On the control of the improved Boussinesq equation

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What is the plan?

1. The improved Boussinesq equation
2. Boundary control: lack of exact controllability
3. Boundary control: approximate controllability
4. Moving control: exact controllability

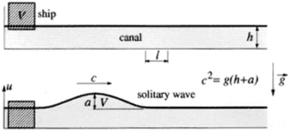
The improved Boussinesq equation – Origin

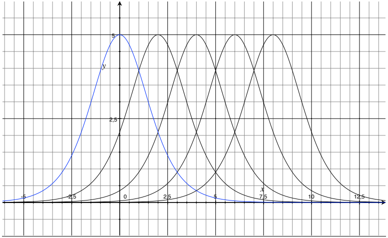
In [Boussinesq, 1872], it is derived the so called “bad” Boussinesq equation:

$$y_{tt} - y_{xx} - y_{xxxx} = (y^2)_{xx}. \qquad (1)$$

This equation describes the flow of shallow water waves with small amplitudes in a flat bottom canal. It is called “bad” because its poor existence and uniqueness properties. There is no local well-posedness results for equation (1). Only solitons-like solutions are known:

$$y(x,t) = \frac{A}{sech^2(kx - \omega t)}$$





The improved Boussinesq equation – Origin

The "good" Boussinesq equation is

$$y_{tt} - y_{xx} + y_{xxxx} = (y^2)_{xx}.$$

This equation, used to study nonlinear strings, is well-posed thanks to the "right" sign in front the of the fourth order operator.

There are control results for the "good" Boussinesq equation:

- Internal controls with periodic boundary conditions by [Zhang, 1998]
- Same result in lower regularity framework by [Cerpa, Rivas, 2016].
- Exact controllability from the boundary by [Lions, 1988] and [Crépeau, 2003].

The improved Boussinesq equation – Origin

In [Makhankov, 1978] it is proved that the "bad" Boussinesq equation can be approached by the following one, called improved Boussinesq equation,

$$y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx}.$$

We have here a well-posed equation [Zhijian, 1998] !

... control properties must be studied !

Boundary control: lack of exact controllability

Boundary control: existence of solutions

Given a time $T > 0$, we consider the linear improved Boussinesq equation posed on a bounded domain $(0,1)$ with a boundary control $h \in H^2(0,T)$ and initial conditions $y^0, y^1 \in L^2(0,1)$:

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = 0,\ y(1,t) = h(t), & t \in (0,T), \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in (0,1). \end{cases}$$

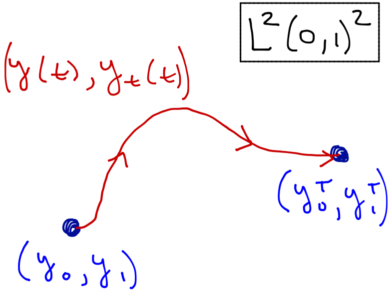
We can prove that the system is well-posed and that the solution satisfies

$$y \in C^1([0,T], L^2(0,1) \times L^2(0,1))$$

and in this way each solution can be seen as a continuous trajectory in the state space

$$(y(t), y_t(t)) \in L^2(0,1) \times L^2(0,1), \qquad \forall t \in [0,T].$$

Boundary control: definition of exact controllability



Definition (Exact controllability)

Given a time $T > 0$, we say that system

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = 0,\ y(1,t) = h(t), & t \in (0,T), \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in (0,1), \end{cases}$$

is exactly controllable if $\forall (y^0, y^1)$, $\forall (y_T^0, y_T^1)$, $\exists h$ such that y satisfies

$$y(T) = y_T^0 \quad \text{and} \quad y_t(T) = y_T^1.$$

Boundary control: spectral analysis

We can rewrite the homogeneous system

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = 0,\ y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in (0,1), \end{cases}$$

as follows,

$$y_{tt} + Ay = 0,$$

where, for $D(A) = H^2 \cap H_0^1(0,1)$, we define the operator

$$A : D(A) \subset L^2(0,1) \longrightarrow L^2(0,1)$$

$$w \longmapsto A(w) = (I - \partial_{xx})^{-1} \partial_{xx} w.$$

Proposition

There exists a basis of $L^2(0,1)$ formed by eigenfunctions $\{f_k\}_{k\in\mathbb{N}^*}$ of the operator A. Moreover, this family is given by $f_k(x) = \sqrt{2}\sin(k\pi x)$ and the corresponding eigenvalues are

$$\lambda_k = \frac{k^2\pi^2}{k^2\pi^2 + 1},$$

for any $k \in \mathbb{N}^*$.

Boundary control: homogeneous solutions

Remark

We can easily remark that the eigenvalues $\lambda_k \in \mathbb{R}^+$ are simple, and $\lim_{k\to+\infty} \lambda_k = 1$. Thus, the spectrum of A admits a finite point of accumulation.

Having in mind

$$H^s(0,1) = \left\{ \sum_{k\geq 1} a_k f_k(x) \Big/ \sum_{k\geq 1} k^{2s} |a_k|^2 \leq \infty \right\},$$

we obtain, for $s \geq 0$, and $y^0 = \sum_{k\geq 1} a_k f_k \in H_0^s(0,1)$, $y^1 = \sum_{k\geq 1} b_k f_k \in H_0^s(0,1)$, that the solution y belongs to $C^1([0,+\infty[, H_0^s(0,1))$ and

$$y(x,t) = \sum_{k\geq 1} \left(a_k \cos(\sqrt{\lambda_k} t) + \frac{b_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \right) f_k(x).$$

Remark

From the cosine and sine functions, we see that there is no gain of regularity for the improved Boussinesq equation. Moreover, from the asymptotic behavior of eigenvalues λ_k, we see the fact that the amplitude and velocity have the same regularity.

Boundary control: lack of exact controllability

Let y be the solution of the direct equation

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = 0,\ y(1,t) = h(t), & t \in (0,T) \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in (0,1). \end{cases} \tag{2}$$

Let z be the solution of the adjoint problem

$$\begin{cases} z_{tt} - z_{xx} - z_{xxtt} = 0, & (x,t) \in (0,1) \times (0,T) \\ z(0,t) = z(1,t) = 0, & t \in (0,T) \\ z(x,T) = z_T^0(x),\ z_t(x,T) = z_T^1(x), & x \in (0,1). \end{cases}$$

Suppose that the system (2) is exactly controllable. Then, there exists a boundary control h such that the solution of (2) with initial data

$$y^0 = \sum_{n\geq 1}^{N} a_n f_n, \quad y^1 = \sum_{n\geq 1}^{N} b_n f_n,$$

satisfies $y(T) = y_t(T) = 0$.

We obtain that if $y(T) = y_t(T) = 0$, then for any z_T^0, z_T^1,

$$\int_0^1 \left[y^1 \Big(z(x,0) - z_{xx}(x,0) \Big) - y^0 \Big(z_t(x,0) - z_{xxt}(x,0) \Big) \right] dx = \int_0^T \Big(h(t) + \ddot{h}(t) \Big) z_x(1,t)\, dt$$

Boundary control: lack of exact controllability

Using that for any z_T^0, z_T^1,

$$\int_0^1 \left[y^1\Big(z(x,0)-z_{xx}(x,0)\Big)-y^0\Big(z_t(x,0)-z_{xxt}(x,0)\Big)\right]dx=\int_0^T \Big(h(t)+\ddot{h}(t)\Big)z_x(1,t)dt$$

with appropriate trajectories:

- First $z(x,t)=e^{i\sqrt{\lambda_n}(T-t)}f_n(x)$
- Then $z(x,t)=e^{-i\sqrt{\lambda_n}(T-t)}f_n(x)$

We see that the control h has to satisfy, for any $n\geq 1$, the following equations

$$\begin{aligned}(1+n^2\pi^2)(i\sqrt{\lambda_n}a_n+b_n)&=\sqrt{2}(n\pi)(-1)^n\int_0^T(h(t)+\ddot{h}(t))e^{-i\sqrt{\lambda_n}t}dt,\\ (1+n^2\pi^2)(-i\sqrt{\lambda_n}a_n+b_n)&=\sqrt{2}(n\pi)(-1)^n\int_0^T(h(t)+\ddot{h}(t))e^{i\sqrt{\lambda_n}t}dt.\end{aligned} \quad (3)$$

Boundary control: lack of exact controllability

Let us define the complex function

$$F(z) := \int_0^T (h(t) + \ddot{h}(t)) e^{izt} dt.$$

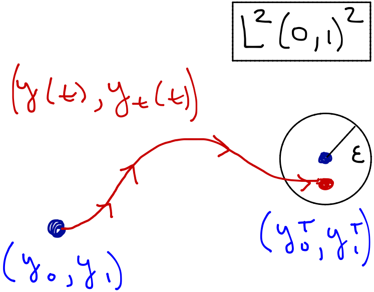
- F is an entire function and it satisfies $F(\pm\sqrt{\lambda_n}) = 0$, for all $n > N$.
- Remember that $\sqrt{\lambda_n} \to 1$ as $n \to \infty$
- Consequently, F vanishes on a set with a finite accumulation point.
- Therefore, we conclude that $F \equiv 0$.
- From (3), we easily obtain that $a_n = b_n = 0$ for each $n \geq 1$.
- That means that the trivial state is the only one which can be steered to zero.

Theorem (EC-Crépeau, 2015)

The linear improved Boussinesq equation is not spectrally controllable in $L^2(0,1)$.

Boundary control: approximate controllability

Boundary control: definition of approximate controllability



Definition (Approximate controllability)

Given a time $T > 0$, we say that system

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = 0,\ y(1,t) = h(t), & t \in (0,T), \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in (0,1), \end{cases}$$

is approximately controllable if $\forall \varepsilon > 0$, $\forall (y^0, y^1)$, $\forall (y_T^0, y_T^1)$, $\exists h$ such that y satisfies

$$\|y(T) - y_T^0\|_{L^2(0,1)} \leq \varepsilon \quad \text{and} \quad \|y_t(T) - y_T^1\|_{L^2(0,1)} \leq \varepsilon.$$

Boundary control: approximate controllability

By linearity, it is enough to consider null initial data.

Let y be the solution of the direct equation

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = 0,\ y(1,t) = h(t), & t \in (0,T) \\ y(x,0) = 0,\ y_t(x,0) = 0, & x \in (0,1). \end{cases} \tag{4}$$

Let us define the map

$$\Lambda : h \in H^2(0,T) \longmapsto (y(T), y_t(T)) \in L^2(0,1)^2.$$

Remark

Notice that system (4) *is approximately controllable if and only if the range of this linear operator* Λ *is dense in* $L^2(0,1)^2$.

Let us assume that it is not the case, i.e., there exist $w_T^0, w_T^1 \in L^2(0,1)\backslash\{0\}$ such that

$$-\int_0^1 y_t(T) w_T^0\, dx + \int_0^1 y(T) w_T^1\, dx = 0. \tag{5}$$

Let us define $z_T^0, z_T^1 \in H^2 \times H_0^1(0,1)$ such that

$$z_T^0 - \partial_{xx} z_T^0 = w_T^0, \quad \text{and} \quad z_T^1 - \partial_{xx} z_T^1 = w_T^1.$$

Boundary control: approximate controllability

Recall that we have null initial data $y(x,0) = y_t(x,0) = 0$.

Let us consider z as the solution of the adjoint equation with initial condition (at $t = T$) given by z_T^0, z_T^1. As before, we use the direct and adjoint equations to get

$$\int_0^1 \Big[-y_t(x,T)\Big(z(x,T) - z_{xx}(x,T)\Big) + y(x,T)\Big(z_t(x,T) - z_{xxt}(x,T)\Big)\Big] dx = \int_0^T \Big(h(t) + \ddot{h}(t)\Big) z_x(1,t) dt$$

Using the orthogonality

$$-\int_0^1 y_t(T) w_T^0 \, dx + \int_0^1 y(T) w_T^1 \, dx = 0,$$

we obtain, for any $h \in H^2(0,T)$, the identity

$$\int_0^T (h(t) + \ddot{h}(t)) z_x(1,t) dt = 0.$$

We prove now that we necessary have $z = 0$. This would imply that $z_T^0 = z_T^1 = 0$ and consequently $w_T^0 = w_T^1 = 0$, which is a contradiction.

Boundary control: approximate controllability

Indeed, let us choose $h(t) = e^{i\frac{2\pi n}{T}t}$, for $n \in \mathbb{Z}$. Then

$$0 = \int_0^T (h(t) + \ddot{h}(t)) z_x(1,t)dt = \left(1 - \left(\frac{2\pi n}{T}\right)^2\right) \int_0^T e^{i\frac{2\pi n}{T}t} z_x(1,t)dt, \; \forall n \in \mathbb{Z}.$$

Thus, $z_x(1,\cdot) \in SPAN\{e^{it}, e^{-it}\}$. As

$$z(x,t) = \sqrt{2} \sum_{n\geq 1} \left[c_n e^{i\sqrt{\lambda_n}(T-t)} + d_n e^{-i\sqrt{\lambda_n}(T-t)} \right] \sin(n\pi x),$$

we see that there exist $\alpha, \beta \in \mathbb{C}$ such that we have,

$$z_x(1,t) = \sum_{n\geq 1} \left[c_n e^{i\sqrt{\lambda_n}(T-t)} + d_n e^{-i\sqrt{\lambda_n}(T-t)} \right] \sqrt{2}(n\pi)(-1)^n = \alpha e^{it} + \beta e^{-it}.$$

From this, we see that

$$\sum_{n\geq 1} \left[c_n e^{i\sqrt{\lambda_n}(T-t)} + d_n e^{-i\sqrt{\lambda_n}(T-t)} \right] \sqrt{2}(n\pi)(-1)^n - \alpha e^{it} - \beta e^{-it} = 0, \qquad (6)$$

for all $t \in (0,T)$. As this function is analytic, it vanishes for any $t \in \mathbb{R}$.

Boundary control: approximate controllability

By using (6), we obtain for any $m \geq 1$ the following

$$\frac{1}{2S}\int_{-S}^{S}\left(\sum_{n\geq 1}\left[c_n e^{i\sqrt{\lambda_n}(T-t)} + d_n e^{-i\sqrt{\lambda_n}(T-t)}\right]\sqrt{2}(n\pi)(-1)^n - \alpha e^{it} - \beta e^{-it}\right) e^{i\sqrt{\lambda_m}t}dt$$

$$\longrightarrow_{S\to\infty} \sqrt{2}(m\pi)c_m e^{i\sqrt{\lambda_m}T} = 0,$$

and

$$\frac{1}{2S}\int_{-S}^{S}\left(\sum_{n\geq 1}\left[c_n e^{i\sqrt{\lambda_n}(T-t)} + d_n e^{-i\sqrt{\lambda_n}(T-t)}\right]\sqrt{2}(n\pi)(-1)^n - \alpha e^{it} - \beta e^{-it}\right) e^{-i\sqrt{\lambda_m}t}dt$$

$$\longrightarrow_{S\to\infty} \sqrt{2}(m\pi)d_m e^{-i\sqrt{\lambda_m}T} = 0.$$

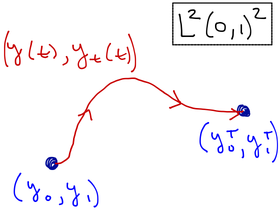
In consequence, for any $m \geq 1$, we obtain that $c_m = d_m = 0$. This fact implies that $z = 0$, which ends the proof.

Theorem (EC-Crépeau, 2015)

The linear improved Boussinesq equation is approximately controllable in $L^2(0,1)$.

Moving control: exact controllability

Moving control: definition of exact controllability



Definition (Exact controllability)

Given a time $T > 0$, we say that system

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx} + b(x+ct)h(t), & x \in \mathbb{T},\ t > 0, \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in \mathbb{T}, \end{cases}$$

is exactly controllable if $\forall (y^0, y^1)$, $\forall (y_T^0, y_T^1)$, $\exists h$ such that y satisfies

$$y(T) = y_T^0 \quad \text{and} \quad y_t(T) = y_T^1.$$

Remark

We consider b *such that* $\int_{\mathbb{T}} b(x)\, dx = 0$.

Moving control: main ideas

KEY IDEAS:

- Linearize around the origin, i.e., drop the nonlinear term $(y^2)_{xx}$.
- Make the following change of variables

$$v(x,t) = y(x - ct, t).$$

Thus, y solves

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx} + b(x+ct)h(t), & x \in \mathbb{T},\ t > 0, \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in \mathbb{T}, \end{cases}$$

if and only if v solves the following equation

$$\begin{cases} v_{tt} + (c^2 - 1)v_{xx} + 2cv_{xt} - v_{xxtt} - c^2 v_{xxxx} - 2cv_{txxx} = b(x)h(t), & x \in \mathbb{T},\ t > 0 \\ v(x,0) = y^0(x),\ v_t(x,0) = -cy^0_x(x) + y^1(x), & x \in \mathbb{T}. \end{cases}$$

The equation for v has much better spectral properties!

(... much better for control purposes!)

Moving control: spectral analysis

We study the spectral properties of the equation

$$v_{tt} - v_{xxtt} + 2cv_{xt} - 2cv_{txxx} + (c^2 - 1)v_{xx} - c^2 v_{xxxx} = 0. \tag{7}$$

Let us denote $w = v_t$ to write equation (7) as

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = A \begin{pmatrix} v \\ w \end{pmatrix} := \begin{pmatrix} w \\ (I - \partial_{xx})^{-1} \left[(1 - c^2) v_{xx} + c^2 v_{xxxx} + 2c(w_{xxx} - w_x) \right] \end{pmatrix}$$

Thus the eigenvalues $i\lambda$ of the linear operator A are solutions of

$$\begin{cases} w = i\lambda v, \\ (1 - c^2) v_{xx} + c^2 v_{xxxx} + 2c(w_{xxx} - w_x) = i\lambda(w - w_{xx}), \end{cases}$$

and we get two families of solutions, for $k \in \mathbb{Z}$,

$$\lambda_k^+ = \left(ck + \frac{|k|}{\sqrt{1 + k^2}} \right) \text{ and } \lambda_k^- = \left(ck - \frac{|k|}{\sqrt{1 + k^2}} \right).$$

If $c > 2$, all the eigenvalues are different and there is a gap Δ between them!

If $c = 4$, the gap is $\Delta = 2$.

Moving control: to a moment problem

Direct equation:

$$v_{tt} - v_{xxtt} + 2cv_{xt} - 2cv_{txxx} + (c^2 - 1)v_{xx} - c^2 v_{xxxx} = b(x)h(t). \qquad (8)$$

Adjoint equation:

$$\varphi_{tt} - \varphi_{xxtt} + 2c\varphi_{xt} - 2c\varphi_{txxx} + (c^2 - 1)\varphi_{xx} - c^2 \varphi_{xxxx} = 0.$$

Let us multiply equation (8) by $\bar{\varphi}$ and integrate by parts on $[0, T] \times \mathbb{T}$ to obtain

$$\int_{\mathbb{T}} [v_t(\bar{\varphi} - \bar{\varphi}_{xx}) + v(-\bar{\varphi}_t - 2c\bar{\varphi}_x + \bar{\varphi}_{xxt} + 2c\bar{\varphi}_{xxx})]_0^T \, dx = \int_0^T \int_{\mathbb{T}} hb\bar{\varphi}dxdt \qquad (9)$$

Using, for $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, $\varphi(x,t) = e^{i\lambda^{\mp}_{-k}(T-t)}e^{ikx} = e^{-i\lambda^{\pm}_k(T-t)}e^{ikx}$, (9) becomes

$$\langle v_t(T) - e^{i\lambda_k^{\pm}T}v_t(0), e^{-ikx}\rangle + i(\lambda_k^{\pm} + 2ck)\langle v(T) - e^{i\lambda_k^{\pm}T}v(0), e^{-ikx}\rangle$$
$$= \frac{1}{1+k^2}\int_0^T h(t)e^{i\lambda_k^{\pm}(T-t)}dt \int_{\mathbb{T}} b(x)e^{-ikx}dx.$$

where $\langle f, e^{ikx}\rangle$ stands for the coordinate f_k in the decomposition $f = \sum_{k\in\mathbb{Z}} f_k e^{ikx}$.

Moving control: to a moment problem

For $k=0$, we take first $\varphi(x,t)=(T-t)$ and then $\tilde{\varphi}(x,t)=1$, to obtain

$$\langle v(T)-v(0),1\rangle - T\langle v_t(0),1\rangle = \int_0^T h(t)(T-t)dt \underbrace{\int_{\mathbb{T}} b(x)dx}_{=0} = 0.$$

$$\langle v_t(T)-v_t(0),1\rangle = \int_0^T h(t)dt \underbrace{\int_{\mathbb{T}} b(x)dx}_{=0} = 0.$$

We suppose that the data satisfy

$$\langle v(0),1\rangle = \langle v(T),1\rangle = \langle v_t(0),1\rangle = \langle v_t(T),1\rangle = 0,$$

and that for any $k\in\mathbb{Z}^*$, we have $b_k := \int_{\mathbb{T}} b(x)e^{ikx} \neq 0$, we can state the problem of finding a control function $h\in L^2(0,T)$ such that

$$\int_0^T h(t)e^{i\lambda_k^{\pm}(T-t)}dt = \frac{1+k^2}{b_k}\langle v_T^1 - e^{i\lambda_k^{\pm}T}v^1, e^{-ikx}\rangle + i(\lambda_k^{\pm}+2ck)\frac{1+k^2}{b_k}\langle v_T^0 - e^{-i\lambda_k^{\pm}T}v^0, e^{-ikx}\rangle, \qquad \forall k\in\mathbb{Z}^*,$$

Moving control: exact controllability, linear case

The previous trigonometric moment problem can be solved thanks to the behavior of the eigenvalues $\lambda_k^{\pm}$, by imposing the right hand side to define a sequence in $l^2(\mathbb{C})$.

To do that, we have to impose an asymptotic behavior for b_k and regularity!

Theorem (EC-Crépeau, 2015)

Let $s \geq 0$ and c be such that $|c| > 2$. Let $b = b(x)$ such that

$$b_k := \int_{\mathbb{T}} b(x) e^{ikx} dx \neq 0, \quad \forall k \in \mathbb{Z}^*, \quad b_0 = 0, \quad \text{and} \quad \left| \frac{k^{2-s}}{b_k} \right| \text{ bounded as } |k| \to \infty. \tag{10}$$

Then, for all $T > \frac{2\pi}{\Delta}$ and all (v^0, v^1), $(v_T^0, v_T^1) \in H^{s+1}(\mathbb{T}) \times H^s(\mathbb{T})$, such that

$$\int_{\mathbb{T}} v^0(x) dx = \int_{\mathbb{T}} v^1(x) dx = \int_{\mathbb{T}} v_T^0(x) dx = \int_{\mathbb{T}} v_T^1(x) dx = 0, \tag{11}$$

there exists a control $h \in L^2(0,T)$ such that the linear problem

$$\begin{cases} v_{tt} + (c^2-1) v_{xx} + 2c v_{xt} - v_{xxtt} - c^2 v_{xxxx} - 2c v_{txxx} = b(x) h(t), & x \in \mathbb{T},\ t > 0 \\ v(x,0) = v^0(x),\ v_t(x,0) = v^1(x), & x \in \mathbb{T}. \end{cases}$$

admits a unique solution $v \in C([0,T], H^{s+1}(\mathbb{T})) \cap C^1([0,T], H^s(\mathbb{T}))$ such that $v(x,T) = v_T^0(x)$ and $v_t(x,T) = v_T^1(x)$.

Moving control: exact controllability, nonlinear case

We decompose the solution of the nonlinear problem,

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx} + b(x+ct)h(t), & x \in \mathbb{T},\ t > 0, \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in \mathbb{T}, \end{cases} \tag{12}$$

in the following way, $y = \alpha + \beta + \gamma$, where α, β, γ satisfy

$$\begin{cases} \alpha_{tt} - \alpha_{xx} - \alpha_{xxtt} = 0, & x \in \mathbb{T},\ t > 0, \\ \alpha(x,0) = y^0(x),\ \alpha_t(x,0) = y^1(x), & x \in \mathbb{T}, \end{cases} \tag{13}$$

$$\begin{cases} \beta_{tt} - \beta_{xx} - \beta_{xxtt} = b(x+ct)h(t), & x \in \mathbb{T},\ t > 0, \\ \beta(x,0) = 0,\ \beta_t(x,0) = 0, & x \in \mathbb{T}, \end{cases} \tag{14}$$

$$\begin{cases} \gamma_{tt} - \gamma_{xx} - \gamma_{xxtt} = F, & x \in \mathbb{T},\ t > 0, \\ \gamma(x,0) = 0,\ \gamma_t(x,0) = 0, & x \in \mathbb{T}. \end{cases} \tag{15}$$

with $F = (y^2)_{xx}$.

Remark

We study the nonlinear problem in the following regularity framework:

$$(y_0, y_1) \in H_0^4(\mathbb{T}) \times H_0^3(\mathbb{T}),\ h \in L^2(0,T),\ F \in L^1(0,T; L_0^2(\mathbb{T})) \text{ and } b \in L^2(\mathbb{T}).$$

Moving control: exact controllability, nonlinear case

Let us consider the following maps, which are well-defined, linear and continuous

- $\psi_0 : (y^0, y^1) \in H_0^4(\mathbb{T}) \times H_0^3(\mathbb{T}) \longmapsto \alpha \in C([0,T], H_0^4(\mathbb{T})) \cap C^1([0,T], H_0^3(\mathbb{T}))$.
- $\psi_1 : h \in L^2(0,T) \longmapsto \beta \in C([0,T], H_0^4(\mathbb{T})) \cap C^1([0,T], H_0^3(\mathbb{T}))$.
- $\psi_2 : F \in L^1(0,T; L_0^2(\mathbb{T})) \longmapsto \gamma \in C([0,T], H_0^4(\mathbb{T})) \cap C^1([0,T], H_0^3(\mathbb{T}))$.
- $\Gamma : (y_T^0, y_T^1) \in H_0^4(\mathbb{T}) \times H_0^3(\mathbb{T}) \longmapsto h \in L^2(0,T)$ where h is the control such that β satisfies $(\beta(T), \beta_t(T)) = (y_T^0, y_T^1)$.

We see that a trajectory $y = y(x,t)$ going from (y^0, y^1) at time $t = 0$ to (y_T^0, y_T^1) at time $t = T$ is a fixed point of

$$\Pi : y \in C([0,T], H_0^4(\mathbb{T})) \cap C^1([0,T], H_0^3(\mathbb{T})) \mapsto \Pi(y) \in C([0,T], H_0^4(\mathbb{T})) \cap C^1([0,T], H_0^3(\mathbb{T}))$$

where $\Pi(y)$ is

$$\psi_0(y^0, y^1) + \psi_1 \circ \Gamma\Big((y_T^0, y_T^1) - \psi_0(y^0, y^1)(T) - (\psi_2((y^2)_{xx})(T), \psi_{2t}((y^2)_{xx})(T))\Big) + \psi_2((y^2)_{xx})$$

We conclude by applying the Banach Fixed Point Theorem!

Moving control: exact controllability, nonlinear case

Theorem (EC-Crépeau, 2015)

Let c be such that $|c| > 2$. Let $b = b(x)$ be such that (10) *holds with $s = 3$. Then, for all $T > \frac{2\pi}{\Delta}$, there exists $\epsilon > 0$ such that for any (y^0, y^1), $(y^0_T, y^1_T) \in H^4(\mathbb{T}) \times H^3(\mathbb{T})$ satisfying* (11) *and*

$$\|(y^0, y^1)\|_{H^4(\mathbb{T})\times H^3(\mathbb{T})} \leq \epsilon, \quad \|(y^0_T, y^1_T)\|_{H^4(\mathbb{T})\times H^3(\mathbb{T})} \leq \epsilon,$$

there exists a control $h \in L^2(0, T)$ such that

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx} + b(x+ct)h(t), & x \in \mathbb{T},\ t > 0, \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & x \in \mathbb{T}, \end{cases}$$

admits a unique solution $y \in C([0,T], H^4(\mathbb{T})) \cap C^1([0,T], H^3(\mathbb{T}))$ such that

$$y(x,T) = y^0_T(x), \quad \text{and} \quad y_t(x,T) = y^1_T(x).$$

Related papers:
[Micu 2001], [Rosier, Zhang 2013] Benjamin-Bona-Mahony equation (1d)
[Martin, Rosier, Rouchon 2013] Wave equation with structural damping (1d)
[Chaves-Silva, Rosier, Zuazua 2014] Multi-d wave equation with viscous Kelvin–Voigt and frictional damping

... that is all...

Thank you for your attention!