = y_{xxx}(t,1) = 0$$

Other Results

Control for a parabolic system coupling KS and heat equations

$$\boxed{\begin{array}{c} u_t + u_{xxxx} + \gamma u_{xx} + u u_x = v_x, \\ v_t - v_{xx} + c v_x = u_x + F 1_\omega, \end{array}}$$

+ Boundary Conditions.

- Boundary control with 3 inputs (EC+Mercado+Pazoto 12)
- Internal control acting on the KS equation (EC+Mercado+Pazoto 15)
- Internal control acting on the heat equation (Carreño+EC 15)

In this last case, we prove

$$(OBS) \quad \exists C > 0, \quad \|\varphi(0,x)\|_{H_0^2(0,1)} + \|\psi(0,x)\|_{H_0^1(0,1)} \leq C \|\varphi\|_{L^2(0,T;L^2(\omega))}$$

for any solution of

$$\begin{cases} -\varphi_t + \varphi_{xxxx} = -\psi_x, \\ -\psi_t - \Gamma \psi_{xx} = -\varphi_x. \end{cases}$$

To conclude

What did we do?

- Introduce the concept of controllability for systems described by ODE or PDE.
- Present some classical results for the control of EDO and one particular PDE: the heat equation.
- Present some recent results on the control of the Kuramoto-Sivashinsky equation.

Some open problems

- Control of the KS equation: higher dimension, discontinuous coefficients, cost of the control, degenerate coefficients, boundary control of the nonlinear KS equation with one control, etc
- Control of the KS system: higher dimension, boundary control with less than 3 controls, huge etc