Control of fourth-order parabolic control systems

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Control of PDEs - CNAM, Paris
April 1st, 2014

Control - Kuramoto-Sivashinsky equation

Let $\gamma > 0$.

$$\begin{aligned} y_t + y_{xxxx} + \gamma y_{xx} + y y_x &= 0, \\ y(t,0) = h_1(t), \quad y(t,1) &= 0, \\ y_x(t,0) = h_2(t), \quad y_x(t,1) &= 0, \\ y(0,x) &= y_0(x). \end{aligned}$$

State: $y(t,.) : [0,1] \to \mathbb{R}$

Controls: $h_1(t), h_2(t) \in \mathbb{R}$

Well-posedness:

$$\begin{array}{l} y_0 \in H^{-2}(0,1) \\ h_1, h_2 \in L^2(0,T) \\ \text{data small} \end{array} \implies \left\{ \begin{array}{l} \exists! \text{ solution} \\ y \in C([0,T], H^{-2}(0,1)) \cap L^2(0,T; L^2(0,1)) \end{array} \right.$$

Control - Linear Kuramoto-Sivashinsky equation

Question: What happens if $h_1 = h_2 = 0$?

Let us consider the linear equation (drop the term yy_x).

$$A : w \in D(A) \subset L^2(0,1) \longmapsto -w'''' - \gamma w'' \in L^2(0,1),$$
$$D(A) := H^4(0,1) \cap H_0^2(0,1).$$

A is self-adjoint with compact resolvent. The spectrum of A is a discrete subset $\{\sigma_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}$ satisfying $\lim_{k\to\infty} \sigma_k = -\infty$

The eigenfunctions $\{\phi_k\}_{k\in\mathbb{N}}$ form a basis of $L^2(0,1)$.

$$\begin{cases} -\gamma\phi_k'' - \phi_k'''' = \sigma_k \phi_k, \\ \phi_k(0) = \phi_k(1) = \phi_k'(0) = \phi_k'(1) = 0. \end{cases}$$

Control - Linear Kuramoto-Sivashinsky equation

The operator A has some positive eigenvalues IFF $\gamma > 4\pi^2$.

If $\gamma < 4\pi^2$, then all the eigenvalues are negative and therefore the linear system is asymptotically stable. The nonlinear KS equation is also asymptotically stable [Liu-Krstic, 2001].

We are interested here in the unstable case $\gamma > 4\pi^2$. We first focus in the linear system dropping the nonlinear term yy_x.

Control - Linear Kuramoto-Sivashinsky equation

Question: [Null controllability]

Let $T > 0$. Given a state $y_0 \in H^{-2}(0,1)$. Does there exist some controls $h_1, h_2 \in L^2(0,T)$ *driving the solution of KS equation from* y_0 *to* 0*?*

Let $z_T \in H_0^2(0,1)$. *Let* z *be the solution of the adjoint equation*

$$\begin{cases} -z_t + z_{xxxx} + \gamma z_{xx} = 0, \\ z(t,0) = 0, \quad z(t,1) = 0, \\ z_x(t,0) = 0, \quad z_x(t,1) = 0, \quad z(T,\cdot) = z_T. \end{cases}$$

We get

$$\langle y_0, z(0)\rangle_{-2,2} = \langle y(T), z_T\rangle_{-2,2} + \int_0^T z_{xx}(t,0)h_2(t) - \int_0^T z_{xxx}(t,0)h_1(t)$$

By duality, the KS equation is null-controllable if we prove

$$\forall z_T, \ \|z(0,x)\|_{H_0^2(0,1)} \leq C \left\{\|z_{xx}(t,0)\|_{L^2(0,T)} + \|z_{xxx}(t,0)\|_{L^2(0,T)}\right\}$$

where z *is the solution of the adjoint equation.*

Control - Linear Kuramoto-Sivashinsky equation

We have to prove the previous inequality.
If $z_T = \phi_k$, then $z(t,x) = \phi_k e^{(T-t)\sigma_k}$.

$$e^{T\sigma_k} \|\phi_k\|_{H^2(0,L)} \leq C \left\{ |\phi_k''(0)| \cdot \|e^{(T-t)\sigma_k}\|_{L^2(0,T)} + |\phi_k'''(0)| \cdot \|e^{(T-t)\sigma_k}\|_{L^2(0,T)} \right\}$$

Lemma

It holds for any ϕ_k with C_k.
Moreover, thanks to the asymptotic behavior of σ_k, then C is independent of k. Thus, it holds for any $z_T \in L^2(0,1)$.

Remark

The inequality does not hold for any γ with one term at the right-hand side. The system is not always null controllable with only one control.

Control - Linear Kuramoto-Sivashinsky equation

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = 0, \\ y(t,0) = 0, \quad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \quad y_x(t,1) = 0, \\ y(0,\cdot) = y_0. \end{cases}$$

The inequality to prove now is

$$e^{T\sigma_k} \|\phi_k\|_{H^2(0,1)} \leq C|\phi_k''(0)| \cdot \|e^{(T-t)\sigma_k}\|_{L^2(0,T)}$$

Counterexample

If $\gamma = 20\pi^2$, we find out

$$\phi = -2\sin(2\pi x) + \sin(4\pi x), \quad \sigma = 64\pi^4.$$

This eigenfunction satisfies $\phi''(0) = 0$.

Control - Linear Kuramoto-Sivashinsky equation

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = 0, \\ y(t,0) = 0, \quad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \quad y_x(t,1) = 0, \\ y(0,\cdot) = y_0. \end{cases}$$

The inequality to prove is now

$$e^{T\sigma_k} \|\phi_k\|_{H^2(0,1)} \le C|\phi_k''(0)| \cdot \|e^{(T-t)\sigma_k}\|_{L^2(0,T)}$$

Theorem (C, 2010)

The linear system is null controllable with one control h_2 if and only if

$$\gamma \notin \{\pi^2(k^2+l^2);\ k,l \in \mathbb{N}, 1 \le k < l,\ (k+l) \text{ is even}\}$$

Control - Linear Kuramoto-Sivashinsky equation

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = 0, \\ y(t,0) = h_1(t), \quad y(t,1) = 0, \\ y_x(t,0) = 0, \quad y_x(t,1) = 0, \\ y(0,\cdot) = y_0. \end{cases}$$

The inequality to prove now is

$$e^{T\sigma_k} \|\phi_k\|_{H^2(0,1)} \leq C|\phi_k'''(0)| \cdot \|e^{(T-t)\sigma_k}\|_{L^2(0,T)}$$

Counterexample

If $\gamma = 4\pi^2$, we find out

$$\phi = 1 - \cos(2\pi x), \quad \sigma = 0.$$

This eigenfunction satisfies $\phi'''(0) = 0$.

Control - Linear Kuramoto-Sivashinsky equation

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = 0, \\ y(t,0) = h_1(t), \quad y(t,1) = 0, \\ y_x(t,0) = 0, \quad y_x(t,1) = 0, \\ y(0,\cdot) = y_0. \end{cases}$$

The inequality to prove is now

$$e^{T\sigma_k}\|\phi_k\|_{H^2(0,L)} \leq C|\phi_k'''(0)|^2 \|e^{(T-t)\sigma_k}\|_{L^2(0,T)}$$

Theorem (C-Guzmán-Mercado, in preparation)

The linear system is null controllable with one control h_1 if and only if

$$\lambda \notin \left\{(j^2+k^2)\pi^2 \ / \ (j,k) \in \mathbb{N}^2 \text{ with the same parity and } j<k\right\} \cup \left\{4l^2\pi^2 \ / \ l \in \mathbb{N}\right\}.$$

Control - Kuramoto-Sivashinsky equation

Remark

This spectral approach allows us to know if the linear KS equation is null controllable with two or one control but it is not good enough to deal with low order terms or non-constant coefficient or nonlinearities.

We have to consider the system

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} + a(x)y = g, \\ y(t,0) = h_1(t), \quad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \quad y_x(t,1) = 0, \\ y(0,\cdot) = y_0. \end{cases}$$

Remark

The source term g is important to put $g = -yy_x$ later.

Control - Kuramoto-Sivashinsky equation

By duality, we have to find spaces Y, F, U_1, U_2 such that

$$\forall z_T,\ \|z(0,x)\|_{H_0^2} + \|z\|_Y \leq C\left\{\|z_{xx}(t,0)\|_{U_1} + \|z_{xxx}(t,0)\|_{U_2} + \|f\|_F\right\}$$

where z is the solution of the adjoint equation

$$\begin{cases} -z_t + z_{xxxx} + \gamma z_{xx} + a(x)z = f, \\ z(t,0) = 0, \quad z(t,1) = 0, \\ z_x(t,0) = 0, \quad z_x(t,1) = 0, \\ z(T,\cdot) = z_T. \end{cases}$$

Control - Kuramoto-Sivashinsky equation

Proposition (C-Mercado, 2011)

There exist $k_1, k_2, C > 0$ *such that*

$$\|ze^{-\frac{k_2}{T-t}}\|_{L^\infty(W^{1,\infty})} + \int_0^1 |z_{xx}(0,x)|^2 \leq C\Big\{ \iint |f|^2 e^{-\frac{2k_1}{T-t}} + \int_0^T |z_{xx}(t,0)|^2 e^{\frac{-2k_1}{T-t}}(T-t)^{-3} + \int_0^T |z_{xxx}(t,0)|^2 e^{\frac{-2k_1}{T-t}}(T-t)^{-1} \Big\}$$

$\forall z_T \in H_0^2$, $\forall f \in L^2_{xt}(e^{\frac{-2k_1}{(T-t)}}) =: F$.

$$\begin{cases} -z_t + z_{xxxx} + \gamma z_{xx} + a(x)z = f, \\ z(t,0) = 0, \quad z(t,1) = 0, \\ z_x(t,0) = 0, \quad z_x(t,1) = 0, \\ z(T,\cdot) = z_T. \end{cases}$$

$L^2(\rho) := \{h \in L^2; \int |h(y)|^2 \rho(y)\, dy < \infty\}$

Control - Kuramoto-Sivashinsky equation

Theorem (C-Mercado, 2011)

For each $g \in \{g; ge^{\frac{k_2}{T-t}} \in L^1(W^{-1,1})\} \subset Y^*$ *and* $y_0 \in H^{-2}(0,1)$ *there exist controls*

$$h_1 \in L^2_t\left(e^{\frac{2k_1}{T-t}}(T-t)\right), \; h_2 \in L^2_t\left(e^{\frac{2k_1}{T-t}}(T-t)^3\right)$$

such that the solution of the linear KS equation satisfies

$$y(T) = 0 \quad \text{and} \;\; y \in L^2_{xt}(e^{\frac{2k_1}{T-t}}).$$

Remark

$ye^{\frac{k_1}{T-t}} \in L^2_{xt}$ *IFF* $y^2e^{\frac{2k_1}{T-t}} \in L^1_{xt}$ *IFF* $(y^2)_xe^{\frac{2k_1}{T-t}} \in L^1(0,T;W^{-1,1})$.

BUT k_1, k_2 *can be chosen such that* $k_2 < 2k_1$.

Thus, if $ye^{\frac{k_1}{T-t}} \in L^2_{xt}$, *then we have* $yy_xe^{\frac{k_2}{T-t}} \in L^1(W^{-1,1})$.

Control - Kuramoto-Sivashinsky equation

By applying an inverse function argument:

Theorem (C-Mercado, 2011)

The KS equation is locally null controllable, i.e. for any $g \in Y^*$ *and* $y_0 \in H^{-2}(0,1)$ *small enough (in norm), there exist controls*

$$h_1 \in L^2_t\left(e^{\frac{2k_1}{T-t}}(T-t)\right), \; h_2 \in L^2_t\left(e^{\frac{2k_1}{T-t}}(T-t)^3\right)$$

such that the solution of

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} + y y_x = g, \\ y(t,0) = h_1(t), \quad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \quad y_x(t,1) = 0, \\ y(0,\cdot) = y_0. \end{cases}$$

satisfies $y(T) = 0$.

Neumann BC - Linear Kuramoto-Sivashinsky equation

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = 0, \\ y_{xx}(t,0) = h_1(t), \;\; y_{xx}(t,1) = 0, \\ y_{xxx}(t,0) = h_2(t), \;\; y_{xxx}(t,1) = 0, \\ y(0,x) = y_0(x) \; . \end{cases}$$

Theorem (C-Guzmán-Mercado, in preparation)

(a) *If $h_2 = 0$ and the control is h_1, then you cannot control any state $u_0 \in L^2(0,1)$ such that*

$$\int_0^1 z_0(x) \cos\left(\sqrt{\gamma} x\right) dx \neq 0.$$

(b) *If $h_1 = 0$ and the control is h_2, then you cannot control any state $u_0 \in L^2(0,1)$ such that*

$$\int_0^1 z_0(x) \sin\left(\sqrt{\gamma} x\right) dx \neq 0.$$

(c) *With two controls h_1, h_2, the system is null-controllable.*

A related Inverse Problem for KS

The KS equation with non-constant coefficients describing the diffusion $\sigma = \sigma(x)$, and the anti-diffusion $\gamma = \gamma(x)$, is given as

$$\begin{cases} y_t + (\sigma(x) y_{xx})_{xx} + \gamma(x) y_{xx} + y y_x = g, \\ y(t,0) = h_1(t), \quad y(t,1) = h_2(t), \\ y_x(t,0) = h_3(t), \quad y_x(t,1) = h_4(t), \\ y(0,x) = y_0(x), \end{cases}$$

where the domain is $(0,T) \times (0,1)$, and the functions y_0, g, h_j are assumed to be known.

We consider the inverse problems of retrieving by separate the diffusion coefficient σ and the anti-diffusion coefficient γ from partial measurements of the solution.

This corresponds for instance to getting information on the instability of a reaction-diffusion media by measuring a single solution, which could represent a flame propagating on the domain.

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A related Inverse Problem for KS

$$\begin{cases} y_t + (\sigma(x)y_{xx})_{xx} + \gamma(x)y_{xx} + yy_x = g, & \forall (t,x) \in Q, \\ y(t,0) = h_1(t), \quad y(t,1) = h_2(t), & \forall t \in (0,T), \\ y_x(t,0) = h_3(t), \quad y_x(t,1) = h_4(t), & \forall t \in (0,T), \\ y(0,x) = y_0(x), & \forall x \in (0,1), \end{cases}$$

Inverse Problem

Can we recover $\sigma = \sigma(x)$ *(or* $\gamma = \gamma(x)$*) from some partial knowledge of* $y = y(x,t)$*?*

Inverse Problem (Uniqueness)

Given some measurements $meas(y)$*, is there a unique* $\sigma = \sigma(x)$ *(or* $\gamma = \gamma(x)$*) associated?*

i.e., $\quad meas(y) = meas(\tilde{y}) \implies \sigma = \tilde{\sigma}?$

Inverse Problem (Stability)

$$\|\sigma - \tilde{\sigma}\|_X \le C\|meas(y) - meas(\tilde{y})\|_Y?$$

$$\|\gamma - \tilde{\gamma}\|_X \le C\|meas(y) - meas(\tilde{y})\|_Y?$$

A related Inverse Problem for KS

$$\begin{cases} y_t + (\sigma(x)y_{xx})_{xx} + \gamma(x)y_{xx} + yy_x = g, & \forall (t,x) \in Q, \\ y(t,0) = h_1(t), \quad y(t,1) = h_2(t), & \forall t \in (0,T), \\ y_x(t,0) = h_3(t), \quad y_x(t,1) = h_4(t), & \forall t \in (0,T), \\ y(0,x) = y_0(x), & \forall x \in (0,1), \end{cases}$$

For parabolic equations, there appears measurements like

$$\|y(x,T_0) - \tilde{y}(x,T_0)\|_{H^m(0,L)}$$

for a given $T_0 \in (0,T)$.

We apply the classical approach:

1. The Bukhgeim-Klibanov-Malinsky method.
2. Carleman estimate for the linearized equation.

A related Inverse Problem for KS

Theorem (Baudouin-C-Crépeau-Mercado, 2013, and Guzmán 2013)

(Under some technical conditions)
$\forall M_k > 0$, $\exists C_k > 0$ *depending on* (T, m_k, M_k), *such that*

- *For every* γ *with* $\|\gamma\|_{L^\infty(0,1)} \leq m_1$,

$$\|\gamma - \tilde{\gamma}\|^2_{L^2(0,1)} \leq C_1 \|y_{xx}(\cdot,0) - \tilde{y}_{xx}(\cdot,0)\|^2_{H^1(0,T)} + C_1 \|y_{xxx}(\cdot,0) - \tilde{y}_{xxx}(\cdot,0)\|^2_{H^1(0,T)} + C_1 \|y(T_0,\cdot) - \tilde{y}(T_0,\cdot)\|^2_{H^4(0,1)}$$

  for all y *satisfying* $\|y\|_{H^1(0,T,H^4(0,L))} \leq M_1$.

- *For every* σ *with* $\|\sigma\|_{H^4(0,1)} \leq m_2$,

$$\|\sigma - \tilde{\sigma}\|^2_{L^2(0,1)} \leq C_2 \|y_{xx}(\cdot,0) - \tilde{y}_{xx}(\cdot,0)\|^2_{H^1(0,T)} + C_2 \|y_{xxx}(\cdot,0) - \tilde{y}_{xxx}(\cdot,0)\|^2_{H^1(0,T)} + C_2 \|y(T_0,\cdot) - \tilde{y}(T_0,\cdot)\|^2_{H^4(0,1)}$$

  for all y *satisfying* $\|y\|_{H^1(0,T,H^4(0,L))} \leq M_2$.

Stabilized Kuramoto-Sivashinsky system

This parabolic system models front propagation in reaction-diffusion phenomena and combines dissipative with dispersive features and simultaneously supports stable solitary-pulse.

$$\begin{cases} u_t + u_{xxxx} + uu_x = v_x + h\mathbf{1}_\omega, \\ v_t - \Gamma v_{xx} = u_x, \end{cases}$$

$$\begin{cases} u(0,t) = 0, \quad u(1,t) = 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, \\ v(0,t) = 0, \quad v(1,t) = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \end{cases}$$

It was proposed in [Malomed-Feng-Kawahara, Phys. Rev. E, 2001]

Stabilized Kuramoto-Sivashinsky system

$$\begin{cases} -\varphi_t + \varphi_{xxxx} = -\psi_x, \\ -\psi_t - \Gamma\psi_{xx} = -\varphi_x. \end{cases}$$

Adding up both Carleman estimates and absorbing cross terms we obtain

$$\|\psi\|^2_{L^2_\rho(H^2)} + \|\varphi\|^2_{L^2_\rho(H^4)} \leq C\left\{\int\int_\omega |\psi|^2 + \int\int_\omega |\varphi|^2\right\}$$

If ω touches the boundary, by Poincaré:

$$\cdots \leq \int\int_\omega |\psi|^2 + \int\int_\omega |\varphi|^2 \leq C\left\{\int\int_\omega |\psi_x|^2 + \int\int_\omega |\varphi|^2\right\}$$

$$\int\int_\omega \psi_x\psi_x = \int\int_\omega \psi_x(\varphi_t - \varphi_{xxxx}) \approx \int\int_\omega \psi_x(\varphi_t - \Gamma\varphi_{xx}) + \int\int_\omega \psi_x(\Gamma\varphi_{xx} - \varphi_{xxxx})$$

$$\cdots \leq \int\int_\omega (\psi_t + \Gamma\psi_{xx})\varphi_x + \int\int_\omega \psi_x(\Gamma\varphi_{xx} - \varphi_{xxxx})$$

$$\cdots \leq \int\int_\omega |\varphi_x|^2 + \epsilon\int\int_\omega |\psi_x|^2 + |\psi_{xx}|^2 + \frac{1}{\epsilon}\int\int_\omega |\varphi_x|^2 + |\varphi_{xx}|^2 + |\varphi_{xxx}|^2$$

Stabilized Kuramoto-Sivashinsky system

If ω does not touch the boundary, we need a Carleman estimate for ψ_x for which we do not have boundary conditions.

We applied [FernándezCara-GonzálezBurgos-Guerrero-Puel, 2006]
(Carleman heat equation without boundary data).

and we closely follow [Guerrero 2007] (parabolic system second-order coupling)

Theorem (C-Mercado-Pazoto, under review)

Then, there exist $\lambda_0, s_0, C > 0$ *such that*

$$s\lambda^2 \iint e^{-2s\alpha}|\psi_{2x}|^2 + s^3\lambda^4 \iint e^{-2s\alpha}|\psi_x|^2 \leq Cs\lambda^2 \iint e^{-2s\alpha}|\varphi_x|^2 + Cs\lambda^2 \iint e^{-2s\alpha}|\varphi_{xx}|^2 + s^3\lambda^4 \iint_\omega e^{-2s\alpha}|\psi_x|^2$$

for every $s \geq s_0,\ \lambda \geq \lambda_0$ *and any solution of* $(-\psi_t - \Gamma\psi_{xx} = -\varphi_x)$.

Stabilized Kuramoto-Sivashinsky system

Theorem (C-Mercado-Pazoto, under review)

Let $T > 0$ *and* ω *any nonempty open subset of* $(0,1)$. *There exists* $\delta > 0$ *such that for any* $(u_0, v_0) \in H^{-2}(0,1) \times H^{-1}(0,1)$ *with*

$$\|u_0\|_{H^{-2}(0,1)} + \|v_0\|_{H^{-1}(0,1)} < \delta,$$

we can find a control $h \in L^2(0,T;L^2(\omega))$ *such that the corresponding solution*

$$(u,v) \in C([0,T], H^{-2}(0,1) \times H^{-1}(0,1)) \cap L^2(0,T; L^2(0,1) \times L^2(0,1))$$

of Stabilized KS system satisfies

$$u(\cdot,T) = v(\cdot,T) = 0.$$

Stabilized Kuramoto-Sivashinsky system

$$\begin{cases} u_t + u_{xxxx} + uu_x = v_x, \\ v_t - \Gamma v_{xx} = u_x + h\mathbf{1}_\omega, \end{cases}$$

$$\begin{cases} u(0,t) = 0, \quad u(1,t) = 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, \\ v(0,t) = 0, \quad v(1,t) = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \end{cases}$$

Key steps. A new Carleman estimate for KS with nonhomogeneous data and then:

Proposition (Carreño-C, in preparation)

Let $\omega \subset (0,1)$. $\exists\ \lambda_0, s_0 > 0$ *and* $C > 0$ *(depending only on* ω, T*) such that* $\forall\ \lambda \geq \lambda_0$, $s \geq s_0$, *any solution* φ *of* $(-\varphi_t + \varphi_{xxxx} = -\psi_x + g_0)$ *satisfies*

$$s^7\lambda^8 \iint_Q \left(e^{-2s\alpha}|\varphi_x|^2 + e^{-2s\alpha^*}|\varphi|^2\right) dx\, dt + \|s^{5/2}\lambda^4 e^{-s\alpha^*}\varphi\|^2_{L^2(0,T;H^4(0,1))}$$

$$\leq C \iint_Q e^{-2s\alpha}\left(s^5\lambda^8|\psi_x|^2 + |\psi_{xx}|^2 + s^5\lambda^8|g_0|^2\right) dx\, dt$$

$$+ Cs^7\lambda^8 \iint_{\omega\times(0,T)} e^{-2s\alpha}|\varphi_x|^2\, dx\, dt.$$

Open Problems

- Nonlinear KS with only one control (h_1 or h_2). Harder if γ is critical
- Inverse problem with two unknown coefficients
- Boundary control for the system with less than 3 control inputs
- KS equation: discontinuous main coefficient, singular optimal control, cost of control, degenerated coefficients, ...
- Dimension higher than 1 for the KS equation
- Dimension higher than 1 for the stabilized KS system

Thank you for your attention!